1. A curve is given by the equation 
\[ \mathbf{r}(t) = \left\langle \frac{5t^2}{2}, \frac{4t^3}{3}, -t^3 \right\rangle, \quad 0 \leq t \leq \sqrt{3}. \]

(a). Calculate the equation of the tangent line to the curve at the point \((\frac{5}{2}, \frac{4}{3}, -1)\).

(b). Calculate the length of the curve.

Solution. (a). \( \mathbf{r}'(t) = \left\langle 5t, 4t^2, -3t^2 \right\rangle. \) At the point given, 
\[ \frac{5t^2}{2} = \frac{5}{2}, \quad \frac{4t^3}{3} = \frac{4}{3}, \quad -t^3 = -1. \]
The unique solution to these three equations is \( t = 1. \) Now \( \mathbf{r}'(1) = \langle 5, 4, -3 \rangle \) and so the equation of the tangent line in vector form is \( \mathbf{r}_0(t) = \langle \frac{5}{2} + 5t, \frac{4}{3} + 4t, -1 - 3t \rangle. \)

(b). \( |\mathbf{r}'(t)| = \sqrt{(5t)^2 + (4t^2)^2 + (3t^2)^2} = 5t\sqrt{1 + t^2}. \) Then the length is
\[ \int_0^{\sqrt{3}} 5t\sqrt{1 + t^2} \, dt = \frac{5}{3}(1 + t^2)^{3/2} \bigg|_0^{\sqrt{3}} = \frac{35}{3}. \]

2. Let \( f(x, y) = e^x(1 + \sin y). \)

(a). Calculate the equation of the tangent plane to the graph \( z = f(x, y) \) at the point \((x, y, z) = (0, \pi, 1)\).

(b). By making a linear approximation of \( f \) at the point \((x, y) = (0, \pi)\), estimate \( f(0.2, \pi - 0.1)\).

Solution. (a).
\[ f_x = e^x(1 + \sin y), \quad f_y = e^x \cos y, \quad f_x(0, \pi) = 1, \quad f_y(0, \pi) = -1. \]

Tangent plane at \((0, \pi, 1)\) is \( z - 1 = (x - 0) - (y - \pi). \)

(b). The value on the graph \( z = f(x, y) \) is approximated by the value of \( z \) on the tangent plane: \( f(0.2, \pi - 0.1) \approx 1 + (0.2 - 0) - (\pi - 0.1 - \pi) = 1 + 0.2 + 0.1 = 1.3. \)
(Alternatively use differentials to get the same answer.)
3. The temperature of a heated plate is given by \( f(x, y) = \frac{x^2}{y} \).

(a). At the point \((1, 1)\), in which direction is the rate of change of temperature greatest?
(b). An ant travels on the plane at unit speed and passes through the point \((1, 1)\) in the direction \(\langle 3, 4 \rangle\). What rate of change of temperature does the ant experience as it passes through \((1, 1)\)?

Solution. (a). Rate is greatest in the direction \(\nabla f(1, 1)\). Now \(\nabla f = \langle 2x/y, -x^2/y^2 \rangle\), and so \(\nabla f(1, 1) = \langle 2, -1 \rangle\).
(b). Unit vector in the direction \(\langle 3, 4 \rangle\) is \(\langle \frac{3}{5}, \frac{4}{5} \rangle\). Rate of change of temperature is 
\[
D_u f(1, 1) = \nabla f(1, 1) \cdot u = \langle 2, -1 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{2}{5}.
\]

4. Find all the critical points of the function \( f(x, y) = x^2 - 2xy + \frac{y^3}{3} \) and determine whether each critical point is a local maximum, a local minimum or a saddle point.

Solution. Solve for critical points
\[
0 = f_x = 2x - 2y \quad \Rightarrow \quad x = y.
0 = f_y = -2x + y^2 \quad \Rightarrow \quad 0 = -2y + y^2 \quad \Rightarrow \quad y(y-2) = 0.
\]
We get the points \((0,0)\) and \((2,2)\). Now
\[
D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ -2 & 2y \end{vmatrix} = 4y - 4.
\]
At \((0,0)\) we have \(D = -4 < 0\) so \((0,0)\) is a saddle. At \((2,2)\) we have \(D = 4 > 0\) and \(f_{xx} = 2 > 0\) so \((2,2)\) is a local minimum.

5. Use Lagrange multipliers to calculate the maximum and minimum values of \(x^4 + y^4\) on the curve \(x^2 + y^4 = 1\).

Solution. \(f = x^4 + y^4, g = x^2 + y^4\). Solve \(\nabla f = \lambda \nabla g\) and \(g = 4\):
\[
\begin{cases}
 4x^3 = \lambda 2x \\
 4y^3 = \lambda 4y^3 \\
 x^2 + y^4 = 1
\end{cases}
\]
The second equation implies \(\lambda = 1\) or \(y = 0\). If \(\lambda = 1\) the the first equation implies \(2x(2x^2 - 1) = 0\) so \(x = 0\) or \(x = \pm 2^{-1/2}\). The constraint gives the points \((x, y) = (0, \pm 1)\) and the four points \((x, y) = (\pm 2^{-1/2}, \pm 2^{-1/4})\). If \(y = 0\) then the constraint gives the points \((\pm 1, 0)\). Evaluating \(f\) at these points,
\[
f(0, \pm 1) = 1, \quad f(\pm 2^{-1/2}, \pm 2^{-1/4}) = 3/4, \quad f(\pm 1, 0) = 1.
\]
The max is 1 and the min is 3/4.