I. (30 pts.) For each natural number $n$, let $A_n = \{-n\} \cup [1/n, 3n + 1]$.

1. Show that for each natural number $n$, we have $3\sqrt{n} \in A_n$.
2. Find $\bigcup_{n \in \mathbb{N}} A_n$ and $\bigcap_{n \in \mathbb{N}} A_n$. Justify your answers.

We will show that
$$\frac{1}{n} \leq 3\sqrt{n} < 3n + 1 \quad \forall n \in \mathbb{N}.$$ 

Since $n \geq 1$, we have
$$\frac{1}{n} \leq \frac{1}{n} \leq 3\sqrt{\frac{1}{n}} \leq 3\sqrt{n} \quad \forall n \in \mathbb{N}.$$ 

Therefore
$$\frac{1}{n} \leq 3\sqrt{n}, \quad \forall n \in \mathbb{N}. \quad (a)$$

We have:

$$3n + 1 - 3\sqrt{n} = (3n + 1 - 2\sqrt{3}\sqrt{n}) + (2\sqrt{3} - 3)\sqrt{n} =$$

$$= (\sqrt{3}\sqrt{n} - 1)^2 + (2\sqrt{3} - 3)\sqrt{n}.$$ 

However, since $2\sqrt{3} > 3$ (because $(2\sqrt{3})^2 > 3^2$), the last equality implies that

$$3n + 1 - 3\sqrt{n} > 0 \quad \forall n \in \mathbb{N} \quad (b)$$

Inequalities $(a)$ and $(b)$ show that
$$\frac{1}{n} \leq 3\sqrt{n} < 3n + 1,$$
thus
$$3\sqrt{n} \in \left[\frac{1}{n}, 3n + 1\right].$$

Therefore $3\sqrt{n} \in A_n, \forall n$. 

(2) We will show that 
\[ \bigcap_{n \in \mathbb{N}} U_{\alpha} = \left( \mathbb{Z} \setminus \{0\} \right) \cup (0, +\infty). \]

**Proof.**

Let \( x \in \bigcap_{n \in \mathbb{N}} U_{\alpha} \). Then \( \exists n \in \mathbb{N} \), s.t. \( x \in U_{\alpha n} \).

Therefore \( x = -n \), in which case \( x \in \mathbb{Z} \setminus \{0\} \),
or \( x \in \left[ \frac{1}{n}, 3n+1 \right) \), in which case \( x \in (0, +\infty) \).

Consequently \( x \in \left( \mathbb{Z} \setminus \{0\} \right) \cup (0, +\infty) \).

Hence \( \bigcap_{n \in \mathbb{N}} U_{\alpha} = \left( \mathbb{Z} \setminus \{0\} \right) \cup (0, +\infty) \). (a)

Let \( x \in \left( \mathbb{Z} \setminus \{0\} \right) \cup (0, +\infty) \). If \( x \in \mathbb{Z} \setminus \{0\} \), then either \( x = -n \), \( n \in \mathbb{N} \), in which case \( x \in U_{\alpha n} \), or \( x = n \), \( n \in \mathbb{N} \), in which case \( \frac{1}{n} \leq x < 3n+1 \) and, consequently \( x \in U_{\alpha n} \). If \( x \in (0, +\infty) \), since \( \lim_{n \to \infty} \frac{1}{n} = 0 \) and \( \lim (3n+1) = +\infty \), \( \exists n \) such that \( \frac{1}{n} < x < 3n+1 \).

Consequently \( x \in U_{\alpha n} \). Therefore \( \left( \mathbb{Z} \setminus \{0\} \right) \cup (0, +\infty) \subseteq \bigcap_{n \in \mathbb{N}} U_{\alpha} \).

We will show that \( \bigcap_{n \in \mathbb{N}} \) for \( n \in \mathbb{N} \)

\[ A_n = \{-n\} \cup \left[ \frac{1}{n}, 3n+1 \right), \forall n. \]

Since \( \frac{1}{n} \leq x < 3n+1, \forall n \), we have

\[ [1, 4) \subseteq \bigcap_{n \in \mathbb{N}} A_n. \]

\[ [1, 4) \subseteq \bigcap_{n \in \mathbb{N}} \left[ \frac{1}{n}, 3n+1 \right) \subseteq A_n, \forall n. \]

Therefore \( [1, 4) \subseteq \bigcap_{n \in \mathbb{N}} A_n \).

Let \( x \in \bigcap_{n \in \mathbb{N}} A_n \). Then \( x \in \bigcap_{n \geq 1} \left[ \frac{1}{n}, 3n+1 \right), \forall n. \)

\[ \frac{1}{n} \leq x < 3n+1, \forall n. \]

In particular, for \( n = 1 \), we obtain \( 1 \leq x < 4 \). Therefore \( x \in [1, 4) \).
II. (40 pts.) Let \( \{a_n\}_{n \in \mathbb{N}} \) be the sequence of real numbers defined recursively by \( a_1 = a_2 = 1 \), and \( a_n = 2a_{n-1} + 3a_{n-2} \), for all \( n \geq 3 \).

(1) Prove that for each natural number \( n \geq 3 \), we have

\[
2 \cdot 3^{n-2} > a_n > 3^{n-2}.
\]

(2) Prove that for each natural number \( n \), we have an equality

\[
a_n = \frac{2}{12} \cdot 3^n - \frac{6}{12} \cdot (-1)^n.
\]

(1) Let \( P(n) : 2 \cdot 3^{n-2} > a_n > 3^{n-2} \), \( n \in \mathbb{N} \). Let

\[
S := \{ n \in \mathbb{N} | P(n) \text{ is true} \}. \]

We will use the extended second principle of math. induction to show that

\[
S = \{ n \in \mathbb{N} | n \geq 3 \}.
\]

Step 1. Check that \( P(3), P(4) \) are true. This shows that \( 3, 4 \in S \).

Step 2. Let \( n \geq 4 \). We will show that \( \{3, 4, \ldots, n\} \subseteq S \).

We will show that \( (n+1) \in S \).

\[
n \in S \quad \Rightarrow \quad P(n) : 2 \cdot 3^{n-2} > a_n > 3^{n-2} \quad \text{holds true}
\]

\[
n-1 \in S \quad \Rightarrow \quad P(n-1) : 2 \cdot 3^{n-3} > a_{n-1} > 3^{n-3} \quad \text{holds true}
\]

Consequently, since \( a_{n+1} = 2a_n + 3a_{n-1} \), we have:

\[
2 \cdot (2 \cdot 3^{n-2} + 3 \cdot 3^{n-3}) > a_{n+1} > 2 \cdot 3^{n-2} + 3 \cdot 3^{n-3}
\]

\[
\Rightarrow \quad 3 \cdot (2 \cdot 3^{n-2}) > a_{n+1} > 3 \cdot 3^{n-2}
\]

\[
\Rightarrow \quad 2 \cdot 3^{n-1} > a_{n+1} > 3^{n-1}
\]

\[
\Rightarrow \quad P(n+1) \text{ is true.}
\]

Thus, \( (n+1) \in S \).

Steps 1 and 2 show that \( S = \{ n \in \mathbb{N} | n \geq 3 \} \).
(2)

Let \( \mathcal{Q}(n) : a_n = \frac{2}{12} \cdot 3^n - \frac{6}{12} \cdot (-1)^n \), \( n \in \mathbb{N} \).

We will apply the second extended second principle of math. induction to show that the set

\( \mathcal{S} = \{ n \mid n \in \mathbb{N}, \mathcal{Q}(n) \text{ is true} \} \)

is equal to \( \mathbb{N} \).

**Step 1.** Check that \( \mathcal{Q}(1) \) and \( \mathcal{Q}(2) \) are true.

This shows that \( 1, 2 \in \mathcal{S} \).

**Step 2.** Let us assume that for a fixed \( n \in \mathbb{N}, n \geq 2 \), we have \( \{1, 2, \ldots, n\} \subseteq \mathcal{S} \). We will show that \( \{n+1\} \subseteq \mathcal{S} \).

\[ a_{n+1} = 2a_n + 3a_{n-1} \]

The equality above combined with \( \mathcal{Q}(n) \) and \( \mathcal{Q}(n-1) \) implies:

\[
a_{n+1} = 2 \left( \frac{2}{12} \cdot 3^n - \frac{6}{12} \cdot (-1)^n \right) + 3 \left( \frac{2}{12} \cdot 3^{n-1} - \frac{6}{12} \cdot (-1)^{n-1} \right) = \frac{3}{12} \cdot 3^n (2+1) - \frac{6}{12} \cdot (-1)^n (2-3) = \frac{3}{12} \cdot 3^{n+1} - \frac{6}{12} \cdot (-1)^{n+1} \]

Therefore \( \mathcal{Q}(n+1) \) is true. Therefore \( \{n+1\} \subseteq \mathcal{S} \).

Steps 1 and 2 show that \( \mathcal{S} = \mathbb{N} \).
III. (30 pts.)

(1) Use the Euclidean Algorithm to find $\gcd(-219, 69)$.

(2) Find integers $m$ and $n$ such that

$$\gcd(-219, 69) = -219 \cdot m + 69 \cdot n.$$ 

(3) **(Bonus, 10pts.)** Show that if the integers $m, n$ and $m', n'$ satisfy the equalities

$$\gcd(-219, 69) = -219 \cdot m + 69 \cdot n = -219 \cdot m' + 69 \cdot n',$$

then $73 \mid (m - m')$ and $23 \mid (n - n')$.

1) We have an equality

$$\gcd(-219, 69) = \gcd(219, 69).$$

We apply the Euclidean algorithm to the pair

$a = 219, \quad b = 69$.

$$219 = 3 \cdot 69 + 12$$

$$69 = 5 \cdot 12 + 9$$

$$12 = 1 \cdot 9 + 3$$

$$9 = 3 \cdot 3 + 0$$

Therefore $\gcd(-219, 69) = 3$.

2) 

$$3 = 12 - 1 \cdot 9 = 12 - 1 \cdot (69 - 5 \cdot 12) =$$

$$= 6 \cdot 12 - 1 \cdot 69 =$$

$$= 6 \cdot (219 - 3 \cdot 69) - 1 \cdot 69 =$$

$$= 6 \cdot 219 - 19 \cdot 69.$$ 

Therefore

$$3 = (-6) \cdot (-219) - 19 \cdot 69,$$

$m = -6, \quad n = -19$.
3) \[-219 \cdot m + 69 \cdot n = -219 \cdot m' + 69 \cdot n' \iff \]

\[-219 \cdot (m - m') = 69 \cdot (n' - n) \iff \]

\[-\frac{219}{3} \cdot (m - m') = \frac{69}{3} \cdot (n' - n) \iff \]

\[-73 \cdot (m - m') = 23 \cdot (n' - n). \]

\[\frac{43}{23} \cdot (n' - n) \iff \frac{43}{23} \mid n' - n. \]

\[\frac{43\text{ prime}}{43 + 23} \]

\[\frac{23}{73} \cdot (m - m') \iff \frac{23}{73} \mid m - m'. \]

\[\frac{23\text{ prime}}{23 + 73} \]