10.5. Let $\mathbb{R}^*$ be the group of nonzero real numbers under multiplication, and let $r$ be a positive integer. Show that the mapping that takes $x$ to $x^r$ is a homomorphism from $\mathbb{R}^*$ to $\mathbb{R}^*$ and determine the kernel. Which values of $r$ yield an isomorphism?

**Solution:** Let $\varphi_r : \mathbb{R}^* \to \mathbb{R}^*$ denote the map $\varphi_r(x) = x^r$. Then for $x, y \in \mathbb{R}^*$, we have

$$\varphi_r(xy) = (xy)^r = x^r y^r = \varphi_r(x) \varphi_r(y).$$

Hence, $\varphi_r$ is a homomorphism. The kernel is the set of elements $x \in \mathbb{R}^*$ such that

$$1 = \varphi_r(x) = x^r.$$

Hence, $\ker \varphi_r = \{-1, 1\}$ if $r$ is even, and $\ker \varphi_r = \{1\}$ if $r$ is odd. This shows that $\varphi_r$ is not an isomorphism if $r$ is even (since $\varphi_r(-1) = 1 = \varphi_r(1)$, $\varphi_r$ is not injective). If $r$ is odd, then $\varphi_r$ is one-to-one by Theorem 10.2.5 since $\ker \varphi_r = \{1\}$. Also, we know from calculus that $\varphi_r$ is surjective in this case (by applying the intermediate value theorem to $\varphi_r$, for example). Hence, $\varphi_r$ is an isomorphism if $r$ is odd.

10.18. Can there be a homomorphism from $\mathbb{Z}_4 \times \mathbb{Z}_4$ onto $\mathbb{Z}_8$? Can there be a homomorphism from $\mathbb{Z}_{16}$ onto $\mathbb{Z}_2 \times \mathbb{Z}_2$? Explain.

**Solution:** Neither of these are possible. First, suppose there exists a homomorphism $\varphi : \mathbb{Z}_4 \times \mathbb{Z}_4 \to \mathbb{Z}_8$ which is surjective. In particular, there is $x \in \mathbb{Z}_4 \times \mathbb{Z}_4$ such that $\varphi(x) = 1 \in \mathbb{Z}_8$. By Theorem 10.1.3, this tells us that $|\varphi(x)| = 8$ divides $|x|$, but we know that $|x| \leq 4$ by Theorem 8.1, a contradiction. Hence, no such homomorphism exists.

Similarly, if there were a surjective homomorphism $\psi : \mathbb{Z}_{16} \to \mathbb{Z}_2 \times \mathbb{Z}_2$, then since $\mathbb{Z}_{16}$ is cyclic, it would follow that $\psi(\mathbb{Z}_{16}) = \mathbb{Z}_2 \times \mathbb{Z}_2$, since $\psi$ is surjective) is cyclic by Theorem 10.2.2, but this contradicts the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not cyclic (by Theorem 8.1 all elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ have order at most 2). Thus, no such homomorphism exists.

10.29. Suppose that there is a homomorphism from a finite group $G$ onto $\mathbb{Z}_{10}$. Prove that $G$ has normal subgroups of indexes 2 and 5.

**Solution:** By assumption, there is a surjective homomorphism $\varphi : G \to \mathbb{Z}_{10}$. By Theorem 10.2.8, $\varphi^{-1}(\langle 2 \rangle)$ and $\varphi^{-1}(\langle 5 \rangle)$ are normal subgroups of $G$ (since $\langle 2 \rangle$ and $\langle 5 \rangle$ are normal subgroups of $\mathbb{Z}_{10}$). We claim that $\varphi^{-1}(\langle 2 \rangle)$ has index 2 in $G$ and $\varphi^{-1}(\langle 5 \rangle)$ has index 5 in $G$. 

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In particular, using the fact that by the First Isomorphism Theorem, we have that $G/\ker \varphi \cong \mathbb{Z}_{10}$. In particular, using the fact that $G$ is finite, we get that

$$|G|/|\ker \varphi| = |G/\ker \varphi| = |\mathbb{Z}_{10}| = 10,$$

so $|G| = 10 \cdot |\ker \varphi|$. If we set $n = |\ker \varphi|$, then by Theorem 10.2.5, $\varphi$ is an $n$-to-1 map. Since $|\langle 2 \rangle| = 5$ (and $\langle 2 \rangle \subseteq \text{Im}(\varphi)$, this implies that $|\varphi^{-1}(\langle 2 \rangle)| = 5n$. Since $|G| = 10n$, we get that $\varphi^{-1}(\langle 2 \rangle)$ has index 2 in $G$. Similar reasoning shows that $\varphi^{-1}(\langle 5 \rangle)$ has index 5 in $G$.

(For a different proof that works even if $G$ is infinite, define the map $\psi : \mathbb{Z}_{10} \to \mathbb{Z}_2$ by $\psi(x) = x \mod 2$ (i.e. $\psi(x)$ is the unique element of $\{0, 1\}$ which is equivalent to $x \mod 2$). Then if we consider the composition

$$\psi \circ \varphi : G \to \mathbb{Z}_2,$$

we see that $\psi \circ \varphi$ is surjective with kernel $\varphi^{-1}(\langle 2 \rangle)$, which implies that $G/(\varphi^{-1}(\langle 2 \rangle)) \cong \mathbb{Z}_2$ by the First Isomorphism Theorem, so $\varphi^{-1}(\langle 2 \rangle)$ has index 2 in $G$. A similar proof shows that $\varphi^{-1}(\langle 5 \rangle)$ has index 5 in $G$.)

10.34. **Prove that there is no homomorphism from $A_4$ onto $\mathbb{Z}_2$.**

**Solution:** Suppose, by way of contradiction, that there is a surjective homomorphism $\varphi : A_4 \to \mathbb{Z}_2$. Then by the First Isomorphism Theorem, $A_4/\ker \varphi \cong \varphi(A_4) = \mathbb{Z}_2$. Thus, $|A_4/\ker \varphi| = |\mathbb{Z}_2| = 2$, so $\ker \varphi$ has index 2 in $A_4$, i.e. $|\ker \varphi| = 6$. However, by example 5 in chapter 7 (page 144), we know that $A_4$ has no subgroups of order 6, so this is a contradiction.

10.37. **Let $H = \{z \in \mathbb{C}^* ||z| = 1\}$. Prove that $\mathbb{C}^*/H \cong \mathbb{R}^+$, the group of positive real numbers under multiplication.**

**Solution:** Define a function $\varphi : \mathbb{C}^* \to \mathbb{R}^+$ by $\varphi(z) = |z|$. Then $\varphi$ is a homomorphism since

$$\varphi(z_1z_2) = |z_1z_2| = |z_1||z_2| = \varphi(z_1)\varphi(z_2)$$

for $z_1, z_2 \in \mathbb{C}^*$. Furthermore, $H = \ker \varphi$ (by the definition of $H$), and $\varphi$ is surjective since for any $x \in \mathbb{R}^+$ we have $x \in \mathbb{C}^*$ as well and $\varphi(x) = |x| = x$. Hence, by Theorem 10.3 we see that

$$\mathbb{C}^*/H = \mathbb{C}^*/\ker \varphi \cong \varphi(\mathbb{C}^*) = \mathbb{R}^+.$$

10.48. **Suppose that $\mathbb{Z}_{10}$ and $\mathbb{Z}_{15}$ are both homomorphic images of a finite group $G$. What can we say about $|G|$? Generalize.**

**Solution:** Saying that $\mathbb{Z}_{10}$ is a homomorphic image of $G$ means that there is a surjective homomorphism $\varphi : G \to \mathbb{Z}_{10}$. By the First Isomorphism Theorem, $G/\ker \varphi \cong \mathbb{Z}_{10}$. This tells us that $10 = |G/\ker \varphi| = |G|/|\ker \varphi| = |G|/|\ker \varphi| = 10 \cdot |\ker \varphi|$, so $10$ divides $|G|$. Similar reasoning shows that the fact that $\mathbb{Z}_{15}$ is a homomorphic image of $G$ implies
that 15 divides \(|G|\). Since 10 and 15 both divide \(|G|\), we see that lcm\((10, 15) = 30\) divides \(|G|\) as well.

A similar proof shows that the following generalization is true - if \(\mathbb{Z}_{n_1}, \ldots, \mathbb{Z}_{n_m}\) are homomorphic images of a finite group \(G\), then lcm\((n_1, \ldots, n_m)\) divides \(|G|\).

**10.49.** Suppose that for each prime \(p\), \(\mathbb{Z}_p\) is the homomorphic image of a group \(G\). What can we say about \(|G|\)? Give an example of such a group.

**Solution:** This implies that \(|G|\) is infinite. To see this, note that if \(|G|\) were finite, then the same argument as in problem 10.48 shows that \(p\) divides \(|G|\) for every prime \(p\), contradicting the fact that \(|G|\) is finite. (In fact, we don’t even need group theory to tell us this, since a set which surjects onto finite sets of arbitrarily large cardinality must be infinite.)

An example of such a group is \(\mathbb{Z}\) - for every prime \(p\), the map \(\phi_p : \mathbb{Z} \rightarrow \mathbb{Z}_p, \phi_p(n) = n \mod p\) (see example 5 in chapter 10) is a surjective homomorphism.

**10.65.** Prove that the map \(\phi : \mathbb{C}^* \rightarrow \mathbb{C}^*\) given by \(\phi(x) = x^2\) is a homomorphism and that \(\mathbb{C}^*/\{1, -1\}\) is isomorphic to \(\mathbb{C}^*\). What happens if \(\mathbb{C}^*\) is replaced by \(\mathbb{R}\)?

**Solution:** For \(x, y \in \mathbb{C}^*\), we have

\[
\phi(xy) = (xy)^2 = x^2y^2 = \phi(x)\phi(y).
\]

Hence, \(\phi\) is a homomorphism. We have that

\[
\ker \phi = \{x \in \mathbb{C}^* | \phi(x) = 1\} = \{x \in \mathbb{C}^* | x^2 = 1\} = \{1, -1\}.
\]

Also, \(\phi\) is surjective by the Fundamental Theorem of Algebra. More specifically, given a \(y \in \mathbb{C}^*\), the Fundamental Theorem of Algebra tells us that the polynomial \(x^2 - y\) (remember \(y\) is fixed here and \(x\) is a variable) has a complex root \(x_0\), i.e. \((x_0)^2 - y = 0\) so that \(\phi(x_0) = (x_0)^2 = y\). Thus, \(\phi\) is surjective and so the First Isomorphism Theorem tells us that

\[
\mathbb{C}^*/\{1, -1\} = \mathbb{C}^*/\ker \phi \cong \text{Im}(\phi) = \mathbb{C}^*.
\]

Now, if we replace \(\mathbb{C}^*\) with \(\mathbb{R}^*\) (we’ll call the map \(\psi : \mathbb{R}^* \rightarrow \mathbb{R}^*, \psi(x) = x^2\) to distinguish it from \(\phi\)) then the same proof as above shows that \(\psi\) is a homomorphism and the kernel is \(\{1, -1\}\). In this case, however, \(\psi\) is not surjective. Instead, the image of \(\psi\) is \(\mathbb{R}_+ = \{x \in \mathbb{R}^* | x > 0\}\) (which can be proven using calculus), so by the First Isomorphism Theorem

\[
\mathbb{R}^*/\ker \psi = \mathbb{R}^*/\{1, -1\} \cong \text{Im}(\psi) = \mathbb{R}_+.
\]

**11.18.** Let \(p_1, p_2, \ldots, p_n\) be distinct primes. Up to isomorphism, how many abelian groups are there of order \(p_1^4p_2^3 \cdots p_n^4\)?
Solution:

First, note that for each $i \in \{1, 2, \ldots, n\}$, we have the following 5 abelian groups of order $p_i^4$:

- $\mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i}$,
- $\mathbb{Z}_{p_i^2} \times \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i}$,
- $\mathbb{Z}_{p_i^2} \times \mathbb{Z}_{p_i^2}$,
- $\mathbb{Z}_{p_i^3} \times \mathbb{Z}_{p_i}$,
- $\mathbb{Z}_{p_i^4}$.

By choosing, for each $i \in \{1, 2, \ldots, n\}$, an abelian group $G_i$ of order $p_i^4$ from the list above, we can construct an abelian group $G_1 \times G_2 \times \cdots \times G_n$ of order $p_1^4 p_2^4 \cdots p_n^4$. Since there are 5 choices for each $G_i$, it follows that there are $5^n$ choices of $G_1 \times G_2 \times \cdots \times G_n$. Furthermore, these $5^n$ groups are pairwise nonisomorphic (i.e. any two of the groups constructed above are not isomorphic; this is part of the statement of the Fundamental Theorem of Finite Abelian Groups, but one can also see this by simply looking at the orders of the elements of each group). Finally, the Fundamental Theorem of Finite Abelian Groups tells us that any finite abelian group of order $p_1^4 p_2^4 \cdots p_n^4$ is isomorphic to one of the groups constructed above. Hence, there are $5^n$ abelian groups of order $p_1^4 p_2^4 \cdots p_n^4$ up to isomorphism.