1)* Let \( R \) be a comm. ring \((0 \neq 1_R)\) and \( S \subseteq R \) a multiplicatively closed set. Let \( \{ M_i \}_{i \in I} \) be a directed system of \( R \)-modules with respect to \( R \)-module morphisms \( \phi_{ij} : M_i \to M_j \), \( i, j \in I \) such that \( i \leq j \).

a) Show that \( \{ S^{-1}M_i \}_{i \in I} \) is a directed system of \( S^{-1}R \)-modules with respect to \( \{ S^{-1}\phi_{ij} \}_{i, j, i \leq j} \).

b) Show that there is a canonical \( S^{-1}R \)-module isomorphism
\[
\lim_{i \in I} S^{-1}M_i \cong \frac{S^{-1}\lim_{i \in I} M_i}{i \in I}
\]

2)* Give an example of an inverse system \( \{ M_i \}_{i \in I} \) of \( R \)-modules, such that
\[
\lim_{i \in I} S^{-1}M_i \ncong \frac{S^{-1}\lim_{i \in I} M_i}{i \in I}
\]
as \( S^{-1}R \)-modules, for some multiplicatively closed subset \( S \subseteq R \).
3*) Let \( n \in \mathbb{N} \), \( p \) prime number, \( S_p := \mathbb{Z}/p\mathbb{Z} \).

a) Show that there is a canonical \( \mathbb{Z}_p \)-module isomorphism

\[
S_p^{-1}(\mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/p^k\mathbb{Z}, & \text{if } p \mid n \\ \mathbb{Z}/n\mathbb{Z}, & \text{otherwise} \end{cases}
\]

where \( p^k \) is the largest power of \( p \) dividing \( n \).

(Note: First show that \( \mathbb{Z}/p^k\mathbb{Z} \cong \mathbb{Z}(p)/p^k\mathbb{Z}(p) \) and therefore \( \mathbb{Z}/p^k\mathbb{Z} \) has a canonical \( \mathbb{Z}(p) \)-module structure.

b) Generalize the result in a) by replacing \( \mathbb{Z} \) with an arbitrary PID \( R \) and \( \mathbb{Z}/n\mathbb{Z} \) with an arbitrary cyclic, torsion \( R \)-module \( M \). (Note: A cyclic \( R \)-module is an \( R \)-module generated by one element.)

4*) Let \( f \in \mathbb{Z}[x] \setminus \mathbb{Z} \), such that \( \gcd(f, f') = 1 \).

(\( f' \) is the formal derivative \( \frac{d}{dx} f \) viewed in \( \mathbb{Z}[x] \).)

Let \( S \) be the set of non-zero divisors in \( \mathbb{Z}[x]/(f) \).

a) Show that \( S^{-1}(\mathbb{Z}[x]/(f)) \) is isomorphic (as a ring) to a finite direct sum of fields.
3. b) Is the condition "gcd \( f, f' \) = 1" imposed in a) necessary? Justify with an example.

4. a) Use the result in 4*) to show that if \( G \) is a finite cyclic group, then the total ring of fractions of \( \mathbb{Z}[G] \) is isomorphic to a direct sum of fields.

b) What can you say about the total ring of fractions of \( \mathbb{Z}[G] \), if \( G \) is a finite direct product of finite cyclic groups?