Math 102 Homework 10 Solutions

6.6

22) $T: P_2 \to P_2$ given by $T(p(x)) = p'(x)$

Let $B = C = \{1, x, x^2\}$, standard basis for P_2

$T(1) = 0$ $\quad [T(1)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$T(x) = 1$ $\quad [T(x)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T(x^2) = 2x$ $\quad [T(x^2)]_B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

Hence $[T]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

This matrix is <u>not</u> invertible, since it is rank 2 $\Rightarrow$ the map T is not invertible

6.6

30) $W = \text{span}(\cos x, \sin x, x\cos x, x\sin x)$

Let D = differentiation

Then $D(\cos x) = -\sin x$

$D(\sin x) = \cos x$

$D(x\cos x) = \cos x + -x\sin x$

$D(x\sin x) = \sin x + x\cos x$

If $B = \{\cos x, \sin x, x\cos x, x\sin x\}$ basis for W,

$$[D]_B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Since $\int$ = integration is the inverse of D = diffenentiation, we find $[D]_B^{-1}$:

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{now reduce}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

So $\int (x\cos x + x\sin x)\,dx$

$$= [D]_B^{-1} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Hence $\int (x\cos x + x\sin x)\,dx =$

$$\cos x + \sin x - x\cos x + x\sin x$$

6.6 # 35) $T: P_1 \to P_1$ by $T(p(x)) = p(x) + x\,p'(x)$

let $B = \{1, x\}$ the standard basis for P_1

We check $[T]_B$ - if it is diagonal, we're done!

$T(1) = 1 + x(0) = 1$ $\qquad [T(1)]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$T(x) = x + x(1) = 2x$ $\qquad [T(x)]_B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Hence $[T]_B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is diagonal

So $B = \{1, x\}$ is such a basis //

4.2 # 60)
$$\begin{aligned} x + y - z &= 1 \\ x + y + z &= 2 \\ x - y \quad &= 3 \end{aligned} \Rightarrow \overset{A}{\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & 3 \end{array}\right]} \ \vec{b}$$

Then
$$x = \frac{\det\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{pmatrix}}{\det A} = \frac{9}{4}$$

$$y = \frac{\det\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{pmatrix}}{\det A} = \frac{-3}{4}, \quad z = \frac{\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix}}{\det A} = \frac{2}{4}$$

So the solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9/4 \\ -3/4 \\ 1/2 \end{bmatrix}$