SOLUTIONS

Exam 2, Mathematics 20D
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Name:
SSN:
Section Number:

Note: There are 4 problems on this exam, worth 25 points each. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted. Good luck !

(25 pts.) I. (1) Solve the initial value problem

$$ty' + 2y = \frac{\cos t}{t}, \qquad y(\pi) = \pi\,.$$

(2) Indicate the interval of definition for the solution to the initial value problem in (1).

(1) Bring the equation to its normal form

$$y' + \frac{2}{t}y = \frac{\cos t}{t^2}$$

This is a linear equation. Apply the method of the integrating factor.

$$\mu(t) = e^{\int \frac{2}{t}\,dt} = e^{2\cdot\ln|t|} = |t|^2 = t^2$$

General solution

$$y(t) = \frac{\int t^2\cdot\frac{\cos t}{t^2}\,dt + C}{t^2} = \frac{\int \cos t\,dt + C}{t^2} =$$

$$= \frac{\sin t}{t^2} + \frac{C}{t^2}$$

Initial value problem

$$y(\pi) = \pi \iff \cancel{\frac{\sin\pi}{\pi^2}} + \frac{C}{\pi^2} = \pi \iff C = \pi^3$$

see verso

$$\boxed{y(t) = \frac{\sin t}{t^2} + \frac{\pi^3}{t^2}}$$

(2) Interval of definition

$(0, +\infty)$.

(25 pts.) II. For the differential equation

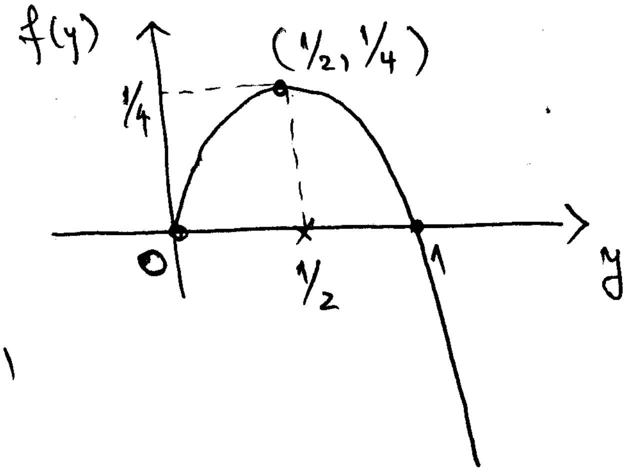
$$\frac{dy}{dt} = f(y),$$

with $f(y) = -y(y-1)$ and $y \geq 0$, do the following.

(1) Plot the graph of $f(y)$ versus y.

(2) Calculate the critical points and classify each of the corresponding equilibrium solutions as asymptotically stable or unstable.

(3) Plot several integral curves, including the ones corresponding to the equilibrium solutions.



(1) $f(y) = -y(y-1)$

$y \geqslant 0$

<u>y- intercepts</u>

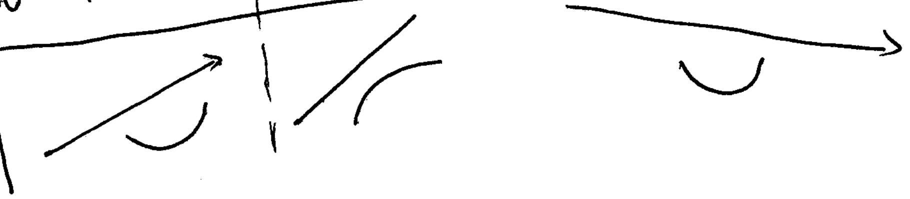
$f(y) = 0 \iff y=0$ or $y=1$

<u>absolute maximum</u>

$(1/2, 1/4)$

(2) equilibrium solutions $y=0$, $y=1$

y	0		1/2		1	
$f(y) = \frac{dy}{dt}$	0	+ +	+	+	0	− − − −
$f'(y)\cdot f(y) = \frac{d^2y}{dt^2}$	0	+ +	0	− −	0	+ + + +
y						



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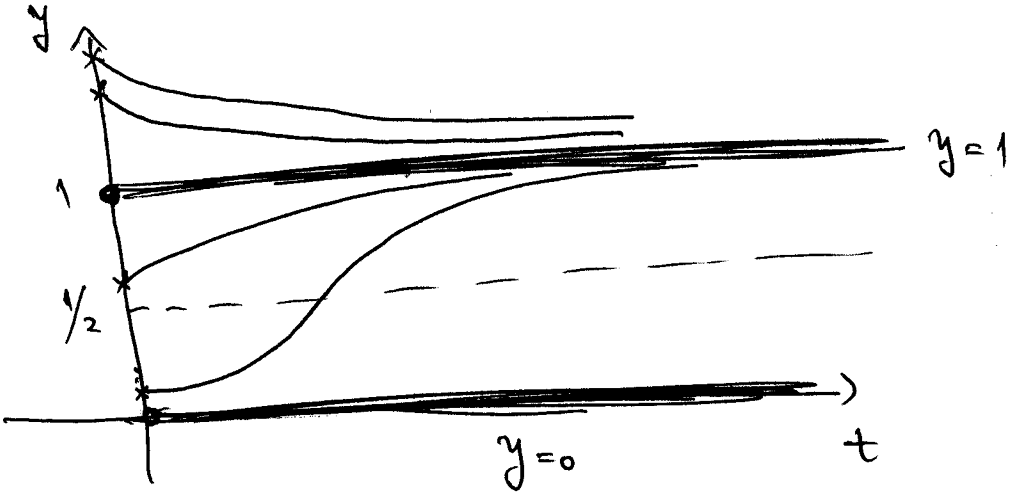
(3)



$y = 0$: unstable equilibrium

$y = 1$: asymptotically stable equilibrium.

(25 pts.) III. (1) Find the explicit solution to the following initial value problem

$$\frac{x}{(x^2+y^2)^{3/2}}dx + \frac{y}{(x^2+y^2)^{3/2}}dy = 0\,, \qquad y(0) = 1\,.$$

(2) Indicate the interval of definition for the solution you found in (1).
(3) Plot the graph of the solution you found in (1).

Let

(1) $M(x,y) = \dfrac{x}{(x^2+y^2)^{3/2}}$, $N(x,y) = \dfrac{y}{(x^2+y^2)^{3/2}}$

$$\frac{\partial M}{\partial y} = -\frac{3xy}{(x^2+y^2)^{5/2}}$$

$$\frac{\partial N}{\partial x} = -\frac{3xy}{(x^2+y^2)^{5/2}}$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation in (1) is exact.

Therefore, there exists a function $\psi(x,y)$ such that:

$$\begin{cases} \dfrac{\partial \psi}{\partial x} = \dfrac{x}{(x^2+y^2)^{3/2}} \\ \dfrac{\partial \psi}{\partial y} = \dfrac{y}{(x^2+y^2)^{3/2}} \end{cases}$$

<u>Determine ψ</u>

$$\psi = \int \frac{x}{(x^2+y^2)^{3/2}}\,dx$$

$$= -\frac{1}{(x^2+y^2)^{1/2}} + h(y).$$

$u = x^2+y^2$, $du = 2x\,dx \ldots$

verso

Therefore:

$$\frac{\partial \psi}{\partial y} = \frac{y}{(x^2+y^2)^{3/2}} + h'(y).$$

Consequently:

$$\frac{y}{(x^2+y^2)^{3/2}} + h'(y) = \frac{y}{(x^2+y^2)^{3/2}} \Longrightarrow$$

$$\Rightarrow \quad h'(y) = 0 \Rightarrow h(y) = C \text{ (constant).}$$

Therefore (take $C = 0$):

$$\psi(x,y) = \frac{1}{(x^2+y^2)^{3/2}}$$

Implicit solution

$$\begin{cases} \dfrac{1}{(x^2+y^2)^{3/2}} = C \\ y(0) = 1 \end{cases} \Leftrightarrow \begin{cases} \dfrac{1}{(x^2+y^2)^{3/2}} = 1 \\ y(0)=1 \end{cases}$$

Explicit solution

$$\left.\begin{matrix} x^2+y^2 = 1, \\ y(0) = 1 \end{matrix} \quad \begin{matrix} y = \pm\sqrt{1-x^2} \\ y(0)=1 \end{matrix}\right\} \Rightarrow \boxed{y = \sqrt{1-x^2}}$$

(2) Interval of definition $x \in (-1, 1)$.

(3) graph ...

(25 pts.) IV. (1) Find the explicit solution to the following initial value problem

$$y' = xy^3(1+x^2)^{-1/2}, \qquad y(0) = 1.$$

(2) Indicate the interval of definition for the solution you found in (1).

(1) The equation is separable

$$\frac{dy}{dx} = \frac{x \cdot (1+x^2)^{-1/2}}{1/y^3}$$

$$\frac{1}{y^3}\,dy = \frac{x}{\sqrt{1+x^2}}\,dx$$

$$\int \frac{1}{y^3}\,dy = \int \frac{x}{\sqrt{1+x^2}}\,dx \quad \Longleftrightarrow$$

$$\Leftrightarrow \left| -\frac{1}{2} y^{-2} = \sqrt{1+x^2} + C\right.$$

However $y(0) = 1$, Therefore $-\frac{1}{2} = 1 + C$.

$C = -3/2$.

$$-1/2\, y^{-2} = -3/2 + \sqrt{1+x^2}$$

$$y^2 = \frac{1}{-2\sqrt{1+x^2}+3}, \quad y(0) = 1$$

verso.

$$y = \pm \frac{1}{\sqrt{-2\sqrt{1+x^2}+3}} \quad | y(0) = 1$$

$$y = \frac{1}{\sqrt{-2\sqrt{1+x^2} + 3}}$$

(2) Interval of definition

$$3 > 2\sqrt{1+x^2}$$

$$9 > 4(1+x^2)$$

$$5/4 > x^2.$$

$$\boxed{x \in \left(-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right)}$$