I. (30 points)

(1) Write formally the following statement: “For all strictly positive real numbers $x$ and $y$, the sum $x/y + y/x$ is at least 2.”

(2) Write the formal negation of the statement in (1) above.

(3) Prove or disprove the statement in (1) above.

Let $x, y \in \mathbb{R}^+$.

Then \[
\frac{x}{y} + \frac{y}{x} - 2 = \frac{x^2 + y^2 - 2xy}{xy} = \frac{(x - y)^2}{xy}.
\]

Since $x, y \in \mathbb{R}^+$, $(x - y)^2 \geq 0$ and $xy > 0$.

Therefore $\frac{(x - y)^2}{xy} \geq 0$. Therefore

\[
\frac{x}{y} + \frac{y}{x} - 2 \geq 0.
\]

Thus $\frac{x}{y} + \frac{y}{x} \geq 2$. \qed
II. (30 points)

(1) Write formally the following statement: "If \( x \) is an irrational real number, then \( x^2 + x \) is an irrational real number."

(2) Prove or disprove the statement in (1) above.

\[
(1) \quad x \in \mathbb{R} \\
(\forall x) (x \notin \mathbb{Q}) \rightarrow (x^2 + x \notin \mathbb{Q}).
\]

(2) The statement in (1) is false. We will show this by proving that its negation is true. The negation of the statement above is

\[
(\exists x) (x \notin \mathbb{Q}) \land (x^2 + x \in \mathbb{Q}).
\]

Proof.

Let \( x = \frac{-1 + \sqrt{5}}{2} \). We claim that \( (x \notin \mathbb{Q}) \) and \( (x^2 + x \notin \mathbb{Q}) \).

Step 1. We show that \( \frac{-1 + \sqrt{5}}{2} \notin \mathbb{Q} \) by contradiction.

Assume that \( \frac{-1 + \sqrt{5}}{2} \in \mathbb{Q} \). Then

\[
\sqrt{5} = 2 \cdot \frac{-1 + \sqrt{5}}{2} + 1 \in \mathbb{Q}.
\]

Therefore, there exist \( m, n \in \mathbb{N} \) with \( \gcd(m, n) = 1 \) such that \( \sqrt{5} = \frac{m}{n} \). Consequently

\[
5n^2 = m^2.
\]

Therefore \( 5 \mid m^2 \Rightarrow 5 \mid m \Rightarrow \exists k \in \mathbb{Z}, m = 5k.\)
Therefore \( 5n^2 = 25k^2 \).

Consequently \( n^2 = 5k^2 \Rightarrow 5|n^2 \Rightarrow 5|n \).

Thus \( 5|m^2 \Rightarrow 5|gcd(m,n) \Rightarrow 5|1 \).

\[ \frac{5}{n} \quad gcd(m,n) = 1 \]

This is false.

Consequently \( \sqrt{5} \neq \frac{-1 + \sqrt{5}}{2} \in \mathbb{Q} \).

**Step 2**

\[
\left( \frac{-1 + \sqrt{5}}{2} \right)^2 + \left( \frac{-1 + \sqrt{5}}{2} \right) = \frac{1 - 2\sqrt{5} + 5}{4} + \frac{-1 + \sqrt{5}}{2} = \]

\[ = \frac{1 - 2\sqrt{5} + 5 + (4\sqrt{5})}{4} = \]

\[ = \frac{4}{4} = 1 \in \mathbb{Q} . \]

\[ \Box \]
III. (40 points)

(1) Show that the following propositional expression is a tautology.

\[ (P \rightarrow Q) \rightarrow ((P \lor Q) \leftrightarrow Q) \]

(2) Use the tautology in (1) above to show that if \( A \) and \( B \) are two subsets of a universal set \( U \) such that \( A \subseteq B \), then \( A \cup B = B \).

(3) Prove or disprove the following statement: "If \( A \) and \( B \) are two subsets of a universal set \( U \), then \( \mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B) \)."

(1) Draw the truth table for

\[ (P \rightarrow Q) \rightarrow ((P \lor Q) \leftrightarrow Q) \]

and conclude that this propositional expression is true for all possible truth values of the propositional variables \( P \) and \( Q \).

(2) Let \( x \in U \). Let \( P(x) \) and \( Q(x) \) be the propositions

\[ P(x) : x \in A \quad \text{and} \quad Q(x) : x \in B. \]

Please note that

\[ (A \subseteq B) \leftrightarrow (\forall x)(P(x) \rightarrow Q(x)). \]

\[ (A \cup B = B) \leftrightarrow (\forall x)(P(x) \lor Q(x)) \leftrightarrow Q(x). \]

Therefore, we have the equivalence:

\[ (A \subseteq B) \rightarrow (A \cup B = B) \leftrightarrow (\forall x)(P(x) \rightarrow Q(x)) \rightarrow ((P(x) \lor Q(x)) \leftrightarrow Q(x)) \]

Now, the right hand side is true as a consequence of (1). Consequently, the left hand side is also true. Therefore:

\[ (A \subseteq B) \rightarrow (A \cup B = B). \]
(3) Formally, the statement is:

\[(\forall A)(\forall B) \notin \mathcal{F}(A \cup B) = \notin \mathcal{F}(A) \cup \mathcal{F}(B)\]

where \(A, B\) are subsets of the universal set \(U\).

The statement above is \underline{false}. We will show this by proving that its negation is \underline{true}. Its negation is equivalent to the following:

\[(\exists A)(\exists B) \notin \mathcal{F}(A \cup B) \neq \notin \mathcal{F}(A) \cup \mathcal{F}(B)\]

**Proof.**

Let \(U = \mathbb{N}\), and let \(A = \{1, 2, 3\}\), \(B = \{4, 5, 6\}\). Then \(A \cup B = \{1, 2, 3, 4, 5, 6\}\). Consequently,

\[\{3, 4\} \subseteq A \cup B \Rightarrow \{3, 4\} \notin \mathcal{F}(A \cup B) \quad (*)\]

However \[\{3, 4\} \notin A\] and \[\{3, 4\} \notin B\]. Therefore \[\{3, 4\} \notin \mathcal{F}(A)\] and \[\{3, 4\} \notin \mathcal{F}(B)\]. Consequently \[\{3, 4\} \notin \mathcal{F}(A) \cup \mathcal{F}(B) \quad (\neq)\]

\[\Rightarrow \mathcal{F}(A \cup B) \neq \mathcal{F}(A) \cup \mathcal{F}(B). \quad (\neq)\]

\[\Rightarrow \mathcal{F}(A \cup B) \neq \mathcal{F}(A) \cup \mathcal{F}(B). \quad \square\]