Final Exam, Mathematics 109
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Name:
Student ID:
Section Number:

Note: There are 5 problems on this exam, worth 40 points each. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted. Good luck !

I. (40 pts.) Let $X = [0,1]$ and let $R \subseteq X \times X$ be the relation on X (i.e. from X to X) defined as follows

$$R = \{(x, y) \mid x, y \in [0,1]\,, \quad x^2 + y^2 = 1\}\,.$$

(1) Is R an equivalence relation on X? Justify.
(2) Show that R is a functional relation on X.
(3) Write the explicit expression of $f(x)$, $x \in X$, for the function

$$f : X \longrightarrow X$$

determined by the functional relation R above.
(4) Show that the function f is bijective.
(5) Determine the inverse $f^{-1} : X \longrightarrow X$ of the bijective function f.

II. (40 pts.)

(1) Solve (i.e. determine the full solution set of) the following system of linear congruences.

$$\begin{aligned} x &\equiv 2 \mod 3 \\ x &\equiv 6 \mod 7 \\ x &\equiv 10 \mod 11 \end{aligned}$$

(2) Show that if $x \in \mathbb{Z}$ is a solution to the system above, then

$$x^2 \equiv 1 \mod 231\,.$$

(**Hint:** $231 = 3 \cdot 7 \cdot 11$.)

III. (40 pts.) Let $\{f_n\}_{n\geq 1}$ be the Fibonacci sequence given recursively by $f_1 = f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$, for all $n \in \mathbb{N}$.

(1) Prove that for each natural number n we have an equality

$$\sum_{i=1}^{n} f_i^2 = f_n \cdot f_{n+1}\,.$$

(2) Prove that for each natural number n, we have

$$f_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \cdot \sqrt{5}}\,.$$

(3) Prove that every natural number greater than 2 can be written as a sum of distinct terms of the Fibonacci sequence.

IV. (40 pts.) The universe $\mathcal{U}$ for all the variables in the statements below is the set of integers $\mathbb{Z}$.

(1) Write the negation of the following statement

$$(\forall\, x)(\exists\, y)(\exists\, z) \quad (x^3 + y^3 = z^3) \wedge (x + z = 0)\,.$$

(2) Prove or disprove the statement in (1) above.

V. (40 pts.)

(1) Prove that if A and B are two subsets of a given universal set $\mathcal{U}$, then

$$\overline{(A \setminus B)} \setminus \overline{(B \setminus A)} = B \setminus A\,.$$

Here, as usual, $\overline{C}$ denotes the universal complement of the set C inside $\mathcal{U}$.

(2) For each real number $x \in \mathbb{R}$, let

$$A_x := \{3, -2\} \cup \{y \in \mathbb{R} \mid y > x\}\,.$$

Determine $\cup_{x\in\mathbb{R}} A_x$ and $\cap_{x\in\mathbb{R}} A_x$.