

Final Exam, Mathematics 20D  
Dr. Cristian D. Popescu  
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Name:  
SSN:  
Section Number:

Note: There are 8 problems on this exam, worth 25 points each. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted. Good luck !

(25 pts.) I. (1) Solve the following initial value problem.

$$y'' + 4y' + 4y = t^{-2} \cdot e^{-2t}, \quad y(0) = 0, \quad y'(0) = 0.$$

(2) Indicate the interval of definition for the solution you found in (1).

(25 pts.) II. **A.** For the differential equation

$$y'' - y = 0,$$

at the regular point  $x_0 = 0$

- (1) determine the recurrence relation satisfied by the coefficients  $\{a_n\}_{n \geq 0}$  of its general power series solution  $\sum_{n \geq 0} a_n \cdot x^n$ .
- (2) find the first four terms in each of the two linearly independent series solutions.

**B.** Determine a lower bound for the radius of convergence of series solutions to the differential equation

$$(x^2 - 2x + 5)y'' + xy' + 4y = 0,$$

about the regular point  $x_0 = 4$ .

(25 pts.) III. (1) Find the radius of convergence and interval of convergence for the following power series.

$$\sum_{n \geq 1} \frac{2^n \cdot (x - 2)^n}{n + 2}.$$

(2) Specify for which points in its interval of convergence the power series in (1) converges absolutely, respectively conditionally.

(25 pts.) IV. Determine which of the following infinite series are convergent and which are divergent.

$$(a) \sum_{n \geq 1} n \cdot \sin 1/n; \quad (b) \sum_{n \geq 1} 1/n \cdot \ln n; \quad (c) \sum_{n \geq 1} (-1)^n 1/n \cdot \ln n.$$

(25 pts.) V. (1) Write down the Taylor series expansion of

$$f(x) = e^{-x^2},$$

about  $a = 0$  and determine its radius of convergence.

(2) Compute the value  $T_2(0.2)$  of the second Taylor polynomial  $T_2(x)$  associated to  $f$  at  $a = 0$  and estimate the error in approximating  $e^{-0.04}$  with  $T_2(0.2)$ . Take into account that  $|f^{(3)}(x)| \leq 0.6$ , for all  $x$  in the interval  $(-0.2, 0.2)$ .

(3) Use the power series in (1) to approximate the integral

$$\int_0^1 e^{-x^2} dx$$

within 0.01.

(25 pts.) VI. Solve the following initial value problem

$$t^3 y' + 4t^2 y = e^{-t}, \quad y(-1) = 0$$

and indicate the maximal interval of definition for its solution.

(25 pts.) VII. Find the explicit solution of the following initial value problem

$$x dx + ye^{-x} dy = 0, \quad y(0) = 1,$$

and indicate the maximal interval of definition for its solution.

(25 pts.) VIII. (1) Show that the differential equation

$$ydx + (2x - ye^y)dy = 0$$

is not exact.

(2) Show that the equation in (1) becomes exact when multiplied by the factor of integration  $\mu(x, y) = y$ .

(3) Solve the equation in (1).