Criteria for convergence/divergence of the series $\sum_{n=0}^{\infty} a_n$

The Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ is divergent.

The Integral Test: Suppose f is a continuous, positive, decreasing function on $[0, \infty)$ such that $a_n = f(n)$. Then the series $\sum a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

The Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $a_n \leq b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ is also convergent. If $a_n \geq b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ is also divergent.

The Limit-Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. Let $c = \lim_{n\to\infty} \frac{a_n}{b_n}$. If $c > 0$ and finite then the series are both convergent or both divergent.

The Alternating Series Test: Suppose $b_n > 0$ for all n and satisfies (i) $b_{n+1} \leq b_n$ for all n, and (ii) $\lim_{n\to\infty} b_n = 0$. Then the series $\sum_{n=0}^{\infty} (-1)^n b_n$ is convergent.

The Ratio Test: Let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$. If $L < 1$ then the series $\sum a_n$ is absolutely convergent (and therefore convergent). If $L > 1$ (or $L = \infty$) then the series $\sum a_n$ is divergent. If $L = 1$ or does not exist, the Ratio Test is inconclusive.

The Root Test: Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$. If $L < 1$ then the series $\sum a_n$ is absolutely convergent (and therefore convergent). If $L > 1$ (or $L = \infty$) then the series $\sum a_n$ is divergent. If $L = 1$ or does not exist, the Root Test is inconclusive.

Selected Taylor series expansions

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n, \text{ for } \quad -1 < x < 1.$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ... = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \text{ for } \quad -1 \leq x < 1.$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \text{ for } \quad -1 \leq x \leq 1.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \text{ for all } x.$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \text{ for all } x.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ for all } x.$$