Solutions to Math 21C Final, Winter 02.

1. Two vectors for two of the sides of the triangle are \( \mathbf{a} = \langle 1, 2, 1 \rangle \) and \( \mathbf{b} = \langle 2, -2, 3 \rangle \). The area of the triangle is \( |\mathbf{a} \times \mathbf{b}|/2 \). We have \( \mathbf{a} \times \mathbf{b} = \ldots = 8\mathbf{i} - \mathbf{j} - 6\mathbf{k} \) and \( |\mathbf{a} \times \mathbf{b}| = \sqrt{8^2 + 1 + 6^2} = \sqrt{101} \).

2. The curves intersect when \( (t, t^2, t^3) = \langle 1 + s, 4s, 8s^2 \rangle \) for some \( t \) and \( s \). This means that \( t = 1 + s \) and \( t^2 = 4s = (t-1) \) so \( t^2 - 4t + 4 = 0 \) and hence \( (t-2)^2 = 0 \), i.e. \( t = 2 \) and hence \( s = 1 \). It is easy to check that also the last equation is satisfied for these values. We have \( \mathbf{r}'_1(t) = \langle 1, 2t, 3t^2 \rangle \) so \( \mathbf{r}'_1(2) = \langle 1, 4, 12 \rangle \) and \( \mathbf{r}'_2(s) = \langle 1, 4, 16s \rangle \) so \( \mathbf{r}'_2(1) = \langle 1, 4, 16 \rangle \). The angle is given by \( \cos \theta = \mathbf{r}'_1(2) \cdot \mathbf{r}'_2(1)/(|\mathbf{r}'_1(2)| |\mathbf{r}'_2(1)|) = 209/(\sqrt{1 + 4^2 + 12^2} \cdot \sqrt{1 + 4^2 + 16^2}) = 209/\sqrt{161^2 \cdot 273}. \)

3. The vector \( \mathbf{n} = \langle 2, 1, 4 \rangle \) between the points is normal to the plane and the point in the middle between the points \( P = (0, 3/2, 1) \) is on the plane. The equation of the plane is therefore \( 2(x - 0) + 1(y - 3/2) + 4(z - 1) = 0 \).

4. \( \mathbf{r}'(t) = 3t^{1/2}\mathbf{i} - 2 \sin 2t \mathbf{j} + 2 \cos 2t \mathbf{k} \). The initial speed is \( |\mathbf{r}'(0)| = |2\mathbf{k}| = 2 \).

5. The tangent plane to \( F(x, y, z) = x^2 + y^2/4 + z^2/9 = 1 \) at a point \( (x_0, y_0, z_0) \) has normal \( \nabla F(x_0, y_0, z_0) = (2x_0, y_0/2, 2z_0/9) \). This vector is parallel to the normal of the plane \( x+y-z=0 \), which is \((1, 1, -1)\), if \( 2x_0 = \lambda, y_0/2 = \lambda \) and \( 2z_0/9 = -\lambda \). Since the point also must lie on the surface we must have \( F(\lambda/2, 2\lambda, -9\lambda/2) = \lambda^2/4 + \lambda^2 + 9\lambda^2/4 = 7\lambda^2/2 = 1 \) so \( \lambda = \pm \sqrt{2/7} \). The point is \((x_0, y_0, z_0) = \pm \sqrt{2/7}(1, 2, -9/2) \).

6. \( f_x(x, y) = 4x^3 - 8y = 0 \) and \( f_y(x, y) = -8x + 4y = 0 \) gives \( y = 2x \) and \( 4x(x^2 - 4) = 0 \). Hence \( x = 0 \) or \( x = \pm 2 \) so the critical points are \((0, 0), (2, 4), (-2, -4)\).

\( f_{xx} = 12x^2, f_{xy} = -8, f_{yy} = 4 \) so \( D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 48x^2 - 64 \). Then \( D(0, 0) < 0 \) so \((0, 0)\) is a saddle point. \( D(2, 4) = 128 > 0 \) and \( f_{xx}(2, 4) = 48 > 0 \) so \((2, 4)\) is local min, \( D(-2, -4) = 128 > 0 \) and \( f_{xx}(-2, -4) = 48 > 0 \) so \((-2, -4)\) is local min.

7. Minimize the area \( A = xy + 2xz + 2yz \), subject to the constraint that the volume is \( V = xyz = 32,000 \). Lagrange multiplies: \( \nabla A(x, y, z) = \langle y+z, x+2z, 2x+y \rangle \) and \( \nabla V(x, y, z) = \langle y, z, x \rangle \) so we must find all \((x, y, z)\) and \( \lambda \) such that \( y + 2z = \lambda yz, x + 2z = \lambda xz, 2x + 2y = \lambda xy \) and \( V(x, y, z) = 32,000 \). Multiplying the first equation by \( x \), the second by \( y \) and the third by \( z \) we get \( x(y + 2z) = y(x + 2z) = z(2x + 2y) \).

Subtracting the first two equations gives \( 2z(x - y) = 0 \) and subtracting the first and third gives \( (x - 2z)y = 0 \). If \( z = 0 \) then \( y = 0 \) or \( x = 0 \). If \( z \neq 0 \) then \( x = y = 0 \) or \( x = y = 2z \). Hence we have the points \((x, 0, 0), (0, y, 0), (0, 0, z)\) and \((2z, 2z, z)\). Only the last one gives \( V \neq 0 \) and we must have \( V(2z, 2z, z) = 4z^3 = 32,000 \) which is equivalent to \( z = 20 \) so \((x, y, z) = (40, 40, 20)\).

8. Let \( D = \{(x, y); 10 - 3x^2 - 3y^2 \geq 4\} = \{(x, y); x^2 + y^2 \leq 2\} \).

The volume is \( \int \int_D z \, dA = \int \int_D (10 - 3x^2 - 3y^2) \, dA = \int_0^{2\pi} \int_0^2 (10 - 3r^2) \, rdr \, d\theta = \int_0^{2\pi} (5r^2 - 3r^4/4) \, d\theta = \int_0^{2\pi} 7 \, d\theta = 14\pi. \)

\( 9. \int \int_T \sqrt{1 + 1 + 4y^2} \, dA = \int_0^1 \int_0^y \sqrt{2 + 4y^2} \, dx \, dy = \int_0^1 \sqrt{2 + 4y^2} \, dx \Big|_{y=0}^y \, dy = \int_0^1 \frac{\sqrt{2 + 4y^2} \, dy}{2} \left(2 + 4y^2\right)^{3/2}/12 \Big|_0^1 = (63/2 - 23/2)/12. \)

\( 10. E = \{(x, y, z); x \geq 0, y \geq 0, z \geq 0, x + y + z/2 \leq 1\} = \{(x, y, z); 0 \leq z \leq 2, 0 \leq y \leq 1 - z/2, 0 \leq x \leq 1 - y - z/2\}. \)

\( \int \int \int_E y \, dV = \int_0^1 \int_0^{1-z/2} \int_0^{y-z/2} y \, dx \, dy \, dz = \int_0^1 \int_0^{1-z/2} \frac{y(1 - y - z/2)}{2} \, dy \, dz = \int_0^1 \int_0^{1-z/2} \frac{y^2}{2} - \frac{y^3}{3} - \frac{zy^2}{4} \, dy \, dz \big|_{y=0}^{1/2} = 1/12. \)