SOLUTIONS

Exam 2, Mathematics 109
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Name:
Student ID:
Section Number:

Note: This exam consists of 3 problems worth a total of 100 points and a bonus question worth 10 points. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted. Good luck !

I. (30 pts.) For each natural number n, let $A_n = \{-n\} \cup [1/n\,,\ 3n+1)$.

(1) Show that for each natural number n, we have $3\sqrt{n} \in A_n$.

(2) Find $\bigcup_{n\in\mathbb{N}} A_n$ and $\bigcap_{n\in\mathbb{N}} A_n$. Justify your answers.

1). We will show that $1/n \leq 3\sqrt{n} < 3n+1$, $\forall n \in \mathbb{N}$.

Since $n \geq 1$, we have

$$1/n \leq \frac{1}{1} \leq 3\cdot\sqrt{1} \leq 3\sqrt{n}, \quad \forall n \in \mathbb{N}.$$

Therefore $1/n \leq 3\sqrt{n}$, $\forall n \in \mathbb{N}$. (a).

We have:

$$3n+1-3\sqrt{n} = \left(3n+1-2\sqrt{3}\sqrt{n}\right) + (2\sqrt{3}-3)\sqrt{n} =$$
$$= \left(\sqrt{3}\cdot\sqrt{n}-1\right)^2 + (2\sqrt{3}-3)\sqrt{n}.$$

However, since $2\sqrt{3} > 3$ (because $(2\sqrt{3})^2 > 3^2$), the last equality implies that

$$3n+1-3\sqrt{n} > 0, \quad \forall n \in \mathbb{N} \quad (b)$$

Inequalities (a) and (b) show that

$$1/n \leq 3\sqrt{n} < 3n+1,$$

therefore $3\sqrt{n} \in [1/n, 3n+1)$. Therefore $3\sqrt{n} \in A_n$, $\forall n$.

(2) I. We will show that

$$\bigcup_{n \in \mathbb{N}} A_n = (\mathbb{Z} \setminus \{0\}) \cup (0, +\infty).$$

Proof.

Let $x \in \bigcup_{n \in \mathbb{N}} A_n$. Then $\exists\, n \in \mathbb{N}$, s.t. $x \in A_n$.

Therefore ~~$x \in$~~ $x = -n$, in which case $x \in \mathbb{Z} \setminus \{0\}$,

or $x \in [1/n, 3n+1)$, in which case $x \in (0, +\infty)$.

Consequently $x \in (\mathbb{Z} \setminus \{0\}) \cup (0, +\infty)$.

Hence $\bigcup_{n \in \mathbb{N}} A_n \subseteq (\mathbb{Z} \setminus \{0\}) \cup (0, +\infty)$. (a)

Let $x \in (\mathbb{Z} \setminus \{0\}) \cup (0, +\infty)$. If $x \in \mathbb{Z} \setminus \{0\}$, then either $x = -n$, $n \in \mathbb{N}$, in which case $x \in A_n$, or $x = n$, $n \in \mathbb{N}$, in which case $1/n \le x < 3n+1$ and, consequently $x \in A_n$. If $x \in (0, +\infty)$, since $\lim_{n \to \infty} 1/n = 0$ and $\lim_{n \to \infty} (3n+1) = +\infty$, $\exists\, n$ such that $1/n < x < 3n+1$. Consequently $x \in A_n$. Therefore $(\mathbb{Z} \setminus \{0\}) \cup (0, +\infty) \subseteq \bigcup_{n \ge 0} A_n$.

II. We will show that $\bigcap_{n \in \mathbb{N}} A_n = [1, 4)$.

"$\supseteq$". $A_n = \{-n\} \cup [1/n, 3n+1)$, $\forall n$.

Since $1/n \le 1 < 4 \le 3n+1$, $\forall n$, we have $[1, 4) \subseteq [1/n, 3n+1) \subseteq A_n$, $\forall n$. Therefore $[1,4) \subseteq \bigcap_{n \in \mathbb{N}} A_n$.

"$\subseteq$" Let $x \in \bigcap_{n \in \mathbb{N}} A_n$. Then $x \in \bigcap_{n \ge 1} [1/n, 3n+1)$. Therefore $1/n \le x < 3n+1$, $\forall n$. In particular, for $n = 1$, we obtain $1 \le x < 4$. Therefore $x \in [1, 4)$

II. (40 pts.) Let $\{a_n\}_{n\in\mathbb{N}}$ be the sequence of real numbers defined recursively by $a_1 = a_2 = 1$, and $a_n = 2a_{n-1} + 3a_{n-2}$, for all $n \geq 3$.

(1) Prove that for each natural number $n \geq 3$, we have

$$2 \cdot 3^{n-2} > a_n > 3^{n-2}.$$

(2) Prove that for each natural number n, we have an equality

$$a_n = \frac{2}{12} \cdot 3^n - \frac{6}{12} \cdot (-1)^n.$$

(1) Let $P(n): 2 \cdot 3^{n-2} > a_n > 3^{n-2}$, $n \in \mathbb{N}$. Let $S := \{n \in \mathbb{N} \mid P(n) \text{ is true}\}$. We will use the extended second principle of math. induction to show that

$$S = \{n \mid n \in \mathbb{N}, n \geq 3\}.$$

Step 1 Check that $P(3)$, $P(4)$ are true. This shows that

$$3, 4 \in S.$$

Step 2. Let $n \geq 4$ ~~We will show that~~ Assume that

$$\{3, 4, \dots, n\} \subseteq S.$$

We will show that $(n+1) \in S$.

$n \in S \Rightarrow P(n): 2 \cdot 3^{n-2} > a_n > 3^{n-2}$ holds true

$n-1 \in S \Rightarrow P(n-1): 2 \cdot 3^{n-3} > a_{n-1} > 3^{n-3}$ holds true.

Consequently, since $a_{n+1} = 2a_n + 3a_{n-1}$, we have:

$$2 \cdot (2 \cdot 3^{n-2}) + 3 \cdot (2 \cdot 3^{n-3}) > a_{n+1} > 2 \cdot 3^{n-2} + 3 \cdot 3^{n-3} \iff$$

$$\iff 3 \cdot (2 \cdot 3^{n-2}) > a_{n+1} > 3 \cdot 3^{n-2}$$

$$\iff 2 \cdot 3^{n-1} > a_{n+1} > 3^{n-1} \iff P(n+1) \text{ is true.}$$

Therefore $(n+1) \in S$.

Steps 1 and 2 show that $S = \{n \mid n \in \mathbb{N}, n \geq 3\}$.

(2)

Let $Q(n): \quad a_n = \frac{2}{12}\cdot 3^n - \frac{6}{12}\cdot(-1)^n, \quad n \in \mathbb{N}$.

We will apply the ~~second ext~~ extended second principle of math. induction to show that the set

$$S = \{ n \mid n \in \mathbb{N},\ Q(n) \text{ is true} \}.$$

is equal to $\mathbb{N}$.

<u>Step 1</u>. Check that $Q(1)$ and $Q(2)$ are true.
This shows that $1, 2 \in S$.

<u>Step 2</u>. Let us assume that for a fixed $n \in \mathbb{N}_{\geq 2}$, we have $\{1, 2, \dots, n\} \subseteq S$. We will show that $(n+1) \in S$.

$$a_{n+1} = 2a_n + 3a_{n-1}.$$

The equality above combined with $Q(n)$ and $Q(n-1)$ implies:

$$a_{n+1} = 2\left(\frac{3}{12}\cdot 3^n - \frac{6}{12}(-1)^n\right) + 3\left(\frac{3}{12}\cdot 3^{n-1} - \frac{6}{12}(-1)^{n-1}\right) =$$

$$= \frac{3}{12}\cdot 3^n(2+1) - \frac{6}{12}\cdot(-1)^n\cdot(2-3) =$$

$$= \frac{3}{12}\cdot 3^{n+1} - \frac{6}{12}\cdot(-1)^{n+1}$$

Therefore $Q(n+1)$ is true. Therefore $(n+1) \in S$.

Steps 1 and 2 show that $S = \mathbb{N}$ ∎

III. (30 pts.)

(1) Use the Euclidean Algorithm to find $\gcd(-219, 69)$.

(2) Find integers m and n such that

$$\gcd(-219, 69) = -219 \cdot m + 69 \cdot n\,.$$

(3) **(Bonus, 10pts.)** Show that if the integers m, n and m', n' satisfy the equalities

$$\gcd(-219, 69) = -219 \cdot m + 69 \cdot n = -219 \cdot m' + 69 \cdot n'\,,$$

then $73 \mid (m - m')$ and $23 \mid (n - n')$.

1) $-\cancel{219}$ We have an equality

$$\gcd(-219, 69) = \gcd(219, 69).$$

We apply ~~the~~ the Euclidian algorithm to the pair $a = 219$, $b = 69$.

$$219 = 3 \cdot 69 + 12$$
$$69 = 5 \cdot 12 + 9$$
$$12 = 1 \cdot 9 + 3$$
$$9 = 3 \cdot 3 + 0$$

Therefore $\gcd(-219, 69) = 3$.

2) $$3 = 12 - 1 \cdot 9 = 12 - 1 \cdot (69 - 5 \cdot 12) =$$
$$= 6 \cdot 12 - 1 \cdot 69 =$$
$$= 6 \cdot (219 - 3 \cdot 69) - 1 \cdot 69 =$$
$$= 6 \cdot 219 - 19 \cdot 69.$$

Therefore

$$3 = (-6) \cdot (-219) - 19 \cdot 69,$$

$m = -6$, $n = -19$.

3)

$$-219\cdot m + 69\cdot n = -219\cdot m' + 69\cdot n' \iff$$

$$\iff -219\cdot(m-m') = 69\cdot(n'-n) \iff$$

$$\iff -\frac{219}{3}(m-m') = \frac{69}{3}(n'-n) \iff$$

$$\iff -73\cdot(m-m') = 23\cdot(n'-n).$$

$$\left.\begin{array}{l} 73 \mid 23\cdot(n'-n) \\ 73 \text{ prime} \\ 73 \nmid 23 \end{array}\right\} \Rightarrow 73 \mid n'-n.$$

$$\left.\begin{array}{l} 23 \mid -73\cdot(m-m') \\ 23 \text{ prime} \\ 23 \nmid 73 \end{array}\right\} \Rightarrow 23 \mid m-m'$$