Math 102 HW 7 Solutions

6.2 #16) $W(x) = \det \begin{bmatrix} \sin x & \sin 2x & \sin 3x \\ \cos x & 2\cos 2x & 3\cos 3x \\ -\sin x & -4\sin 2x & -9\sin 3x \end{bmatrix}$

Instead of computing this, let's just plug in some points.

If $x = \frac{\pi}{2}$, $W(x) = \begin{vmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & 9 \end{vmatrix}$

$= 1(-2 \cdot 9) + 0 - 1(-2) = -18 + 2$

$= -16 \neq 0$

So W is not identically zero

$\therefore$ $\{\sin x, \sin 2x, \sin 3x\}$ is a set of linearly independent functions

43) a) $U \times V = \{(\vec{u}, \vec{v}) \mid \vec{u} \text{ in } U, \vec{v} \text{ in } V\}$

Suppose $\dim U = n$, and $\{\vec{u}_1, \dots, \vec{u}_n\}$ is a basis for U

Suppose $\dim V = m$, $\{\vec{v}_1, \dots, \vec{v}_m\}$ basis for V

Claim: $B = \{(\vec{u}_1, \vec{0}), \dots, (\vec{u}_n, \vec{0}), (\vec{0}, \vec{v}_1), \dots, (\vec{0}, \vec{v}_m)\}$ is a basis for $U \times V$

Pf: (linear independence)

Suppose $a_1(\vec{u}_1, \vec{0}) + \dots + a_n(\vec{u}_n, \vec{0}) + b_1(\vec{0}, \vec{v}_1) + \dots + b_m(\vec{0}, \vec{v}_m) = (\vec{0}, \vec{0})$

then $(a_1\vec{u}_1, \vec{0}) + \dots + (a_n\vec{u}_n, \vec{0}) + (\vec{0}, b_1\vec{v}_1) + \dots + (\vec{0}, b_m\vec{v}_m)$

$= (a_1\vec{u}_1 + \dots + a_n\vec{u}_n, \vec{0}) + (\vec{0}, b_1\vec{v}_1 + \dots + b_m\vec{v}_m)$

$= (a_1\vec{u}_1 + \dots + a_n\vec{u}_n, b_1\vec{v}_1 + \dots + b_m\vec{v}_m) = (\vec{0}, \vec{0})$

then $a_1\vec{u}_1 + \dots + a_n\vec{u}_n = \vec{0}$ in U,

since $\{\vec{u}_1, \dots, \vec{u}_n\}$ are a basis for U,

they are linearly independent $\Rightarrow a_1 = \dots = a_n = 0$

then $b_1\vec{v}_1 + \dots + b_m\vec{v}_m = \vec{0}$ in V,

$\{\vec{v}_1, \dots, \vec{v}_m\}$ basis for $V \Rightarrow b_1 = b_2 = \dots = b_m = 0$

Hence $a_1 = \dots = a_n = b_1 = \dots = b_m = 0$ are all zero;

(Spanning Set)

Let $(\vec{u}, \vec{v}) \in U \times V$ be given

Then $\vec{u}$ in $U \Rightarrow \vec{u} = c_1\vec{u}_1 + \dots + c_n\vec{u}_n$ for some scalars c_i

since $\{u_1, \dots, u_n\}$ is a basis for U

$\vec{v}$ in $V \Rightarrow \vec{v} = d_1\vec{v}_1 + \dots + d_m\vec{v}_m$ for some scalars $d_1, \dots, d_m$ since $\{\vec{v}_1, \dots, \vec{v}_m\}$ basis for V $\rightarrow$

43 a) cont

Then $c_1(\vec{u}_1, \vec{0}) + \dots + c_n(\vec{u}_n, \vec{0}) +$

$d_1(\vec{0}, \vec{v}_1) + \dots + d_m(\vec{0}, \vec{v}_m)$

$= (c_1\vec{u}_1 + \dots + c_n\vec{u}_n,\ d_1\vec{v}_1 + \dots + d_m\vec{v}_m)$

$= (\vec{u}, \vec{v})$

Therefore B is both a spanning set and a set of linearly independent vectors for $U \times V$ $\Rightarrow$ B is a basis for $U \times V$

So $\dim(U \times V) = \#$ elements in B

$= n + m = \dim(U) + \dim(V)$ //

59 b) $$P_j(x) = \frac{(x-a_0)\cdots(x-a_{j-1})(x-a_{j+1})\cdots(x-a_n)}{(a_j-a_0)\cdots(a_j-a_{j-1})(a_j-a_{j+1})\cdots(a_j-a_n)}$$

$$P_j(a_j) = \frac{(a_j-a_0)\cdots(a_j-a_n)}{(a_j-a_0)\cdots(a_j-a_n)} = 1$$

since the denominator + numerator are equal

$$P_j(a_i) = \frac{(a_i-a_0)\cdots\overset{\to 0}{(a_i-a_i)}\cdots(a_i-a_n)}{(a_j-a_0)\cdots(a_j-a_i)\cdots(a_j-a_n)}$$

$i \neq j$

$$= \frac{0}{(a_j-a_0)\cdots(a_j-a_n)} = 0$$

60)

a) $\{p_0(x), \ldots, p_n(x)\} = B$

<u>WTS</u>: B is a linearly independent set of polynomials in P_n

Suppose $c_0 p_0(x) + \ldots + c_n p_n(x) = 0$

Use formula from 59(b)

$$p_j(a_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Plug in a_i into the above:

$$c_0 \underbrace{p_0(a_i)}_{0} + \ldots + c_i \underbrace{p_i(a_i)}_{1} + \ldots + c_n \underbrace{p_n(a_i)}_{0}$$

$$= 0 + 0 + \ldots + c_i \cdot 1 + 0 + \ldots + 0 = 0$$

$\Rightarrow c_i = 0$ for each i, $1 \leq i \leq n$.

b) B is a set of $n+1$ linearly independent polynomials in P_n

We know $\dim P_n = n+1$

Hence by Thm 6.10 (c)

B is a basis for P_n