$\underline{I}$. Let R be a commutative ring with 1 and $I \trianglelefteq R$ an ideal in R. Let

$$\pi : R \longrightarrow R/_{I}, \qquad \pi(x) = \hat{x} := x + I$$

be the canonical surjective ring morphism.

$$\begin{tikzcd} R \arrow[r, "\pi"] \arrow[d, "\varphi"'] & R/_I \arrow[dl, dashed, "\hat{\varphi}"] \\ S & \end{tikzcd}$$

Let $\varphi : R \longrightarrow S$ be a morphism of commutative rings. Then the following hold:

1) $\exists\ \hat{\varphi} : R/_{I} \longrightarrow S$ ring morphism, such that $\hat{\varphi} \circ \pi = \varphi$ if and only if
$$I \subseteq \ker \varphi.$$

2) If $\hat{\varphi}$ as above exists, then it is unique and it is given by
$$\hat{\varphi}(\hat{x}) = \varphi(x), \ \forall x \in R.$$

Assume that $\hat{\varphi}$ as above exists. Then

3) $\hat{\varphi}$ is injective if and only if $\operatorname{Ker}\varphi = I$.

4) $\hat{\varphi}$ is surjective if and only if φ is surjective.

II. Let ~~R be~~ $R \xrightarrow{\varphi} R'$ be a morphism of commutative rings with 1. Let

$$R \xhookrightarrow{i} R[X]$$

be the usual ring morphism taking $\alpha \in R$ to the constant polynomial $\alpha \in R[X]$.

Let $a \in R'$. Then:

1) There exists a unique ring morphism

$$R[X] \xrightarrow{\varphi_a} R'$$

with the properties that

$$\varphi_a \circ i = \varphi$$
$$\varphi_a(X) = a$$

$$\begin{tikzcd} R \arrow[r, hook, "i"] \arrow[d, "\varphi"'] & R[X] \arrow[dl, dashed, "\varphi_a"] \\ a \in R' & \end{tikzcd}$$

2) φ_a is given by

$$\varphi_a(a_0 + a_1 X + \ldots + a_n X^n) =$$
$$= \varphi(a_0) + \varphi(a_1)a + \ldots + \varphi(a_n)a^n,$$

for all $(a_0 + \ldots + a_n X^n) \in R[X]$.

III. Let R be a commutative ring with 1 and $S \subseteq R$ be a multiplicatively closed subset, such that $1 \in S$.

Let $i : R \longrightarrow S^{-1}R$ be the canonical ring morphism, given by

$$i(r) = \frac{r}{1}, \quad \forall\, r \in R.$$

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d, "\varphi"'] & S^{-1}R \arrow[dl, dashed, "\tilde{\varphi}"] \\ R' & \end{tikzcd}$$

Let $\varphi : R \longrightarrow R'$ be a morphism of commutative rings with 1.

Then:

1) $\exists\, \tilde{\varphi} : S^{-1}R \longrightarrow R'$ morphism of rings such that $\tilde{\varphi} \circ i = \varphi$ if and only if $\varphi(S) \subseteq R'^{\times}$.

2) If $\tilde{\varphi}$ as above exists, then it is unique and it is given by:

$$\tilde{\varphi}\left(\frac{r}{s}\right) = \varphi(r)\,\varphi(s)^{-1},$$

for all $r \in R$, $s \in S$.