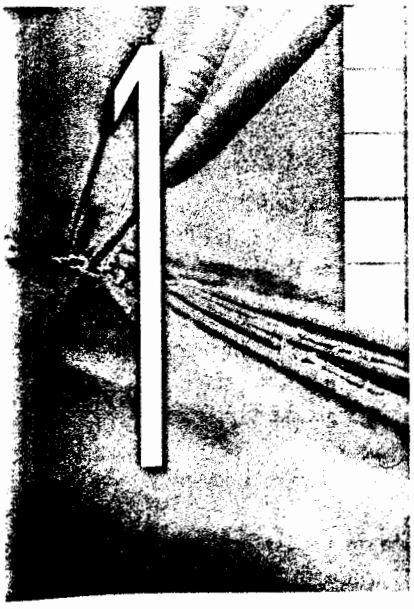

Vectors

Here they come pouring out of the blue. Little arrows for me and for you.

—Albert Hammond and Mike Hazelwood
Little Arrows
Dutchess Music/BMI, 1968

1.0 Introduction: The Racetrack Game

Many measurable quantities, such as length, area, volume, mass, and temperature, can be completely described by specifying their magnitude. Other quantities, such as velocity, force, and acceleration, require both a magnitude and a direction for their description. These quantities are *vectors.* For example, wind velocity is a vector consisting of wind speed and direction, such as 10 km/h southwest. Geometrically, vectors are often represented as arrows or directed line segments.

Although the idea of a vector was introduced in the 19th century, its usefulness in applications, particularly those in the physical sciences, was not realized until the 20th century. More recently, vectors have found applications in computer science, statistics, economics, and the life and social sciences. We will consider some of these many applications throughout this book.

This chapter introduces vectors and begins to consider some of their geometric and algebraic properties. We will also consider one nongeometric application where vectors are useful. We begin, though, with a simple game that introduces some of the key ideas. [You may even wish to play it with a friend during those (very rare!) dull moments in linear algebra class.]

The game is played on graph paper. A track, with a starting line and a finish line, is drawn on the paper. The track can be of any length and shape, so long as it is wide enough to accommodate all of the players. For this example, we will have two players (let's call them Ann and Bert) who use different colored pens to represent their cars or bicycles or whatever they are going to race around the track. (Let's think of Ann and Bert as cyclists.)

Ann and Bert each begin by drawing a dot on the starting line at a grid point on the graph paper. They take turns moving to a new grid point, subject to the following rules:

1. Each new grid point and the line segment connecting it to the previous grid point must lie entirely within the track.
2. No two players may occupy the same grid point on the same turn. (This is the "no collisions" rule.)
3. Each new move is related to the previous move as follows: If a player moves a units horizontally and b units vertically on one move, then on the next move

The Irish mathematician William Rowan Hamilton (1805–1865) used vector concepts in his study of complex numbers and their generalization, the quaternions.

he or she must move between $a - 1$ and $a + 1$ units horizontally and between $b - 1$ and $b + 1$ units vertically. In other words, if the second move is c units horizontally and d units vertically, then $|a - c| \leq 1$ and $|b - d| \leq 1$. (This is the "acceleration/deceleration" rule.) Note that this rule forces the first move to be 1 unit vertically and/or 1 unit horizontally.

A player who collides with another player or leaves the track is eliminated. The winner is the first player to cross the finish line. If more than one player crosses the finish line on the same turn, the one who goes farthest past the finish line is the winner.

In the sample game shown in Figure 1.1, Ann was the winner. Bert accelerated too quickly and had difficulty negotiating the turn at the top of the track.

To understand rule 3, consider Ann's third and fourth moves. On her third move, she went 1 unit horizontally and 3 units vertically. On her fourth move, her options were to move 0 to 2 units horizontally and 2 to 4 units vertically. (Notice that some of these combinations would have placed her outside the track.) She chose to move 2 units in each direction.

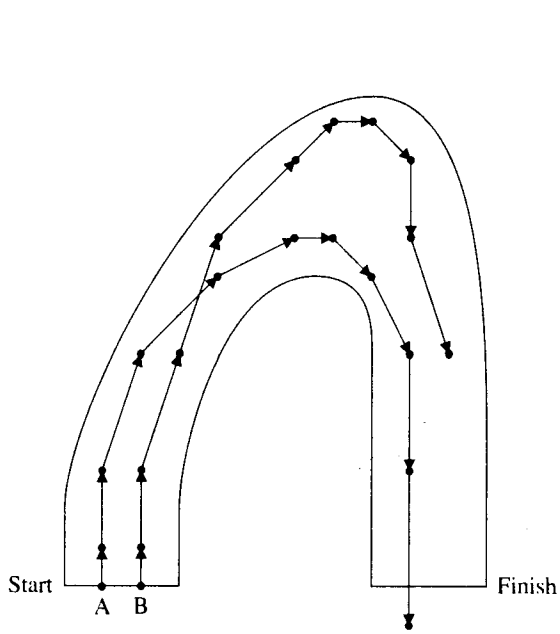

Figure 1.1
A sample game of racetrack

Problem 1 Play a few games of racetrack.

Problem 2 Is it possible for Bert to win this race by choosing a different sequence of moves?

Problem 3 Use the notation $[a, b]$ to denote a move that is a units horizontally and b units vertically. (Either a or b or both may be negative.) If move $[3, 4]$ has just been made, draw on graph paper all the grid points that could possibly be reached on the next move.

Problem 4 What is the *net* effect of two successive moves? In other words, if you move $[a, b]$ and then $[c, d]$, how far horizontally and vertically will you have moved altogether?

The Cartesian
after the Frenc
mathematician
(1596–1650),
of coordinates
problems to be
algebraic techn

The word *vect*
Latin root mea
vector is forme
displaced—or
given distance
Viewed anothe
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Some people p
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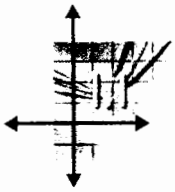
The word *com*
from the Latin
"together with
ing "to put." Th
together" out o

Problem 5 Write out Ann's sequence of moves using the $[a, b]$ notation. Suppose she begins at the origin $(0, 0)$ on the coordinate axes. Explain how you can find the coordinates of the grid point corresponding to each of her moves *without looking at the graph paper.* If the axes were drawn differently, so that Ann's starting point was not the origin but the point $(2, 3)$, what would the coordinates of her final point be?

Although simple, this game introduces several ideas that will be useful in our study of vectors. The next three sections consider vectors from geometric and algebraic viewpoints, beginning, as in the racetrack game, in the plane.



The Geometry and Algebra of Vectors

Vectors in the Plane

We begin by considering the Cartesian plane with the familiar x- and y-axes. A ***vector*** is a *directed line segment* that corresponds to a *displacement* from one point A to another point B; see Figure 1.2.

The Cartesian plane is named after the French philosopher and mathematician René Descartes (1596–1650), whose introduction of coordinates allowed *geometric* problems to be handled using *algebraic* techniques.

The vector from A to B is denoted by $\overrightarrow{AB}$; the point A is called its ***initial point,*** or ***tail,*** and the point B is called its ***terminal point,*** or ***head.*** Often, a vector is simply denoted by a single boldface, lowercase letter such as **v**.

The set of all points in the plane corresponds to the set of all vectors whose tails are at the origin O. To each point A, there corresponds the vector $\mathbf{a} = \overrightarrow{OA}$; to each vector **a** with tail at O, there corresponds its head A. (Vectors of this form are sometimes called *position vectors.*)

It is natural to represent such vectors using coordinates. For example, in Figure 1.3, $A = (3, 2)$ and we write the vector $\mathbf{a} = \overrightarrow{OA} = [3, 2]$ using square brackets. Similarly, the other vectors in Figure 1.3 are

$$\mathbf{b} = [-1, 3] \quad \text{and} \quad \mathbf{c} = [2, -1]$$

The individual coordinates (3 and 2 in the case of **a**) are called the ***components*** of the vector. A vector is sometimes said to be an *ordered pair* of real numbers. The order is important since, for example, $[3, 2] \neq [2, 3]$. In general, two vectors are equal if and only if their corresponding components are equal. Thus, $[x, y] = [1, 5]$ implies that $x = 1$ and $y = 5$.

The word *vector* comes from the Latin root meaning "to carry." A vector is formed when a point is displaced—or "carried off"—a given distance in a given direction. Viewed another way, vectors "carry" two pieces of information: their length and their direction.

When writing vectors by hand, it is difficult to indicate boldface. Some people prefer to write $\vec{v}$ for the vector denoted in print by **v**, but in most cases it is fine to use an ordinary lowercase v. It will usually be clear from the context when the letter denotes a vector.

It is frequently convenient to use ***column vectors*** instead of (or in addition to) ***row vectors.*** Another representation of $[3, 2]$ is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. (The important point is that the

The word *component* is derived from the Latin words *co,* meaning "together with," and *ponere,* meaning "to put." Thus, a vector is "put together" out of its components.

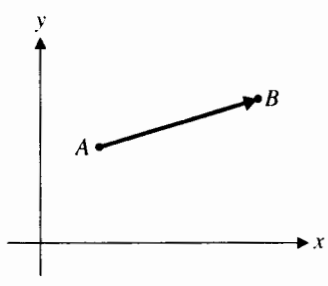


Figure 1.2

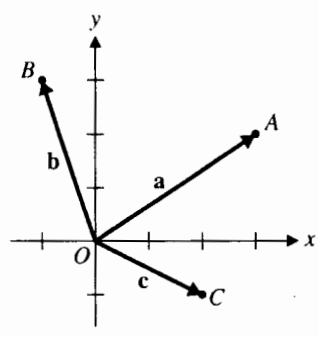


Figure 1.3

components are *ordered.*) In later chapters, you will see that column vectors are somewhat better from a computational point of view; for now, try to get used to both representations.

It may occur to you that we cannot really draw the vector $[0, 0] = \overrightarrow{OO}$ from the origin to itself. Nevertheless, it is a perfectly good vector and has a special name: the ***zero vector.*** The zero vector is denoted by $\mathbf{0}$.

$\mathbb{R}^2$ is pronounced "r two."

The set of all vectors with two components is denoted by $\mathbb{R}^2$ (where $\mathbb{R}$ denotes the set of real numbers from which the components of vectors in $\mathbb{R}^2$ are chosen). Thus, $[-1, 3.5]$, $[\sqrt{2}, \pi]$, and $[\frac{5}{3}, 4]$ are all in $\mathbb{R}^2$.

Thinking back to the racetrack game, let's try to connect all of these ideas to vectors whose tails are *not* at the origin. The etymological origin of the word *vector* in the verb "to carry" provides a clue. The vector $[3, 2]$ may be interpreted as follows: Starting at the origin O, travel 3 units to the right, then 2 units up, finishing at P. The same displacement may be applied with other initial points. Figure 1.4 shows two equivalent displacements, represented by the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

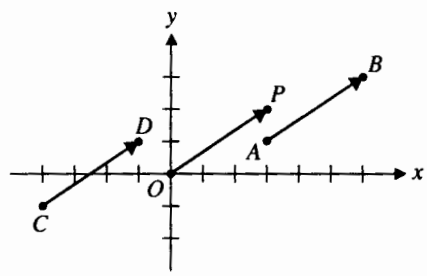


Figure 1.4

We define two vectors as *equal* if they have the same length and the same direction. Thus, $\overrightarrow{AB} = \overrightarrow{CD}$ in Figure 1.4. (Even though they have different initial and terminal points, they represent the same displacement.) Geometrically, two vectors are equal if one can be obtained by sliding (or *translating*) the other parallel to itself until the two vectors coincide. In terms of components, in Figure 1.4 we have $A = (3, 1)$ and $B = (6, 3)$. Notice that the vector $[3, 2]$ that records the displacement is just the difference of the respective components:

When vectors are referred to by their coordinates, they are being considered *analytically.*

$$\overrightarrow{AB} = [3, 2] = [6 - 3, 3 - 1]$$

Similarly,
$$\overrightarrow{CD} = [-1 - (-4), 1 - (-1)] = [3, 2]$$

and thus $\overrightarrow{AB} = \overrightarrow{CD}$, as expected.

A vector such as $\overrightarrow{OP}$ with its tail at the origin is said to be in ***standard position.*** The foregoing discussion shows that every vector can be drawn as a vector in standard position. Conversely, a vector in standard position can be redrawn (by translation) so that its tail is at any point in the plane.

Example 1.1

If $A = (-1, 2)$ and $B = (3, 4)$, find $\overrightarrow{AB}$ and redraw it (a) in standard position and (b) with its tail at the point $C = (2, -1)$.

Solution We compute $\overrightarrow{AB} = [3 - (-1), 4 - 2] = [4, 2]$. If $\overrightarrow{AB}$ is then translated to $\overrightarrow{CD}$, where $C = (2, -1)$, then we must have $D = (2 + 4, -1 + 2) = (6, 1)$. (See Figure 1.5.)

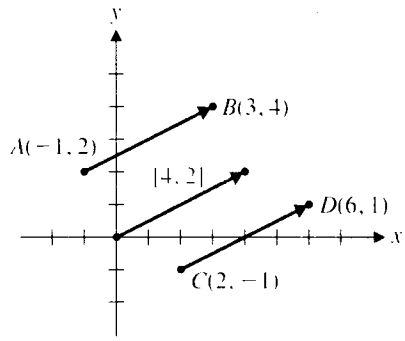


Figure 1.5

New Vectors from Old

As in the racetrack game, we often want to "follow" one vector by another. This leads to the notion of ***vector addition,*** the first basic vector operation.

If we follow **u** by **v**, we can visualize the total displacement as a third vector, denoted by **u** + **v**. In Figure 1.6, **u** = [1, 2] and **v** = [2, 2], so the net effect of following **u** by **v** is

$$[1 + 2, 2 + 2] = [3, 4]$$

which gives **u** + **v**. In general, if **u** = $[u_1, u_2]$ and **v** = $[v_1, v_2]$, then their ***sum*** **u** + **v** is the vector

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2]$$

It is helpful to visualize **u** + **v** geometrically. The following rule is the geometric version of the foregoing discussion.

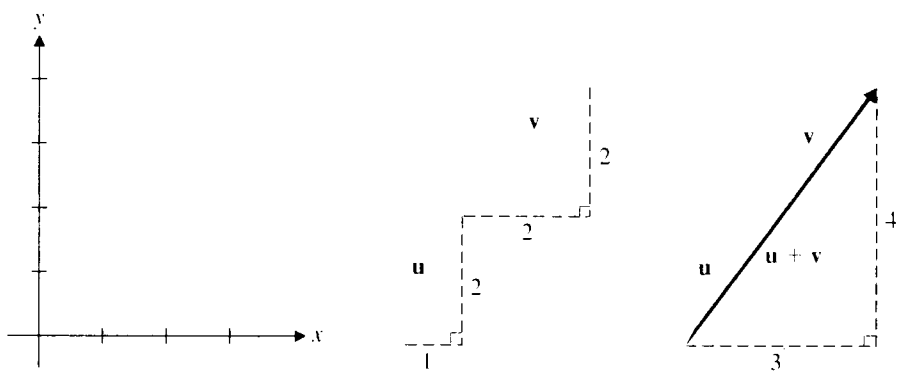


Figure 1.6
Vector addition

The Head-to-Tail Rule

Given vectors **u** and **v** in $\mathbb{R}^2$, translate **v** so that its tail coincides with the head of **u**. The ***sum*** **u** + **v** of **u** and **v** is the vector from the tail of **u** to the head of **v**. (See Figure 1.7.)

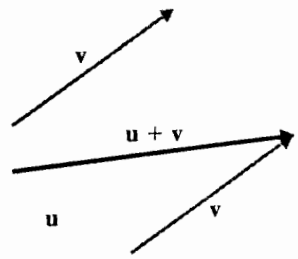


Figure 1.7
The head-to-tail rule

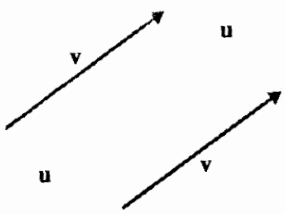


Figure 1.8
The parallelogram determined by **u** and **v**

By translating **u** and **v** parallel to themselves, we obtain a parallelogram, as shown in Figure 1.8. This parallelogram is called the *parallelogram determined by* **u** *and* **v**. It leads to an equivalent version of the head-to-tail rule for vectors in standard position.

The Parallelogram Rule

Given vectors **u** and **v** in $\mathbb{R}^2$ (in standard position), their ***sum*** **u** + **v** is the vector in standard position along the diagonal of the parallelogram determined by **u** and **v**. (See Figure 1.9.)

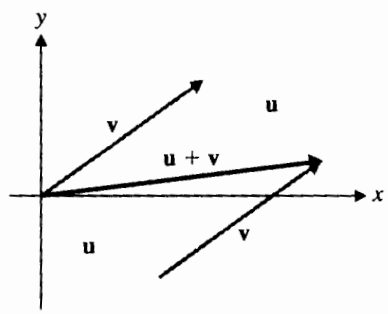


Figure 1.9
The parallelogram rule

Example 1.2

If $\mathbf{u} = [3, -1]$ and $\mathbf{v} = [1, 4]$, compute and draw **u** + **v**.

Solution We compute $\mathbf{u} + \mathbf{v} = [3 + 1, -1 + 4] = [4, 3]$. This vector is drawn using the head-to-tail rule in Figure 1.10(a) and using the parallelogram rule in Figure 1.10(b).

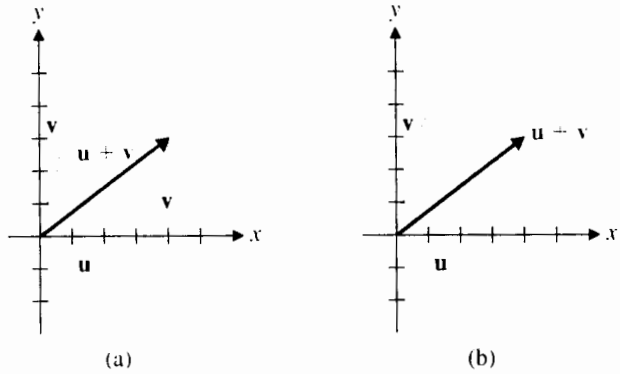


Figure 1.10

The second basic vector operation is ***scalar multiplication.*** Given a vector **v** and a real number c, the ***scalar multiple*** $c\mathbf{v}$ is the vector obtained by multiplying each component of **v** by c. For example, $3[-2, 4] = [-6, 12]$. In general,

$$c\mathbf{v} = c\,[v_1, v_2] = [cv_1, cv_2]$$

Geometrically, $c\mathbf{v}$ is a "scaled" version of **v**.

Example 1.3 If $\mathbf{v} = [-2, 4]$, compute and draw $2\mathbf{v}$, $\frac{1}{2}\mathbf{v}$, and $-2\mathbf{v}$.

Solution We calculate as follows:

$$\begin{aligned} 2\mathbf{v} &= [2(-2), 2(4)] = [-4, 8] \\ \tfrac{1}{2}\mathbf{v} &= [\tfrac{1}{2}(-2), \tfrac{1}{2}(4)] = [-1, 2] \\ -2\mathbf{v} &= [-2(-2), -2(4)] = [4, -8] \end{aligned}$$

These vectors are shown in Figure 1.11.

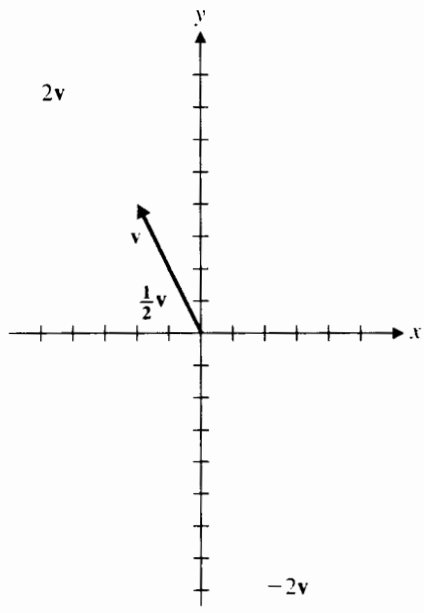


Figure 1.11

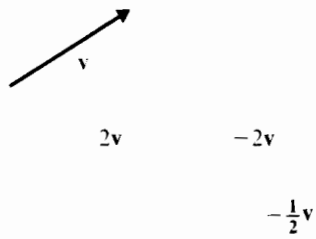


Figure 1.12

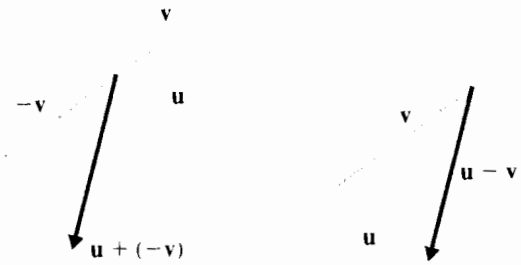


Figure 1.13
Vector subtraction

The term *scalar* comes from the Latin word *scala*, meaning "ladder." The equally spaced rungs on a ladder suggest a scale, and in vector arithmetic, multiplication by a constant changes only the scale (or length) of a vector. Thus, constants became known as scalars.

Observe that $c\mathbf{v}$ has the same direction as $\mathbf{v}$ if $c > 0$ and the opposite direction if $c < 0$. We also see that $c\mathbf{v}$ is $|c|$ times as long as $\mathbf{v}$. For this reason, in the context of vectors, constants (that is, real numbers) are referred to as ***scalars.*** As Figure 1.12 shows, when translation of vectors is taken into account, two vectors are scalar multiples of each other if and only if they are ***parallel.***

A special case of a scalar multiple is $(-1)\mathbf{v}$, which is written as $-\mathbf{v}$ and is called the ***negative of*** $\mathbf{v}$. We can use it to define ***vector subtraction:*** The ***difference*** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} - \mathbf{v}$ defined by

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

Figure 1.13 shows that $\mathbf{u} - \mathbf{v}$ corresponds to the "other" diagonal of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

Example 1.4

If $\mathbf{u} = [1, 2]$ and $\mathbf{v} = [-3, 1]$, then $\mathbf{u} - \mathbf{v} = [1 - (-3), 2 - 1] = [4, 1]$.

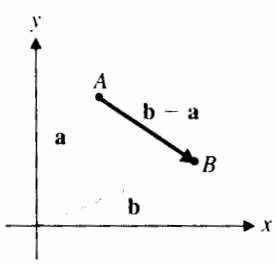


Figure 1.14

The definition of subtraction in Example 1.4 also agrees with the way we calculate a vector such as $\overrightarrow{AB}$. If the points A and B correspond to the vectors $\mathbf{a}$ and $\mathbf{b}$ in standard position, then $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, as shown in Figure 1.14. [Observe that the head-to-tail rule applied to this diagram gives the equation $\mathbf{a} + (\mathbf{b} - \mathbf{a}) = \mathbf{b}$. If we had accidentally drawn $\mathbf{b} - \mathbf{a}$ with its head at A instead of at B, the diagram would have read $\mathbf{b} + (\mathbf{b} - \mathbf{a}) = \mathbf{a}$, which is clearly wrong! More will be said about algebraic expressions involving vectors later in this section.]

Vectors in $\mathbb{R}^3$

Everything we have just done extends easily to three dimensions. The set of all *ordered triples* of real numbers is denoted by $\mathbb{R}^3$. Points and vectors are located using three mutually perpendicular coordinate axes that meet at the origin O. A point such as $A = (1, 2, 3)$ can be located as follows: First travel 1 unit along the x-axis, then move 2 units parallel to the y-axis, and finally move 3 units parallel to the z-axis. The corresponding vector $\mathbf{a} = [1, 2, 3]$ is then $\overrightarrow{OA}$, as shown in Figure 1.15.

Another way to visualize vector $\mathbf{a}$ in $\mathbb{R}^3$ is to construct a box whose six sides are determined by the three coordinate planes (the xy-, xz-, and yz-planes) and by three planes through the point $(1, 2, 3)$ parallel to the coordinate planes. The vector $[1, 2, 3]$ then corresponds to the diagonal from the origin to the opposite corner of the box (see Figure 1.16).

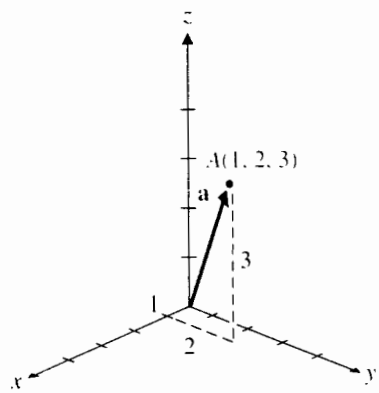


Figure 1.15

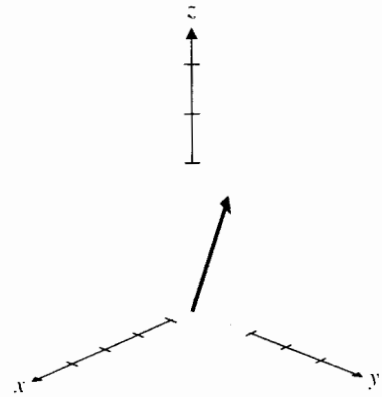


Figure 1.16

The "componentwise" definitions of vector addition and scalar multiplication are extended to $\mathbb{R}^3$ in an obvious way.

Vectors in $\mathbb{R}^n$

In general, we define $\mathbb{R}^n$ as the set of all *ordered n-tuples* of real numbers written as row or column vectors. Thus, a vector $\mathbf{v}$ in $\mathbb{R}^n$ is of the form

$$[v_1, v_2, \ldots, v_n] \quad \text{or} \quad \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The individual entries of $\mathbf{v}$ are its components; v_i is called the ith component.

We extend the definitions of vector addition and scalar multiplication to $\mathbb{R}^n$ in the obvious way: If $\mathbf{u} = [u_1, u_2, \ldots, u_n]$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, the ith component of $\mathbf{u} + \mathbf{v}$ is $u_i + v_i$ and the ith component of $c\mathbf{v}$ is just cv_i.

Since in $\mathbb{R}^n$ we can no longer draw pictures of vectors, it is important to be able to calculate with vectors. We must be careful not to assume that vector arithmetic will be similar to the arithmetic of real numbers. Often it is, and the algebraic calculations we do with vectors are similar to those we would do with scalars. But, in later sections, we will encounter situations where vector algebra is quite *unlike* our previous experience with real numbers. So it is important to verify any algebraic properties before attempting to use them.

One such property is *commutativity* of addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for vectors $\mathbf{u}$ and $\mathbf{v}$. This is certainly true in $\mathbb{R}^2$. Geometrically, the head-to-tail rule shows that both $\mathbf{u} + \mathbf{v}$ and $\mathbf{v} + \mathbf{u}$ are the main diagonals of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$. (The parallelogram rule also reflects this symmetry; see Figure 1.17.)

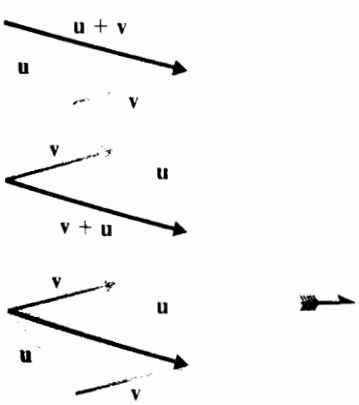


Figure 1.17
$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

Note that Figure 1.17 is simply an illustration of the property $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. It is not a proof, since it does not cover every possible case. For example, we must also include the cases where $\mathbf{u} = \mathbf{v}$, $\mathbf{u} = -\mathbf{v}$, and $\mathbf{u} = \mathbf{0}$. (What would diagrams for these cases look like?) For this reason, an algebraic proof is needed. However, it is just as easy to give a proof that is valid in $\mathbb{R}^n$ as to give one that is valid in $\mathbb{R}^2$.

The following theorem summarizes the algebraic properties of vector addition and scalar multiplication in $\mathbb{R}^n$. The proofs follow from the corresponding properties of real numbers.

Theorem 1.1 **Algebraic Properties of Vectors in $\mathbb{R}^n$**

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c and d be scalars. Then

a. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutativity
b. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associativity
c. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	
d. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	
e. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributivity
f. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributivity
g. $c(d\mathbf{u}) = (cd)\mathbf{u}$	
h. $1\mathbf{u} = \mathbf{u}$	

Remarks

- Properties (c) and (d) together with the commutativity property (a) imply that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ and $-\mathbf{u} + \mathbf{u} = \mathbf{0}$ as well.
- If we read the distributivity properties (e) and (f) from right to left, they say that we can *factor* a common scalar or a common vector from a sum.

The word *theorem* is derived from the Greek word *theorema*, which in turn comes from a word meaning "to look at." Thus, a theorem is based on the insights we have when we look at examples and extract from them properties that we try to prove hold in general. Similarly, when we understand something in mathematics—the proof of a theorem, for example—we often say, "I see."

Proof We prove properties (a) and (b) and leave the proofs of the remaining properties as exercises. Let $\mathbf{u} = [u_1, u_2, \ldots, u_n]$, $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, and $\mathbf{w} = [w_1, w_2, \ldots, w_n]$.

(a)
$$\begin{aligned}\mathbf{u} + \mathbf{v} &= [u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]\\ &= [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n]\\ &= [v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n]\\ &= [v_1, v_2, \ldots, v_n] + [u_1, u_2, \ldots, u_n]\\ &= \mathbf{v} + \mathbf{u}\end{aligned}$$

The second and fourth equalities are by the definition of vector addition, and the third equality is by the commutativity of addition of real numbers.

(b) Figure 1.18 illustrates associativity in $\mathbb{R}^2$. Algebraically, we have

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= ([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) + [w_1, w_2, \ldots, w_n]\\ &= [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] + [w_1, w_2, \ldots, w_n]\\ &= [(u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \ldots, (u_n + v_n) + w_n]\\ &= [u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n)]\\ &= [u_1, u_2, \ldots, u_n] + [v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n]\\ &= [u_1, u_2, \ldots, u_n] + ([v_1, v_2, \ldots, v_n] + [w_1, w_2, \ldots, w_n])\\ &= \mathbf{u} + (\mathbf{v} + \mathbf{w})\end{aligned}$$

The fourth equality is by the associativity of addition of real numbers. Note the careful use of parentheses.

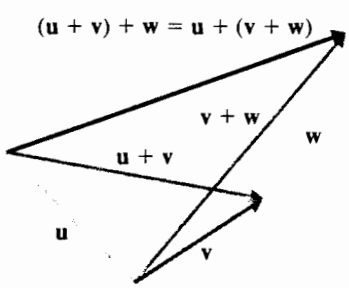


Figure 1.18

By property (b) of Theorem 1.1, we may unambiguously write $\mathbf{u} + \mathbf{v} + \mathbf{w}$ without parentheses, since we may group the summands in whichever way we please. By (a), we may also rearrange the summands—for example, as $\mathbf{w} + \mathbf{u} + \mathbf{v}$—if we choose. Likewise, sums of four or more vectors can be calculated without regard to order or grouping. In general, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are vectors in $\mathbb{R}^n$, we will write such sums without parentheses:

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k$$

The next example illustrates the use of Theorem 1.1 in performing algebraic calculations with vectors.

Example 1.5 Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{x}$ denote vectors in $\mathbb{R}^n$.
(a) Simplify $3\mathbf{a} + (5\mathbf{b} - 2\mathbf{a}) + 2(\mathbf{b} - \mathbf{a})$.
(b) If $5\mathbf{x} - \mathbf{a} = 2(\mathbf{a} + 2\mathbf{x})$, solve for $\mathbf{x}$ in terms of $\mathbf{a}$.

Solution We will give both solutions in detail, with reference to all of the properties in Theorem 1.1 that we use. It is good practice to justify all steps the first few times you do this type of calculation. Once you are comfortable with the vector properties, though, it is acceptable to leave out some of the intermediate steps to save time and space.

(a) We begin by inserting parentheses.

$$\begin{aligned}
3\mathbf{a} + (5\mathbf{b} - 2\mathbf{a}) + 2(\mathbf{b} - \mathbf{a}) &= (3\mathbf{a} + (5\mathbf{b} - 2\mathbf{a})) + 2(\mathbf{b} - \mathbf{a}) \\
&= (3\mathbf{a} + (-2\mathbf{a} + 5\mathbf{b})) + (2\mathbf{b} - 2\mathbf{a}) && \text{(a), (e)} \\
&= ((3\mathbf{a} + (-2\mathbf{a})) + 5\mathbf{b}) + (2\mathbf{b} - 2\mathbf{a}) && \text{(b)} \\
&= ((3 + (-2))\mathbf{a} + 5\mathbf{b}) + (2\mathbf{b} - 2\mathbf{a}) && \text{(f)} \\
&= (1\mathbf{a} + 5\mathbf{b}) + (2\mathbf{b} - 2\mathbf{a}) \\
&= ((\mathbf{a} + 5\mathbf{b}) + 2\mathbf{b}) - 2\mathbf{a} && \text{(b), (h)} \\
&= (\mathbf{a} + (5\mathbf{b} + 2\mathbf{b})) - 2\mathbf{a} && \text{(b)} \\
&= (\mathbf{a} + (5 + 2)\mathbf{b}) - 2\mathbf{a} && \text{(f)} \\
&= (7\mathbf{b} + \mathbf{a}) - 2\mathbf{a} && \text{(a)} \\
&= 7\mathbf{b} + (\mathbf{a} - 2\mathbf{a}) && \text{(b)} \\
&= 7\mathbf{b} + (1 - 2)\mathbf{a} && \text{(f), (h)} \\
&= 7\mathbf{b} + (-1)\mathbf{a} \\
&= 7\mathbf{b} - \mathbf{a}
\end{aligned}$$

You can see why we will agree to omit some of these steps! In practice, it is acceptable to simplify this sequence of steps as

$$\begin{aligned}
3\mathbf{a} + (5\mathbf{b} - 2\mathbf{a}) + 2(\mathbf{b} - \mathbf{a}) &= 3\mathbf{a} + 5\mathbf{b} - 2\mathbf{a} + 2\mathbf{b} - 2\mathbf{a} \\
&= (3\mathbf{a} - 2\mathbf{a} - 2\mathbf{a}) + (5\mathbf{b} + 2\mathbf{b}) \\
&= -\mathbf{a} + 7\mathbf{b}
\end{aligned}$$

or even to do most of the calculation mentally.

(b) In detail, we have

$$
\begin{aligned}
5\mathbf{x} - \mathbf{a} &= 2(\mathbf{a} + 2\mathbf{x}) && \\
5\mathbf{x} - \mathbf{a} &= 2\mathbf{a} + 2(2\mathbf{x}) && \text{(e)} \\
5\mathbf{x} - \mathbf{a} &= 2\mathbf{a} + (2 \cdot 2)\mathbf{x} && \text{(g)} \\
5\mathbf{x} - \mathbf{a} &= 2\mathbf{a} + 4\mathbf{x} && \\
(5\mathbf{x} - \mathbf{a}) - 4\mathbf{x} &= (2\mathbf{a} + 4\mathbf{x}) - 4\mathbf{x} && \\
(-\mathbf{a} + 5\mathbf{x}) - 4\mathbf{x} &= 2\mathbf{a} + (4\mathbf{x} - 4\mathbf{x}) && \text{(a), (b)} \\
-\mathbf{a} + (5\mathbf{x} - 4\mathbf{x}) &= 2\mathbf{a} + \mathbf{0} && \text{(b), (d)} \\
-\mathbf{a} + (5 - 4)\mathbf{x} &= 2\mathbf{a} && \text{(f), (c)} \\
-\mathbf{a} + (1)\mathbf{x} &= 2\mathbf{a} && \\
\mathbf{a} + (-\mathbf{a} + \mathbf{x}) &= \mathbf{a} + 2\mathbf{a} && \text{(h)} \\
(\mathbf{a} + (-\mathbf{a})) + \mathbf{x} &= (1 + 2)\mathbf{a} && \text{(b), (f)} \\
\mathbf{0} + \mathbf{x} &= 3\mathbf{a} && \text{(d)} \\
\mathbf{x} &= 3\mathbf{a} && \text{(c)}
\end{aligned}
$$

Again, we will usually omit most of these steps.

Linear Combinations and Coordinates

A vector that is a sum of scalar multiples of other vectors is said to be a *linear combination* of those vectors. The formal definition follows.

Definition A vector $\mathbf{v}$ is a ***linear combination*** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if there are scalars $c_1, c_2, \ldots, c_k$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$. The scalars $c_1, c_2, \ldots, c_k$ are called the ***coefficients*** of the linear combination.

Example 1.6 The vector $\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}$, since

$$3\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

Remark Determining whether a given vector is a linear combination of other vectors is a problem we will address in Chapter 2.

In $\mathbb{R}^2$, it is possible to depict linear combinations of two (nonparallel) vectors quite conveniently.

Example 1.7 Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. We can use $\mathbf{u}$ and $\mathbf{v}$ to locate a new set of axes (in the same way that $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ locate the standard coordinate axes). We can use

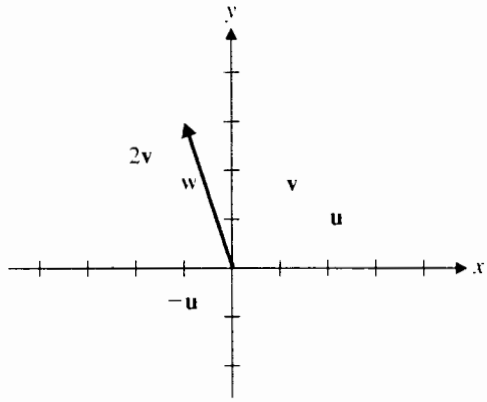


Figure 1.19

these new axes to determine a ***coordinate grid*** that will let us easily locate linear combinations of **u** and **v**.

As Figure 1.19 shows, **w** can be located by starting at the origin and traveling $-\mathbf{u}$ followed by $2\mathbf{v}$. That is,

$$\mathbf{w} = -\mathbf{u} + 2\mathbf{v}$$

We say that the coordinates of **w** with respect to **u** and **v** are -1 and 2. (Note that this is just another way of thinking of the coefficients of the linear combination.) It follows that

$$\mathbf{w} = -\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(Observe that -1 and 3 are the coordinates of **w** with respect to $\mathbf{e}_1$ and $\mathbf{e}_2$.)

Switching from the standard coordinate axes to alternative ones is a useful idea. It has applications in chemistry and geology, since molecular and crystalline structures often do not fall onto a rectangular grid. It is an idea that we will encounter repeatedly in this book.

Binary Vectors and Modular Arithmetic

We will also encounter a type of vector that has no geometric interpretation—at least not using Euclidean geometry. Computers represent data in terms of 0s and 1s (which can be interpreted as off/on, closed/open, false/true, or no/yes). ***Binary vectors*** are vectors each of whose components is a 0 or a 1. As we will see in Section 1.4, such vectors arise naturally in the study of many types of codes.

In this setting, the usual rules of arithmetic must be modified, since the result of each calculation involving scalars must be a 0 or a 1. The modified rules for addition and multiplication are given below.

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

The only curiosity here is the rule that $1 + 1 = 0$. This is not as strange as it appears; if we replace 0 with the word "even" and 1 with the word "odd," these tables simply

summarize the familiar *parity rules* for the addition and multiplication of even and odd integers. For example, $1 + 1 = 0$ expresses the fact that the sum of two odd integers is an even integer. With these rules, our set of scalars $\{0, 1\}$ is denoted by $\mathbb{Z}_2$ and is called the set of ***integers modulo 2.***

Example 1.8

In $\mathbb{Z}_2$, $1 + 1 + 0 + 1 = 1$ and $1 + 1 + 1 + 1 = 0$. (These calculations illustrate the parity rules again: The sum of three odds and an even is odd; the sum of four odds is even.)

We are using the term *length* differently from the way we used it in $\mathbb{R}^n$. This should not be confusing, since there is no *geometric* notion of length for binary vectors.

With $\mathbb{Z}_2$ as our set of scalars, we now extend the above rules to vectors. The set of all n-tuples of 0s and 1s (with all arithmetic performed modulo 2) is denoted by $\mathbb{Z}_2^n$. The vectors in $\mathbb{Z}_2^n$ are called ***binary vectors of length n.***

Example 1.9

The vectors in $\mathbb{Z}_2^2$ are $[0, 0]$, $[0, 1]$, $[1, 0]$, and $[1, 1]$. (How many vectors does $\mathbb{Z}_2^n$ contain, in general?)

Example 1.10

Let $\mathbf{u} = [1, 1, 0, 1, 0]$ and $\mathbf{v} = [0, 1, 1, 1, 0]$ be two binary vectors of length 5. Find $\mathbf{u} \cdot \mathbf{v}$.

Solution The calculation of $\mathbf{u} \cdot \mathbf{v}$ takes place in $\mathbb{Z}_2$, so we have

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 + 0 \cdot 0 \\ &= 0 + 1 + 0 + 1 + 0 \\ &= 0 \end{aligned}$$

It is possible to generalize what we have just done for binary vectors to vectors whose components are taken from a finite set $\{0, 1, 2, \ldots, k\}$ for $k \geq 2$. To do so, we must first extend the idea of binary arithmetic.

Example 1.11

The ***integers modulo 3*** is the set $\mathbb{Z}_3 = \{0, 1, 2\}$ with addition and multiplication given by the following tables:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Observe that the result of each addition and multiplication belongs to the set $\{0, 1, 2\}$; we say that $\mathbb{Z}_3$ is ***closed*** with respect to the operations of addition and multiplication. It is perhaps easiest to think of this set in terms of a 3-hour clock with 0, 1, and 2 on its face, as shown in Figure 1.20.

The calculation $1 + 2 = 0$ translates as follows: 2 hours after 1 o'clock, it is 0 o'clock. Just as 24:00 and 12:00 are the same on a 12-hour clock, so 3 and 0 are equivalent on this 3-hour clock. Likewise, all multiples of 3—positive and negative—are equivalent to 0 here; 1 is equivalent to any number that is 1 more than a multiple of 3 (such as -2, 4, and 7); and 2 is equivalent to any number that is 2 more than a

multiple of 3 (such as -1, 5, and 8). We can visualize the number line as wrapping around a circle, as shown in Figure 1.21.

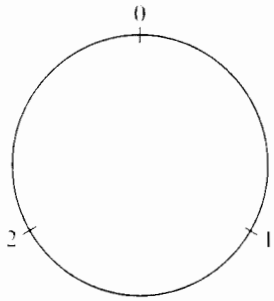


Figure 1.20
Arithmetic modulo 3

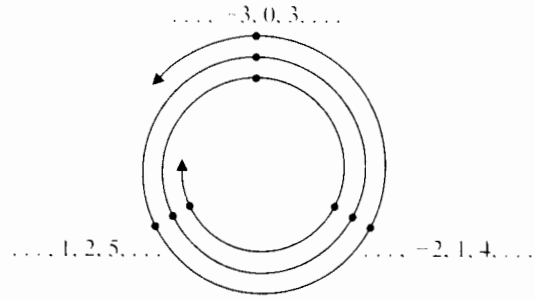


Figure 1.21

Example 1.12 To what is 3548 equivalent in Z_3?

Solution This is the same as asking where 3548 lies on our 3-hour clock. The key is to calculate how far this number is from the nearest (smaller) multiple of 3; that is, we need to know the *remainder* when 3548 is divided by 3. By long division, we find that $3548 = 3 \cdot 1182 + 2$, so the remainder is 2. Therefore, 3548 is equivalent to 2 in $\mathbb{Z}_3$.

In courses in abstract algebra and number theory, which explore this concept in greater detail, the above equivalence is often written as $3548 = 2 \pmod 3$ or $3548 \equiv 2 \pmod 3$, where $\equiv$ is read "is congruent to." We will not use this notation or terminology here.

Example 1.13 In $\mathbb{Z}_3$, calculate $2 + 2 + 1 + 2$.

Solution 1 We use the same ideas as in Example 1.12. The ordinary sum is $2 + 2 + 1 + 2 = 7$, which is 1 more than 6, so division by 3 leaves a remainder of 1. Thus, $2 + 2 + 1 + 2 = 1$ in $\mathbb{Z}_3$.

Solution 2 A better way to perform this calculation is to do it step by step entirely in $\mathbb{Z}_3$.

$$\begin{aligned} 2 + 2 + 1 + 2 &= (2 + 2) + 1 + 2 \\ &= 1 + 1 + 2 \\ &= (1 + 1) + 2 \\ &= 2 + 2 \\ &= 1 \end{aligned}$$

Here we have used parentheses to group the terms we have chosen to combine. We could speed things up by simultaneously combining the first two and the last two terms:

$$\begin{aligned} (2 + 2) + (1 + 2) &= 1 + 0 \\ &= 1 \end{aligned}$$

Repeated multiplication can be handled similarly. The idea is to use the addition and multiplication tables to reduce the result of each calculation to 0, 1, or 2.

Extending these ideas to vectors is straightforward.

Example 1.14

In $\mathbb{Z}_3^5$, let $\mathbf{u} = [2, 2, 0, 1, 2]$ and $\mathbf{v} = [1, 2, 2, 2, 1]$. Then

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 2\cdot 1 + 2\cdot 2 + 0\cdot 2 + 1\cdot 2 + 2\cdot 1\\ &= 2 + 1 + 0 + 2 + 2\\ &= 1\end{aligned}$$

Vectors in $\mathbb{Z}_3^5$ are referred to as ***ternary vectors of length* 5.**

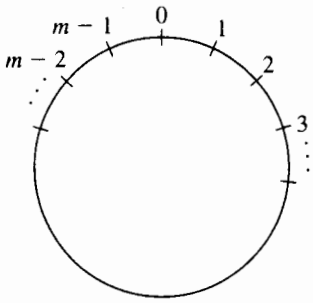


Figure 1.22
Arithmetic modulo m

In general, we have the set $\mathbb{Z}_m = \{0, 1, 2, \ldots, m-1\}$ of ***integers modulo m*** (corresponding to an m-hour clock, as shown in Figure 1.22). A vector of length n whose entries are in $\mathbb{Z}_m$ is called an ***m-ary vector of length n.*** The set of all m-ary vectors of length n is denoted by $\mathbb{Z}_m^n$.

Exercises 1.1

1. Draw the following vectors in standard position in $\mathbb{R}^2$:

   (a) $\mathbf{a} = \begin{bmatrix}3\\0\end{bmatrix}$ (b) $\mathbf{b} = \begin{bmatrix}2\\3\end{bmatrix}$

   (c) $\mathbf{c} = \begin{bmatrix}-2\\3\end{bmatrix}$ (d) $\mathbf{d} = \begin{bmatrix}3\\-2\end{bmatrix}$

2. Draw the vectors in Exercise 1 with their tails at the point $(1, -3)$.

3. Draw the following vectors in standard position in $\mathbb{R}^3$:

   (a) $\mathbf{a} = [0, 2, 0]$ (b) $\mathbf{b} = [3, 2, 1]$
   (c) $\mathbf{c} = [1, -2, 1]$ (d) $\mathbf{d} = [-1, -1, -2]$

4. If the vectors in Exercise 3 are translated so that their heads are at the point $(1, 2, 3)$, find the points that correspond to their tails.

5. For each of the following pairs of points, draw the vector $\overrightarrow{AB}$. Then compute and redraw $\overrightarrow{AB}$ as a vector in standard position.

   (a) $A = (1, -1)$, $B = (4, 2)$
   (b) $A = (0, -2)$, $B = (2, -1)$
   (c) $A = (2, \frac{3}{2})$, $B = (\frac{1}{2}, 3)$
   (d) $A = (\frac{1}{3}, \frac{1}{3})$, $B = (\frac{1}{6}, \frac{1}{2})$

6. A hiker walks 4 km north and then 5 km northeast. Draw displacement vectors representing the hiker's trip and draw a vector that represents the hiker's net displacement from the starting point.

Exercises 7–10 refer to the vectors in Exercise 1. Compute the indicated vectors and also show how the results can be obtained geometrically.

7. $\mathbf{a} + \mathbf{b}$ 8. $\mathbf{b} + \mathbf{c}$

9. $\mathbf{d} - \mathbf{c}$ 10. $\mathbf{a} - \mathbf{d}$

Exercises 11 and 12 refer to the vectors in Exercise 3. Compute the indicated vectors.

11. $2\mathbf{a} + 3\mathbf{c}$ 12. $2\mathbf{c} - 3\mathbf{b} - \mathbf{d}$

13. Find the components of the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are as shown in Figure 1.23.

14. In Figure 1.24, A, B, C, D, E, and F are the vertices of a regular hexagon centered at the origin.

    Express each of the following vectors in terms of $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$:

    (a) $\overrightarrow{AB}$ (b) $\overrightarrow{BC}$

    (c) $\overrightarrow{AD}$ (d) $\overrightarrow{CF}$

    (e) $\overrightarrow{AC}$ (f) $\overrightarrow{BC} + \overrightarrow{DE} + \overrightarrow{FA}$

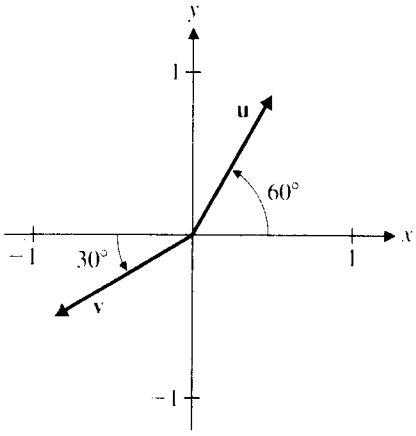


Figure 1.23

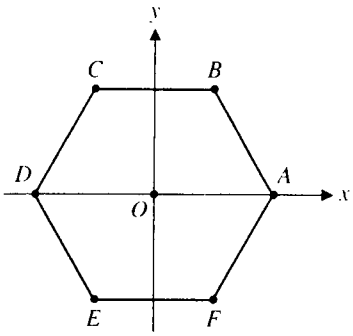


Figure 1.24

In Exercises 15 and 16, simplify the given vector expression. Indicate which properties in Theorem 1.1 you use.

15. $2(\mathbf{a} - 3\mathbf{b}) + 3(2\mathbf{b} + \mathbf{a})$

16. $-3(\mathbf{a} - \mathbf{c}) + 2(\mathbf{a} + 2\mathbf{b}) + 3(\mathbf{c} - \mathbf{b})$

In Exercises 17 and 18, solve for the vector **x** *in terms of the vectors* **a** *and* **b**.

17. $\mathbf{x} - \mathbf{a} = 2(\mathbf{x} - 2\mathbf{a})$

18. $\mathbf{x} + 2\mathbf{a} - \mathbf{b} = 3(\mathbf{x} + \mathbf{a}) - 2(2\mathbf{a} - \mathbf{b})$

In Exercises 19 and 20, draw the coordinate axes relative to **u** *and* **v** *and locate* **w**.

19. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{w} = 2\mathbf{u} + 3\mathbf{v}$

20. $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{w} = -\mathbf{u} - 2\mathbf{v}$

In Exercises 21 and 22, draw the standard coordinate axes on the same diagram as the axes relative to **u** *and* **v**. *Use these to find* **w** *as a linear combination of* **u** *and* **v**.

21. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

22. $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$

23. Draw diagrams to illustrate properties (d) and (e) of Theorem 1.1.

24. Give algebraic proofs of properties (d) through (g) of Theorem 1.1.

In Exercises 25–28, **u** *and* **v** *are binary vectors. Find* **u** + **v** *and* **u** · **v** *in each case.*

25. $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **26.** $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

27. $\mathbf{u} = [1, 0, 1, 1], \mathbf{v} = [1, 1, 1, 1]$

28. $\mathbf{u} = [1, 1, 0, 1, 0], \mathbf{v} = [0, 1, 1, 1, 0]$

29. Write out the addition and multiplication tables for $\mathbb{Z}_4$.

30. Write out the addition and multiplication tables for $\mathbb{Z}_5$.

In Exercises 31–43, perform the indicated calculations.

31. $2 + 2 + 2$ in $\mathbb{Z}_3$ **32.** $2 \cdot 2 \cdot 2$ in $\mathbb{Z}_3$

33. $2(2 + 1 + 2)$ in $\mathbb{Z}_3$ **34.** $3 + 1 + 2 + 3$ in $\mathbb{Z}_4$

35. $2 \cdot 3 \cdot 2$ in $\mathbb{Z}_4$ **36.** $3(3 + 3 + 2)$ in $\mathbb{Z}_4$

37. $2 + 1 + 2 + 2 + 1$ in $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_5$

38. $(3 + 4)(3 + 2 + 4 + 2)$ in $\mathbb{Z}_5$

39. $8(6 + 4 + 3)$ in $\mathbb{Z}_9$ **40.** 2^{100} in $\mathbb{Z}_{11}$

41. $[2, 1, 2] + [2, 0, 1]$ in $\mathbb{Z}_3^3$ **42.** $[2, 1, 2] \cdot [2, 2, 1]$ in $\mathbb{Z}_3^3$

43. $[2, 0, 3, 2] \cdot ([3, 1, 1, 2] + [3, 3, 2, 1])$ in $\mathbb{Z}_4^4$ and in $\mathbb{Z}_5^4$

In Exercises 44–55, solve the given equation or indicate that there is no solution.

44. $x + 3 = 2$ in $\mathbb{Z}_5$ **45.** $x + 5 = 1$ in $\mathbb{Z}_6$

46. $2x = 1$ in $\mathbb{Z}_3$ **47.** $2x = 1$ in $\mathbb{Z}_4$

48. $2x = 1$ in $\mathbb{Z}_5$ **49.** $3x = 4$ in $\mathbb{Z}_5$

50. $3x = 4$ in $\mathbb{Z}_6$ **51.** $6x = 5$ in $\mathbb{Z}_8$

52. $8x = 9$ in $\mathbb{Z}_{11}$ **53.** $2x + 3 = 2$ in $\mathbb{Z}_5$

54. $4x + 5 = 2$ in $\mathbb{Z}_6$ **55.** $6x + 3 = 1$ in $\mathbb{Z}_8$

56. (a) For which values of a does $x + a = 0$ have a solution in $\mathbb{Z}_5$?
(b) For which values of a and b does $x + a = b$ have a solution in $\mathbb{Z}_6$?
(c) For which values of a, b, and m does $x + a = b$ have a solution in $\mathbb{Z}_m$?

57. (a) For which values of a does $ax = 1$ have a solution in $\mathbb{Z}_5$?
(b) For which values of a does $ax = 1$ have a solution in $\mathbb{Z}_6$?
(c) For which values of a and m does $ax = 1$ have a solution in $\mathbb{Z}_m$?

Length and Angle: The Dot Product

It is quite easy to reformulate the familiar geometric concepts of length, distance, and angle in terms of vectors. Doing so will allow us to use these important and powerful ideas in settings more general than $\mathbb{R}^2$ and $\mathbb{R}^3$. In subsequent chapters, these simple geometric tools will be used to solve a wide variety of problems arising in applications—even when there is no geometry apparent at all!

The Dot Product

The vector versions of length, distance, and angle can all be described using the notion of the dot product of two vectors.

Definition If

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then the ***dot product*** $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

In words, $\mathbf{u} \cdot \mathbf{v}$ is the sum of the products of the corresponding components of $\mathbf{u}$ and $\mathbf{v}$. It is important to note a couple of things about this "product" that we have just defined: First, $\mathbf{u}$ and $\mathbf{v}$ must have the same number of components. Second, the dot product $\mathbf{u} \cdot \mathbf{v}$ is a *number,* not another vector. (This is why $\mathbf{u} \cdot \mathbf{v}$ is sometimes called the ***scalar product*** of $\mathbf{u}$ and $\mathbf{v}$.) The dot product of vectors in $\mathbb{R}^n$ is a special and important case of the more general notion of ***inner product,*** which we will explore in Chapter 7.

Example 1.15

Compute $\mathbf{u} \cdot \mathbf{v}$ when $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$.

Solution $\mathbf{u} \cdot \mathbf{v} = 1 \cdot (-3) + 2 \cdot 5 + (-3) \cdot 2 = 1$

Notice that if we had calculated $\mathbf{v} \cdot \mathbf{u}$ in Example 1.15, we would have computed

$$\mathbf{v} \cdot \mathbf{u} = (-3) \cdot 1 + 5 \cdot 2 + 2 \cdot (-3) = 1$$

That $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ in general is clear, since the individual products of the components commute. This commutativity property is one of the properties of the dot product that we will use repeatedly. The main properties of the dot product are summarized in Theorem 1.2.

Theorem 1.2 Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c be a scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
b. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
d. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

We prove (a) and (c) and leave proof of the remaining properties for the exercises.

(a) Applying the definition of dot product to $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$, we obtain

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \\ &= v_1 u_1 + v_2 u_2 + \cdots + v_n u_n \\ &= \mathbf{v} \cdot \mathbf{u} \end{aligned}$$

where the middle equality follows from the fact that multiplication of real numbers is commutative.

(c) Using the definitions of scalar multiplication and dot product, we have

$$\begin{aligned} (c\mathbf{u}) \cdot \mathbf{v} &= [cu_1, cu_2, \ldots, cu_n] \cdot [v_1, v_2, \ldots, v_n] \\ &= cu_1 v_1 + cu_2 v_2 + \cdots + cu_n v_n \\ &= c(u_1 v_1 + u_2 v_2 + \cdots + u_n v_n) \\ &= c(\mathbf{u} \cdot \mathbf{v}) \end{aligned}$$

Property (b) can be read from right to left, in which case it says that we can factor out a common vector $\mathbf{u}$ from a sum of dot products. This property also has a "right-handed" analogue that follows from properties (b) and (a) together: $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$.

Property (c) can be extended to give $\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$ (Exercise 58). This extended version of (c) essentially says that in taking a scalar multiple of a dot product of vectors, the scalar can first be combined with whichever vector is more convenient. For example,

$$(\tfrac{1}{2}[-1, -3, 2]) \cdot [6, -4, 0] = [-1, -3, 2] \cdot (\tfrac{1}{2}[6, -4, 0]) = [-1, -3, 2] \cdot [3, -2, 0] = 3$$

With this approach we avoid introducing fractions into the vectors, as the original grouping would have.

The second part of (d) uses the logical connective *if and only if*. Appendix A discusses this phrase in more detail, but for the moment let us just note that the wording signals a *double implication*—namely,

$$\text{if } \mathbf{u} = \mathbf{0}, \text{ then } \mathbf{u} \cdot \mathbf{u} = 0$$

and

$$\text{if } \mathbf{u} \cdot \mathbf{u} = 0, \text{ then } \mathbf{u} = \mathbf{0}$$

Theorem 1.2 shows that aspects of the algebra of vectors resemble the algebra of numbers. The next example shows that we can sometimes find vector analogues of familiar identities.

Example 1.16 Prove that $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$.

Solution

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) &= (\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} + (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v}\end{aligned}$$

(Identify the parts of Theorem 1.2 that were used at each step.)

Length

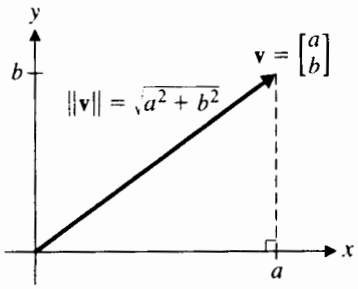


Figure 1.25

To see how the dot product plays a role in the calculation of lengths, recall how lengths are computed in the plane. The Theorem of Pythagoras is all we need.

In $\mathbb{R}^2$, the length of the vector $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ is the distance from the origin to the point (a, b), which, by Pythagoras' Theorem, is given by $\sqrt{a^2 + b^2}$, as in Figure 1.25. Observe that $a^2 + b^2 = \mathbf{v} \cdot \mathbf{v}$. This leads to the following definition.

Definition The ***length*** (or ***norm***) of a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$ is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

In words, the length of a vector is the square root of the sum of the squares of its components. Note that the square root of $\mathbf{v} \cdot \mathbf{v}$ is always defined, since $\mathbf{v} \cdot \mathbf{v} \geq 0$ by Theorem 1.2(d). Note also that the definition can be rewritten to give $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$, which will be useful in proving further properties of the dot product and lengths of vectors.

Example 1.17

$$\|[2, 3]\| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

Theorem 1.3 lists some of the main properties of vector length.

Theorem 1.3 Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ and let c be a scalar. Then

a. $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$

b. $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$

Proof Property (a) follows immediately from Theorem 1.2(d). To show (b), we have

$$\|c\mathbf{v}\|^2 = (c\mathbf{v}) \cdot (c\mathbf{v}) = c^2(\mathbf{v} \cdot \mathbf{v}) = c^2\|\mathbf{v}\|^2$$

using Theorem 1.2(c). Taking square roots of both sides, using the fact that $\sqrt{c^2} = |c|$ for any real number c, gives the result.

A vector of length 1 is called a ***unit vector.*** In $\mathbb{R}^2$, the set of all unit vectors can be identified with the *unit circle*, the circle of radius 1 centered at the origin (see Figure 1.26). Given any nonzero vector $\mathbf{v}$, we can always find a unit vector in the same direction as $\mathbf{v}$ by dividing $\mathbf{v}$ by its own length (or, equivalently, *multiplying* by $1/\|\mathbf{v}\|$). We can show this algebraically by using property (b) of Theorem 1.3 above: If $\mathbf{u} = (1/\|\mathbf{v}\|)\mathbf{v}$, then

$$\|\mathbf{u}\| = \|(1/\|\mathbf{v}\|)\mathbf{v}\| = |1/\|\mathbf{v}\|| \, \|\mathbf{v}\| = (1/\|\mathbf{v}\|)\|\mathbf{v}\| = 1$$

and $\mathbf{u}$ is in the same direction as $\mathbf{v}$, since $1/\|\mathbf{v}\|$ is a positive scalar. Finding a unit vector in the same direction is often referred to as ***normalizing*** a vector (see Figure 1.27).

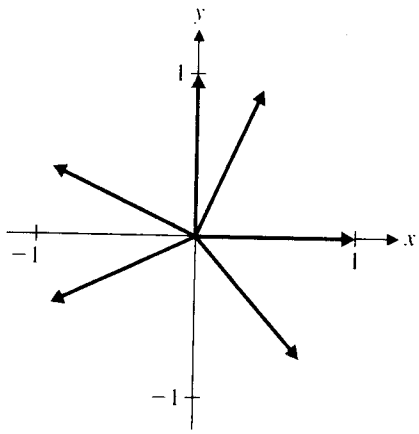


Figure 1.26
Unit vectors in $\mathbb{R}^2$

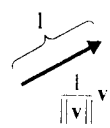


Figure 1.27
Normalizing a vector

Example 1.18

In $\mathbb{R}^2$, let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors, since the sum of the squares of their components is 1 in each case. Similarly, in $\mathbb{R}^3$, we can construct unit vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Observe in Figure 1.28 that these vectors serve to locate the positive coordinate axes in $\mathbb{R}^2$ and $\mathbb{R}^3$.

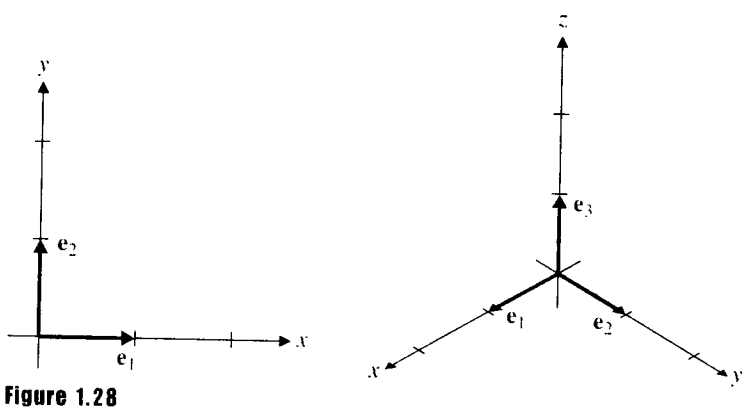


Figure 1.28
Standard unit vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

In general, in $\mathbb{R}^n$, we define unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, where $\mathbf{e}_i$ has 1 in its ith component and zeros elsewhere. These vectors arise repeatedly in linear algebra and are called the ***standard unit vectors.***

Example 1.19

Normalize the vector $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

Solution $\|\mathbf{v}\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$, so a unit vector in the same direction as $\mathbf{v}$ is given by

$$\mathbf{u} = (1/\|\mathbf{v}\|)\mathbf{v} = (1/\sqrt{14})\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{14} \\ -1/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$$

Since property (b) of Theorem 1.3 describes how length behaves with respect to scalar multiplication, natural curiosity suggests that we ask whether length and vector addition are compatible. It would be nice if we had an identity such as $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$, but for almost any choice of vectors $\mathbf{u}$ and $\mathbf{v}$ this turns out to be false. [See Exercise 52(a).] However, all is not lost, for it turns out that if we replace the = sign by $\leq$, the resulting inequality is true. The proof of this famous and important result—the Triangle Inequality—relies on another important inequality—the Cauchy-Schwarz Inequality—which we will prove and discuss in more detail in Chapter 7.

Theorem 1.4 **The Cauchy-Schwarz Inequality**

For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

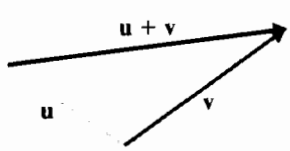


Figure 1.29
The Triangle Inequality

See Exercises 71 and 72 for algebraic and geometric approaches to the proof of this inequality.

In $\mathbb{R}^2$ or $\mathbb{R}^3$, where we can use geometry, it is clear from a diagram such as Figure 1.29 that $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ for all vectors $\mathbf{u}$ and $\mathbf{v}$. We now show that this is true more generally.

Theorem 1.5 **The Triangle Inequality**

For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

Proof Since both sides of the inequality are nonnegative, showing that the *square* of the left-hand side is less than or equal to the *square* of the right-hand side is equivalent to proving the theorem. (Why?) We compute

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\
&= \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v} && \text{By Example 1.9} \\
&\leq \|\mathbf{u}\|^2 + 2|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2 \\
&\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 && \text{By Cauchy-Schwarz} \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2
\end{aligned}$$

as required. ∎

Distance

The distance between two vectors is the direct analogue of the distance between two points on the real number line or two points in the Cartesian plane. On the number line (Figure 1.30), the distance between the numbers a and b is given by $|a-b|$. (Taking the absolute value ensures that we do not need to know which of a or b is larger.) This distance is also equal to $\sqrt{(a-b)^2}$, and its two-dimensional generalization is the familiar formula for the distance d between points (a_1, a_2) and (b_1, b_2)—namely, $d = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2}$.

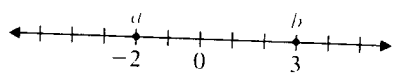


Figure 1.30
$d = |a-b| = |-2-3| = 5$

In terms of vectors, if $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, then d is just the length of $\mathbf{a}-\mathbf{b}$, as shown in Figure 1.31. This is the basis for the next definition.

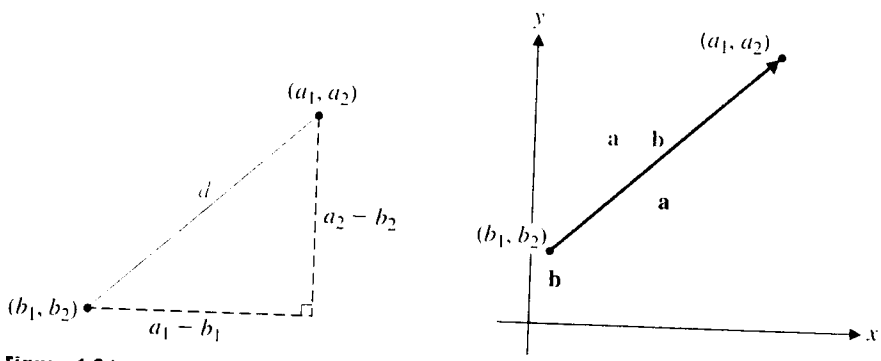


Figure 1.31
$d = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2} = \|\mathbf{a}-\mathbf{b}\|$

Definition The ***distance*** $\mathrm{d}(\mathbf{u}, \mathbf{v})$ between vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is defined by

$$\mathrm{d}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$$

Example 1.20

Find the distance between $\mathbf{u} = \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$.

Solution We compute $\mathbf{u} - \mathbf{v} = \begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix}$, so

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(\sqrt{2})^2 + (-1)^2 + 1^2} = \sqrt{4} = 2$$

Angles

The dot product can also be used to calculate the angle between a pair of vectors. In $\mathbb{R}^2$ or $\mathbb{R}^3$, the angle between the nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ will refer to the angle θ determined by these vectors that satisfies $0 \leq \theta \leq 180°$ (see Figure 1.32).

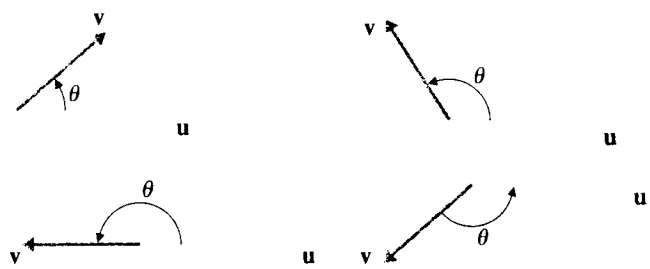


Figure 1.32
The angle between $\mathbf{u}$ and $\mathbf{v}$

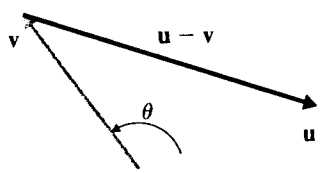


Figure 1.33

In Figure 1.33, consider the triangle with sides $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Applying the law of cosines to this triangle yields

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$$

Expanding the left-hand side and using $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$ several times, we obtain

$$\|\mathbf{u}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$$

which, after simplification, leaves us with $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$. From this we obtain the following formula for the cosine of the angle θ between nonzero vectors $\mathbf{u}$ and $\mathbf{v}$. We state it as a definition.

Definition For nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

Example 1.21

Compute the angle between the vectors $\mathbf{u} = [2, 1, -2]$ and $\mathbf{v} = [1, 1, 1]$.

Solution We calculate $\mathbf{u}\cdot\mathbf{v} = 2\cdot 1 + 1\cdot 1 + (-2)\cdot 1 = 1$, $\|\mathbf{u}\| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$, and $\|\mathbf{v}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$. Therefore, $\cos\theta = 1/3\sqrt{3}$, so $\theta = \cos^{-1}(1/3\sqrt{3}) \approx 1.377$ radians, or 78.9°.

Example 1.22 Compute the angle between the diagonals on two adjacent faces of a cube.

Solution The dimensions of the cube do not matter, so we will work with a cube with sides of length 1. Orient the cube relative to the coordinate axes in $\mathbb{R}^3$, as shown in Figure 1.34, and take the two side diagonals to be the vectors $[1, 0, 1]$ and $[0, 1, 1]$. Then angle θ between these vectors satisfies

$$\cos\theta = \frac{1\cdot 0 + 0\cdot 1 + 1\cdot 1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

from which it follows that the required angle is $\pi/3$ radians, or 60°.

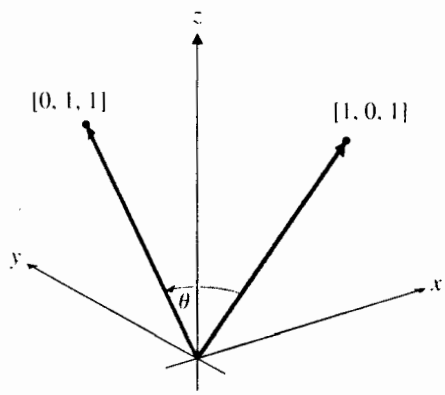


Figure 1.34

(Actually, we don't need to do any calculations at all to get this answer. If we draw a third side diagonal joining the vertices at $(1, 0, 1)$ and $(0, 1, 1)$, we get an equilateral triangle, since all of the side diagonals are of equal length. The angle we want is one of the angles of this triangle and therefore measures 60°. Sometimes a little insight can save a lot of calculation; in this case, it gives a nice check on our work!)

Remarks

- As this discussion shows, we usually will have to settle for an approximation to the angle between two vectors. However, when the angle is one of the so-called special angles (0°, 30°, 45°, 60°, 90°, or an integer multiple of these), we should be able to recognize its cosine (Table 1.1) and thus give the corresponding angle exactly. In all other cases, we will use a calculator or computer to approximate the desired angle by means of the inverse cosine function.

Table 1.1 Cosines of Special Angles

θ	0°	30°	45°	60°	90°
$\cos\theta$	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{0}}{2} = 0$

- The derivation of the formula for the cosine of the angle between two vectors is valid only in $\mathbb{R}^2$ or $\mathbb{R}^3$, since it depends on a geometric fact: the law of cosines. In $\mathbb{R}^n$, for $n > 3$, the formula can be taken as a *definition* instead. This makes sense, since the Cauchy-Schwarz Inequality implies that $\left|\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right| \leq 1$, so $\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$ ranges from -1 to 1, just as the cosine function does.

Orthogonal Vectors

The word *orthogonal* is derived from the Greek words *orthos*, meaning "upright," and *gonia*, meaning "angle." Hence, orthogonal literally means "right-angled." The Latin equivalent is *rectangular*.

The concept of perpendicularity is fundamental to geometry. Anyone studying geometry quickly realizes the importance and usefulness of right angles. We now generalize the idea of perpendicularity to vectors in $\mathbb{R}^n$, where it is called ***orthogonality***.

In $\mathbb{R}^2$ or $\mathbb{R}^3$, two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if the angle θ between them is a right angle—that is, if $\theta = \pi/2$ radians, or 90°. Thus, $\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \cos 90° = 0$, and it follows that $\mathbf{u}\cdot\mathbf{v} = 0$. This motivates the following definition.

Definition Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are ***orthogonal*** to each other if $\mathbf{u}\cdot\mathbf{v} = 0$.

Since $\mathbf{0}\cdot\mathbf{v} = 0$ for every vector $\mathbf{v}$ in $\mathbb{R}^n$, the zero vector is orthogonal to every vector.

Example 1.23 In $\mathbb{R}^3$, $\mathbf{u} = [1, 1, -2]$ and $\mathbf{v} = [3, 1, 2]$ are orthogonal, since $\mathbf{u}\cdot\mathbf{v} = 3 + 1 - 4 = 0$.

Using the notion of orthogonality, we get an easy proof of Pythagoras' Theorem, valid in $\mathbb{R}^n$.

Theorem 1.6 **Pythagoras' Theorem**

For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Proof From Example 1.16, we have $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. It follows immediately that $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $\mathbf{u}\cdot\mathbf{v} = 0$. See Figure 1.35. ∎

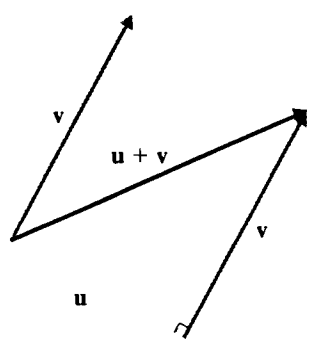


Figure 1.35

The concept of orthogonality is one of the most important and useful in linear algebra, and it often arises in surprising ways. Chapter 5 contains a detailed treatment of the topic, but we will encounter it many times before then. One problem in which it clearly plays a role is finding the distance from a point to a line, where "dropping a perpendicular" is a familiar step.

Projections

We now consider the problem of finding the distance from a point to a line in the context of vectors. As you will see, this technique leads to an important concept: the projection of a vector onto another vector.

As Figure 1.36 shows, the problem of finding the distance from a point B to a line ℓ (in $\mathbb{R}^2$ or $\mathbb{R}^3$) reduces to the problem of finding the length of the perpendicular line segment $\overline{PB}$ or, equivalently, the length of the vector $\overrightarrow{PB}$. If we choose a point A on ℓ, then, in the right-angled triangle ΔAPB, the other two vectors are the leg $\overrightarrow{AP}$ and the hypotenuse $\overrightarrow{AB}$. $\overrightarrow{AP}$ is called the *projection* of $\overrightarrow{AB}$ onto the line ℓ. We will now look at this situation in terms of vectors.

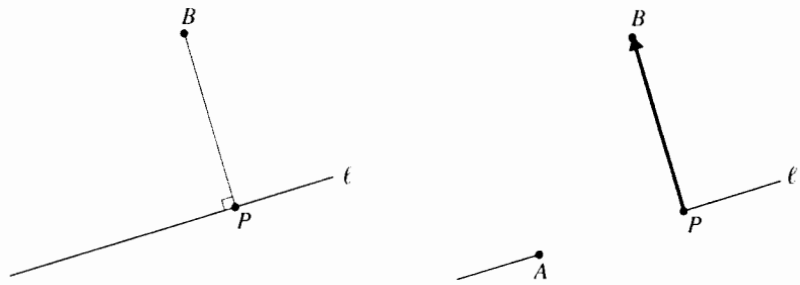


Figure 1.36
The distance from a point to a line

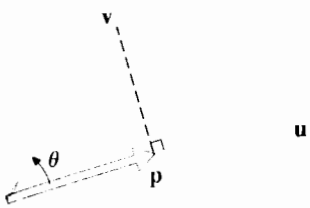


Figure 1.37
The projection of **v** onto **u**

Consider two nonzero vectors **u** and **v**. Let **p** be the vector obtained by dropping a perpendicular from the head of **v** onto **u** and let θ be the angle between **u** and **v**, as shown in Figure 1.37. Then clearly $\mathbf{p} = \|\mathbf{p}\|\hat{\mathbf{u}}$, where $\hat{\mathbf{u}} = (1/\|\mathbf{u}\|)\mathbf{u}$ is the unit vector in the direction of **u**. Moreover, elementary trigonometry gives $\|\mathbf{p}\| = \|\mathbf{v}\| \cos\theta$, and we know that $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$. Thus, after substitution, we obtain

$$\begin{aligned}\mathbf{p} &= \|\mathbf{v}\|\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)\left(\frac{1}{\|\mathbf{u}\|}\right)\mathbf{u}\\ &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\right)\mathbf{u}\\ &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\end{aligned}$$

This is the formula we want, and it is the basis of the following definition for vectors in $\mathbb{R}^n$.

Definition If **u** and **v** are vectors in $\mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{0}$, then the ***projection of*** **v** ***onto*** **u** is the vector $\text{proj}_{\mathbf{u}}(\mathbf{v})$ defined by

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$$

An alternative way to derive this formula is described in Exercise 73.

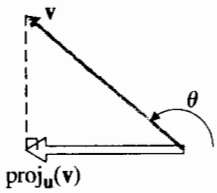


Figure 1.38

Remarks

- The term *projection* comes from the idea of projecting an image onto a wall (with a slide projector, for example). Imagine a beam of light with rays parallel to each other and perpendicular to **u** shining down on **v**. The projection of **v** onto **u** is just the shadow cast, or projected, by **v** onto **u**.
- It may be helpful to think of $\text{proj}_{\mathbf{u}}(\mathbf{v})$ as a function with variable **v**. Then the variable **v** occurs only once on the right-hand side of the definition. Also, it is helpful to remember Figure 1.38, which reminds us that $\text{proj}_{\mathbf{u}}(\mathbf{v})$ is a scalar multiple of the vector **u** (*not* **v**).
- Although in our derivation of the definition of $\text{proj}_{\mathbf{u}}(\mathbf{v})$ we required **v** as well as **u** to be nonzero (why?), it is clear from the geometry that the projection of the zero vector onto **u** is **0**. The definition is in agreement with this, since $\left(\dfrac{\mathbf{u}\cdot\mathbf{0}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = 0\mathbf{u} = \mathbf{0}$.
- If the angle between **u** and **v** is obtuse, as in Figure 1.38, then $\text{proj}_{\mathbf{u}}(\mathbf{v})$ will be in the opposite direction from **u**; that is, $\text{proj}_{\mathbf{u}}(\mathbf{v})$ will be a *negative* scalar multiple of **u**.
- If **u** is a unit vector then $\text{proj}_{\mathbf{u}}(\mathbf{v}) = (\mathbf{u}\cdot\mathbf{v})\mathbf{u}$. (Why?)

Example 1.24

Find the projection of **v** onto **u** in each case.

(a) $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (b) $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{u} = \mathbf{e}_3$

(c) $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}$

Solution

(a) We compute $\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 3 \end{bmatrix} = 1$ and $\mathbf{u}\cdot\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5$, so

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \frac{1}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 1/5 \end{bmatrix}$$

(b) Since $\mathbf{e}_3$ is a unit vector,

$$\text{proj}_{\mathbf{e}_3}(\mathbf{v}) = (\mathbf{e}_3\cdot\mathbf{v})\mathbf{e}_3 = 3\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

(c) We see that $\|\mathbf{u}\| = \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{2}} = 1$. Thus,

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = (\mathbf{u}\cdot\mathbf{v})\mathbf{u} = \left(\frac{1}{2}+1+\frac{3}{\sqrt{2}}\right)\begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix} = \frac{3(1+\sqrt{2})}{2}\begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{3(1+\sqrt{2})}{4}\begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}$$

Exercises 1.2

In Exercises 1–6, find $\mathbf{u} \cdot \mathbf{v}$.

1. $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ **2.** $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$

3. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ CAS **4.** $\mathbf{u} = \begin{bmatrix} 1.5 \\ 0.4 \\ -2.1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3.0 \\ 5.2 \\ -0.6 \end{bmatrix}$

5. $\mathbf{u} = [1, \sqrt{2}, \sqrt{3}, 0], \mathbf{v} = [4, -\sqrt{2}, 0, -5]$

CAS **6.** $\mathbf{u} = [1.12, -3.25, 2.07, -1.83]$,
$\mathbf{v} = [-2.29, 1.72, 4.33, -1.54]$

In Exercises 7–12, find $\|\mathbf{u}\|$ *for the given exercise, and give a unit vector in the direction of* $\mathbf{u}$.

7. Exercise 1 **8.** Exercise 2 **9.** Exercise 3

CAS **10.** Exercise 4 **11.** Exercise 5 CAS **12.** Exercise 6

In Exercises 13–16, find the distance $\mathrm{d}(\mathbf{u}, \mathbf{v})$ *between* $\mathbf{u}$ *and* $\mathbf{v}$ *in the given exercise.*

13. Exercise 1 **14.** Exercise 2

15. Exercise 3 CAS **16.** Exercise 4

17. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, $n \geq 2$, and c is a scalar, explain why the following expressions make no sense:

(a) $\|\mathbf{u} \cdot \mathbf{v}\|$ (b) $\mathbf{u} \cdot \mathbf{v} + \mathbf{w}$
(c) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$ (d) $c \cdot (\mathbf{u} + \mathbf{w})$

In Exercises 18–23, determine whether the angle between $\mathbf{u}$ *and* $\mathbf{v}$ *is acute, obtuse, or a right angle.*

18. $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ **19.** $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$

20. $\mathbf{u} = [5, 4, -3], \mathbf{v} = [1, -2, -1]$

CAS **21.** $\mathbf{u} = [0.9, 2.1, 1.2], \mathbf{v} = [-4.5, 2.6, -0.8]$

22. $\mathbf{u} = [1, 2, 3, 4], \mathbf{v} = [-3, 1, 2, -2]$

23. $\mathbf{u} = [1, 2, 3, 4], \mathbf{v} = [5, 6, 7, 8]$

In Exercises 24–29, find the angle between $\mathbf{u}$ *and* $\mathbf{v}$ *in the given exercise.*

24. Exercise 18 **25.** Exercise 19 **26.** Exercise 20

CAS **27.** Exercise 21 CAS **28.** Exercise 22 CAS **29.** Exercise 23

30. Let $A = (-3, 2)$, $B = (1, 0)$, and $C = (4, 6)$. Prove that ΔABC is a right-angled triangle.

31. Let $A = (1, 1, -1)$, $B = (-3, 2, -2)$, and $C = (2, 2, -4)$. Prove that ΔABC is a right-angled triangle.

CAS **32.** Find the angle between a diagonal of a cube and an adjacent edge.

33. A cube has four diagonals. Show that no two of them are perpendicular.

In Exercises 34–39, find the projection of $\mathbf{v}$ *onto* $\mathbf{u}$. *Draw a sketch in Exercises 34 and 35.*

34. A parallelogram has diagonals determined by the vectors

$$\mathbf{d}_1 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \text{ and } \mathbf{d}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Show that the parallelogram is a rhombus (all sides of equal length) and determine the side length.

35. The rectangle $ABCD$ has vertices at $A = (1, 2, 3)$, $B = (3, 6, -2)$, and $C = (0, 5, -4)$. Determine the coordinates of vertex D.

36. An airplane heading due east has a velocity of 200 miles per hour. A wind is blowing from the north at 40 miles per hour. What is the resultant velocity of the airplane?

37. A boat heads north across a river at a rate of 4 miles per hour. If the current is flowing east at a rate of 3 miles per hour, find the resultant velocity of the boat.

38. Ann is driving a motorboat across a river that is 2 km wide. The boat has a speed of 20 km/h in still water, and the current in the river is flowing at 5 km/h. Ann heads out from one bank of the river for a dock directly across from her on the opposite bank. She drives the boat in a direction perpendicular to the current.

(a) How far downstream from the dock will Ann land?
(b) How long will it take Ann to cross the river?

39. Bert can swim at a rate of 2 miles per hour in still water. The current in a river is flowing at a rate of 1 mile per hour. If Bert wants to swim across the river to a point directly opposite, at what angle to the bank of the river must he swim?

In Exercises 40–45, find the projection of **v** *onto* **u**. *Draw a sketch in Exercises 40 and 41.*

40. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ **41.** $\mathbf{u} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

42. $\mathbf{u} = \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$ **43.** $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ -1 \\ -2 \end{bmatrix}$

44. $\mathbf{u} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2.1 \\ 1.2 \end{bmatrix}$

45. $\mathbf{u} = \begin{bmatrix} 3.01 \\ -0.33 \\ 2.52 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1.34 \\ 4.25 \\ -1.66 \end{bmatrix}$

Figure 1.39 suggests two ways in which vectors may be used to compute the area of a triangle. The area $\mathcal{A}$ of

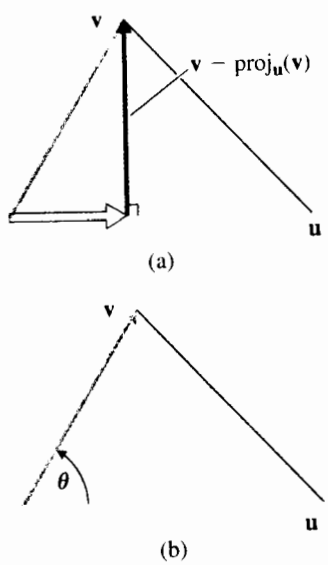


Figure 1.39

the triangle in part (a) is given by $\frac{1}{2}\|\mathbf{u}\|\|\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})\|$, and part (b) suggests the trigonometric form of the area of a triangle: $\mathcal{A} = \frac{1}{2}\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$ (We can use the identity $\sin\theta = \sqrt{1 - \cos^2\theta}$ to find $\sin\theta$.)

In Exercises 46 and 47, compute the area of the triangle with the given vertices using both methods.

46. $A = (1, -1), B = (2, 2), C = (4, 0)$

47. $A = (3, -1, 4), B = (4, -2, 6), C = (5, 0, 2)$

In Exercises 48 and 49, find all values of the scalar k for which the two vectors are orthogonal.

48. $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} k+1 \\ k-1 \end{bmatrix}$ **49.** $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} k^2 \\ k \\ -3 \end{bmatrix}$

50. Describe all vectors $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

51. Describe all vectors $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$.

52. Under what conditions are the following true for vectors **u** and **v** in $\mathbb{R}^2$ or $\mathbb{R}^3$?

(a) $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$ (b) $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$

53. Prove Theorem 1.2(b).

54. Prove Theorem 1.2(d).

In Exercises 55–57, prove the stated property of distance between vectors.

55. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$ for all vectors **u** and **v**

56. $d(\mathbf{u}, \mathbf{w}) \leq d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w})$ for all vectors **u**, **v**, and **w**

57. $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

58. Prove that $\mathbf{u} \cdot c\mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$ for all vectors **u** and **v** in $\mathbb{R}^n$ and all scalars c.

59. Prove that $\|\mathbf{u} - \mathbf{v}\| \geq \|\mathbf{u}\| - \|\mathbf{v}\|$ for all vectors **u** and **v** in $\mathbb{R}^n$. [*Hint:* Replace **u** by $\mathbf{u} - \mathbf{v}$ in the Triangle Inequality.]

60. Suppose we know that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$. Does it follow that $\mathbf{v} = \mathbf{w}$? If it does, give a proof that is valid in $\mathbb{R}^n$; otherwise, give a *counterexample* (that is, a *specific* set of vectors **u**, **v**, and **w** for which $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ but $\mathbf{v} \neq \mathbf{w}$).

61. Prove that $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ for all vectors **u** and **v** in $\mathbb{R}^n$.

62. (a) Prove that $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$ for all vectors **u** and **v** in $\mathbb{R}^n$.

(b) Draw a diagram showing **u**, **v**, $\mathbf{u} + \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ in $\mathbb{R}^2$ and use (a) to deduce a result about parallelograms.

63. Prove that $\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2$ for all vectors **u** and **v** in $\mathbb{R}^n$.

64. (a) Prove that $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(b) Draw a diagram showing $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ in $\mathbb{R}^2$ and use (a) to deduce a result about parallelograms.

65. (a) Prove that $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal in $\mathbb{R}^n$ if and only if $\|\mathbf{u}\| = \|\mathbf{v}\|$.

(b) Draw a diagram showing $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ in $\mathbb{R}^2$ and use (a) to deduce a result about parallelograms.

66. If $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = \sqrt{3}$, and $\mathbf{u} \cdot \mathbf{v} = 1$, find $\|\mathbf{u} + \mathbf{v}\|$.

67. Show that there are no vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\|\mathbf{u}\| = 1$, $\|\mathbf{v}\| = 2$, and $\mathbf{u} \cdot \mathbf{v} = 3$.

68. (a) Prove that if $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}$ is orthogonal to $\mathbf{v} + \mathbf{w}$.

(b) Prove that if $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}$ is orthogonal to $s\mathbf{v} + t\mathbf{w}$ for all scalars s and t.

69. Prove that $\mathbf{u}$ is orthogonal to $\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, where $\mathbf{u} \neq \mathbf{0}$.

70. (a) Prove that $\text{proj}_{\mathbf{u}}(\text{proj}_{\mathbf{u}}(\mathbf{v})) = \text{proj}_{\mathbf{u}}(\mathbf{v})$.

(b) Prove that $\text{proj}_{\mathbf{u}}(\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})) = \mathbf{0}$.

(c) Explain (a) and (b) geometrically.

71. The Cauchy-Schwarz Inequality $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|$ is equivalent to the inequality we get by squaring both sides: $(\mathbf{u} \cdot \mathbf{v})^2 \leq \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2$.

(a) In $\mathbb{R}^2$, with $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, this becomes

$$(u_1v_1 + u_2v_2)^2 \leq (u_1^2 + u_2^2)(v_1^2 + v_2^2)$$

Prove this algebraically. [*Hint:* Subtract the left-hand side from the right-hand side and show that the difference must necessarily be nonnegative.]

(b) Prove the analogue of (a) in $\mathbb{R}^3$.

72. Another approach to the proof of the Cauchy-Schwarz Inequality is suggested by Figure 1.40, which shows that in $\mathbb{R}^2$ or $\mathbb{R}^3$, $\|\text{proj}_{\mathbf{u}}(\mathbf{v})\| \leq \|\mathbf{v}\|$. Show that this is equivalent to the Cauchy-Schwarz Inequality.

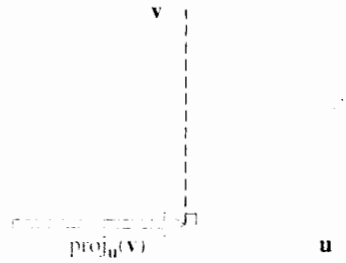


Figure 1.40

73. Use the fact that $\text{proj}_{\mathbf{u}}(\mathbf{v}) = c\mathbf{u}$ for some scalar c, together with Figure 1.41, to find c and thereby derive the formula for $\text{proj}_{\mathbf{u}}(\mathbf{v})$.

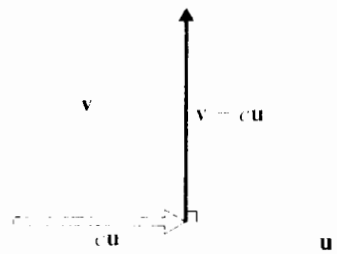
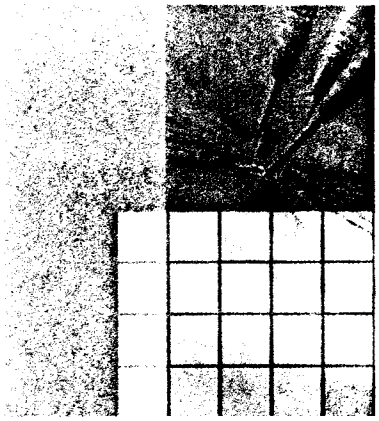


Figure 1.41

74. Using mathematical induction, prove the following generalization of the Triangle Inequality:

$$\|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \cdots + \|\mathbf{v}_n\|$$

for all $n \geq 1$.



Exploration

Vectors and Geometry

Many results in plane Euclidean geometry can be proved using vector techniques. For example, in Example 1.24, we used vectors to prove Pythagoras' Theorem. In this exploration, we will use vectors to develop proofs for some other theorems from Euclidean geometry.

As an introduction to the notation and the basic approach, consider the following easy example.

Example 1.25 Give a vector description of the midpoint M of a line segment $\overline{AB}$.

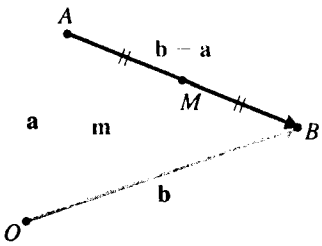


Figure 1.42
The midpoint of $\overline{AB}$

Solution We first convert everything to vector notation. If O denotes the origin and P is a point, let $\mathbf{p}$ be the vector $\overrightarrow{OP}$. In this situation, $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{m} = \overrightarrow{OM}$, and $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$ (Figure 1.42).

Now, since M is the midpoint of $\overline{AB}$, we have

$$\mathbf{m} - \mathbf{a} = \overrightarrow{AM} = \tfrac{1}{2}\overrightarrow{AB} = \tfrac{1}{2}(\mathbf{b} - \mathbf{a})$$

so

$$\mathbf{m} = \mathbf{a} + \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) = \tfrac{1}{2}(\mathbf{a} + \mathbf{b})$$

1. Give a vector description of the point P that is one-third of the way from A to B on the line segment $\overline{AB}$. Generalize.

2. Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long. (In vector notation, prove that $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AB}$ in Figure 1.43.)

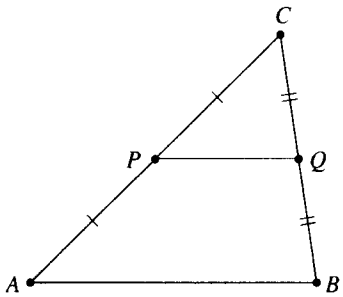


Figure 1.43

3. Prove that the quadrilateral $PQRS$ (Figure 1.44), whose vertices are the midpoints of the sides of an arbitrary quadrilateral $ABCD$, is a parallelogram.

4. A ***median*** of a triangle is a line segment from a vertex to the midpoint of the opposite side (Figure 1.45). Prove that the three medians of any triangle are *concurrent* (i.e., they have a common point of intersection) at a point G that is two-thirds of the distance from each vertex to the midpoint of the opposite side. [*Hint:* In Figure 1.46, show that the point that is two-thirds of the distance from A to P is given by $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$. Then show that $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ is two-thirds of the distance from B to Q and two-thirds of the distance from C to R.] The point G in Figure 1.46 is called the ***centroid*** of the triangle.

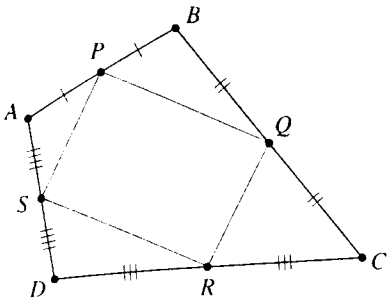


Figure 1.44

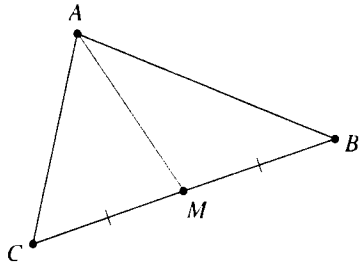


Figure 1.45
A median

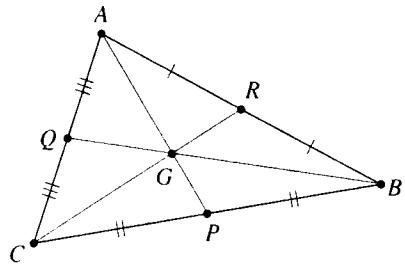


Figure 1.46
The centroid

5. An ***altitude*** of a triangle is a line segment from a vertex that is perpendicular to the opposite side (Figure 1.47). Prove that the three altitudes of a triangle are concurrent. [*Hint:* Let H be the point of intersection of the altitudes from A and B in Figure 1.48. Prove that $\overrightarrow{CH}$ is orthogonal to $\overrightarrow{AB}$.] The point H in Figure 1.48 is called the ***orthocenter*** of the triangle.

6. A ***perpendicular bisector*** of a line segment is a line through the midpoint of the segment, perpendicular to the segment (Figure 1.49). Prove that the perpendicular bisectors of the three sides of a triangle are concurrent. [*Hint:* Let K be the point of intersection of the perpendicular bisectors of $\overline{AC}$ and $\overline{BC}$ in Figure 1.50. Prove that $\overrightarrow{RK}$ is orthogonal to $\overrightarrow{AB}$.] The point K in Figure 1.50 is called the ***circumcenter*** of the triangle.

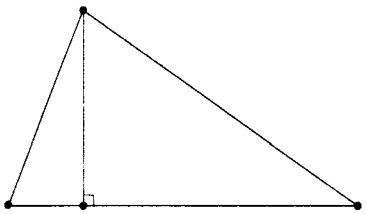


Figure 1.47
An altitude

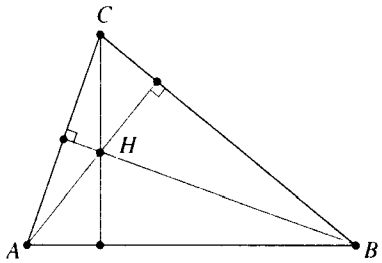


Figure 1.48
The orthocenter

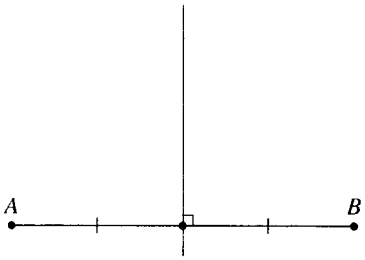


Figure 1.49
A perpendicular bisector

7. Let A and B be the endpoints of a diameter of a circle. If C is any point on the circle, prove that $\angle ACB$ is a right angle. [*Hint:* In Figure 1.51, let O be the center of the circle. Express everything in terms of **a** and **c** and show that $\overrightarrow{AC}$ is orthogonal to $\overrightarrow{BC}$.]

8. Prove that the line segments joining the midpoints of opposite sides of a quadrilateral bisect each other (Figure 1.52).

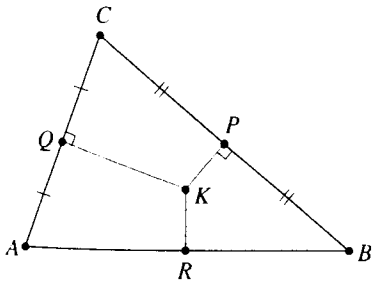


Figure 1.50
The circumcenter

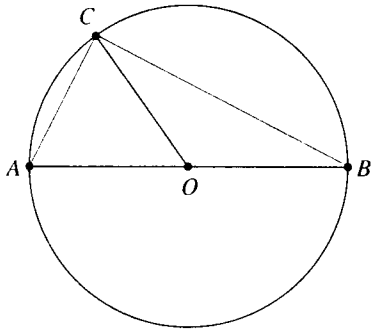


Figure 1.51

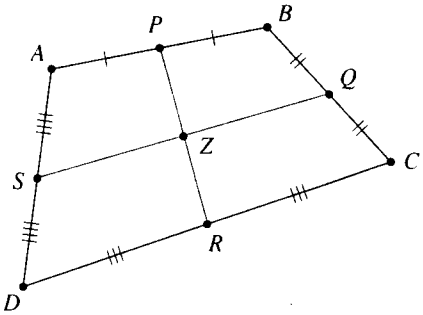


Figure 1.52

Lines and Planes

We are all familiar with the equation of a line in the Cartesian plane. We now want to consider lines in $\mathbb{R}^2$ from a vector point of view. The insights we obtain from this approach will allow us to generalize to lines in $\mathbb{R}^3$ and then to planes in $\mathbb{R}^3$. Much of the linear algebra we will consider in later chapters has its origins in the simple geometry of lines and planes; the ability to visualize these and to think geometrically about a problem will serve you well.

Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$

In the xy-plane, the general form of the equation of a line is $ax + by = c$. If $b \neq 0$, then the equation can be rewritten as $y = -(a/b)x + c/b$, which has the form $y = mx + k$. [This is the slope-intercept form; m is the slope of the line, and the point with coordinates $(0, k)$ is its y-intercept.] To get vectors into the picture, let's consider an example.

Example 1.26

The line ℓ with equation $2x + y = 0$ is shown in Figure 1.53. It is a line with slope -2 passing through the origin. The left-hand side of the equation is in the form of a dot product; in fact, if we let $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, then the equation becomes $\mathbf{n} \cdot \mathbf{x} = 0$. The vector $\mathbf{n}$ is perpendicular to the line—that is, it is *orthogonal* to any vector $\mathbf{x}$ that is parallel to the line (Figure 1.54)—and it is called a ***normal vector*** to the line. The equation $\mathbf{n} \cdot \mathbf{x} = 0$ is the *normal form* of the equation of ℓ.

Another way to think about this line is to imagine a particle moving along the line. Suppose the particle is initially at the origin at time $t = 0$ and it moves along the line in such a way that its x-coordinate changes 1 unit per second. Then at $t = 1$ the particle is at $(1, -2)$, at $t = 1.5$ it is at $(1.5, -3)$, and, if we allow negative values of t (that is, we consider where the particle was in the past), at $t = -2$ it is (or was) at

The Latin word *norma* refers to a carpenter's square, used for drawing right angles. Thus, a *normal* vector is one that is perpendicular to something else, usually a plane.

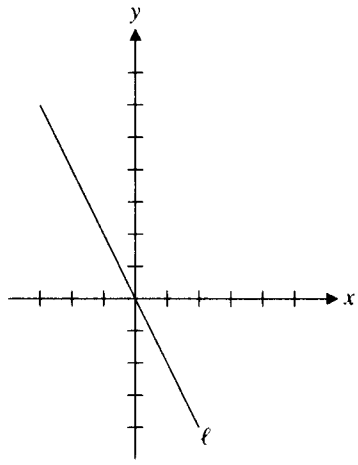


Figure 1.53
The line $2x + y = 0$

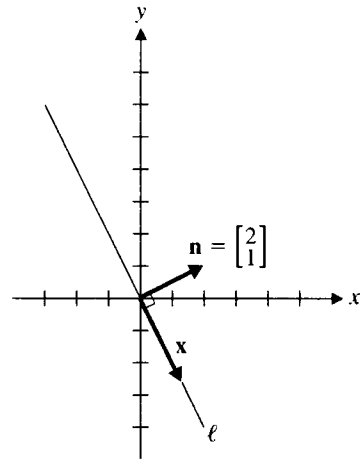


Figure 1.54
A normal vector $\mathbf{n}$

$(-2, 4)$. This movement is illustrated in Figure 1.55. In general, if $x = t$, then $y = -2t$, and we may write this relationship in vector form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ -2t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

What is the significance of the vector $\mathbf{d} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$? It is a particular vector parallel to ℓ, called a ***direction vector*** for the line. As shown in Figure 1.56, we may write the equation of ℓ as $\mathbf{x} = t\mathbf{d}$. This is the *vector form* of the equation of the line.

If the line does not pass through the origin, then we must modify things slightly.

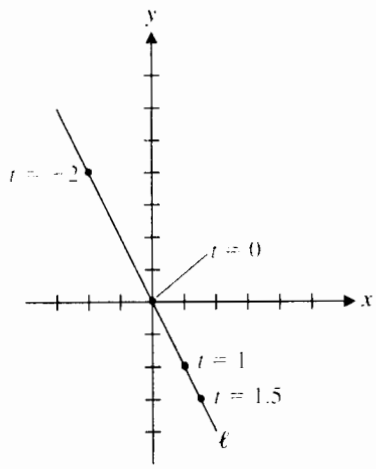


Figure 1.55

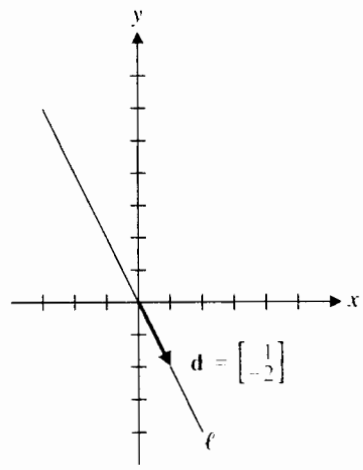


Figure 1.56
A direction vector **d**

Example 1.27

Consider the line ℓ with equation $2x + y = 5$ (Figure 1.57). This is just the line from Example 1.26 shifted upward 5 units. It also has slope -2, but its y-intercept is the point $(0, 5)$. It is clear that the vectors **d** and **n** from Example 1.26 are, respectively, a direction vector and a normal vector for this line too.

Thus, **n** is orthogonal to every vector that is parallel to ℓ. The point $P = (1, 3)$ is on ℓ. If $X = (x, y)$ represents a general point on ℓ, then the vector $\overrightarrow{PX} = \mathbf{x} - \mathbf{p}$ is parallel to ℓ and $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ (see Figure 1.58). Simplified, we have $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$. As a check, we compute

$$\mathbf{n} \cdot \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 2x + y \quad \text{and} \quad \mathbf{n} \cdot \mathbf{p} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 5$$

Thus, the normal form $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ is just a different representation of the general form of the equation of the line. (Note that in Example 1.26, **p** was the zero vector, so $\mathbf{n} \cdot \mathbf{p} = 0$ gave the right-hand side of the equation.)

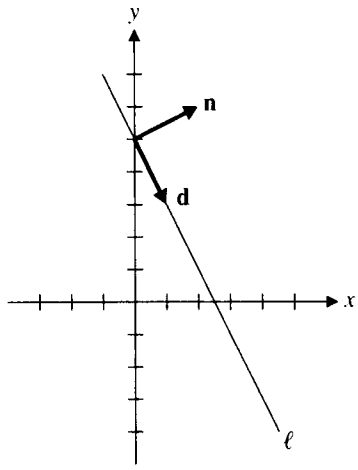


Figure 1.57
The line $2x + y = 5$

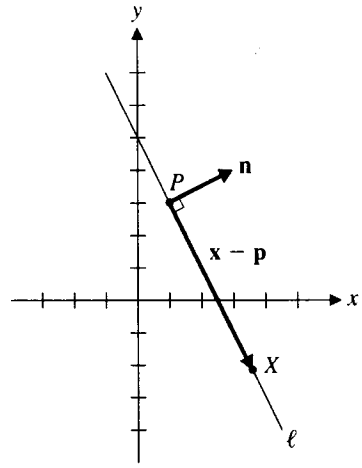


Figure 1.58
$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$

These results lead to the following definition.

Definition The ***normal form of the equation of a line*** ℓ in $\mathbb{R}^2$ is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \quad \text{or} \quad \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$$

where $\mathbf{p}$ is a specific point on ℓ and $\mathbf{n} \neq \mathbf{0}$ is a normal vector for ℓ.

The ***general form of the equation of*** ℓ is $ax + by = c$, where $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a normal vector for ℓ.

Continuing with Example 1.27, let us now find the vector form of the equation of ℓ. Note that, for each choice of $\mathbf{x}$, $\mathbf{x} - \mathbf{p}$ must be parallel to—and thus a multiple of—the direction vector $\mathbf{d}$. That is, $\mathbf{x} - \mathbf{p} = t\mathbf{d}$ or $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ for some scalar t. In terms of components, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{1}$$

or

$$\begin{aligned} x &= 1 + t \\ y &= 3 - 2t \end{aligned} \tag{2}$$

Equation (1) is the vector form of the equation of ℓ, and the componentwise equations (2) are called *parametric equations* of the line. The variable t is called a *parameter*.

How does all of this generalize to $\mathbb{R}^3$? Observe that the vector and parametric forms of the equations of a line carry over perfectly. The notion of the slope of a line in $\mathbb{R}^2$—which is difficult to generalize to three dimensions—is replaced by the more convenient notion of a direction vector, leading to the following definition.

The word *parameter* and the corresponding adjective *parametric* come from the Greek words *para*, meaning "alongside," and *metron*, meaning "measure." Mathematically speaking, a parameter is a variable in terms of which other variables are expressed—a new "measure" placed alongside old ones.

Definition The ***vector form of the equation of a line*** ℓ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is

$$\mathbf{x} = \mathbf{p} + t\mathbf{d}$$

where $\mathbf{p}$ is a specific point on ℓ and $\mathbf{d} \neq \mathbf{0}$ is a direction vector for ℓ.

The equations corresponding to the components of the vector form of the equation are called ***parametric equations*** of ℓ.

We will often abbreviate this terminology slightly, referring simply to the general, normal, vector, and parametric equations of a line or plane.

Example 1.28 Find vector and parametric equations of the line in $\mathbb{R}^3$ through the point $P = (1, 2, -1)$, parallel to the vector $\mathbf{d} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$.

Solution The vector equation $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$$

The parametric form is

$$\begin{aligned} x &= 1 + 5t \\ y &= 2 - t \\ z &= -1 + 3t \end{aligned}$$

Remarks

- The vector and parametric forms of the equation of a given line ℓ are not unique—in fact, there are infinitely many, since we may use any point on ℓ to determine $\mathbf{p}$ and any direction vector for ℓ. However, all direction vectors are clearly multiples of each other.

In Example 1.28, (6, 1, 2) is another point on the line (take $t = 1$), and $\begin{bmatrix} 10 \\ -2 \\ 6 \end{bmatrix}$ is another direction vector. Therefore,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} + s\begin{bmatrix} 10 \\ -2 \\ 6 \end{bmatrix}$$

gives a different (but equivalent) vector equation for the line. The relationship between the two parameters s and t can be found by comparing the parametric equations: For a given point (x, y, z) on ℓ, we have

$$\begin{aligned} x &= 1 + 5t = 6 + 10s \\ y &= 2 - t = 1 - 2s \\ z &= -1 + 3t = 2 + 6s \end{aligned}$$

implying that

$$\begin{aligned} -10s + 5t &= 5 \\ 2s - t &= -1 \\ -6s + 3t &= 3 \end{aligned}$$

Each of these equations reduces to $t = 1 + 2s$.

• Intuitively, we know that a line is a *one-dimensional* object. The idea of "dimension" will be clarified in Chapters 3 and 6, but for the moment observe that this idea appears to agree with the fact that the vector form of the equation of a line requires *one* parameter.

Example 1.29 One often hears the expression "two points determine a line." Find a vector equation of the line ℓ in $\mathbb{R}^3$ determined by the points $P = (-1, 5, 0)$ and $Q = (2, 1, 1)$.

Solution We may choose any point on ℓ for $\mathbf{p}$, so we will use P (Q would also be fine).

A convenient direction vector is $\mathbf{d} = \overrightarrow{PQ} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$ (or any scalar multiple of this). Thus, we obtain

$$\begin{aligned} \mathbf{x} &= \mathbf{p} + t\mathbf{d} \\ &= \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} \end{aligned}$$

Planes in $\mathbb{R}^3$

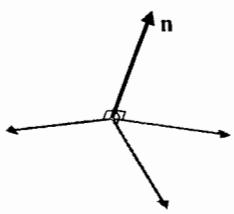


Figure 1.59
n is orthogonal to infinitely many vectors

The next question we should ask ourselves is, How does the general form of the equation of a line generalize to $\mathbb{R}^3$? We might reasonably guess that if $ax + by = c$ is the general form of the equation of a line in $\mathbb{R}^2$, then $ax + by + cz = d$ might represent a line in $\mathbb{R}^3$. In normal form, this equation would be $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$, where $\mathbf{n}$ is a normal vector to the line and $\mathbf{p}$ corresponds to a point on the line.

To see if this is a reasonable hypothesis, let's think about the special case of the equation $ax + by + cz = 0$. In normal form, it becomes $\mathbf{n} \cdot \mathbf{x} = 0$, where $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. However, the set of all vectors $\mathbf{x}$ that satisfy this equation is the set of all vectors orthogonal to $\mathbf{n}$. As shown in Figure 1.59, vectors in infinitely many directions have this property, determining a family of parallel *planes*. So our guess was incorrect: It appears that $ax + by + cz = d$ is the equation of a plane—not a line—in $\mathbb{R}^3$.

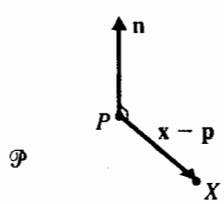


Figure 1.60
$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$

Let's make this finding more precise. Every plane $\mathcal{P}$ in $\mathbb{R}^3$ can be determined by specifying a point $\mathbf{p}$ on $\mathcal{P}$ and a nonzero vector $\mathbf{n}$ normal to $\mathcal{P}$ (Figure 1.60). Thus, if $\mathbf{x}$ represents an arbitrary point on $\mathcal{P}$, we have $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ or $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$. If $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then, in terms of components, the equation becomes $ax + by + cz = d$ (where $d = \mathbf{n} \cdot \mathbf{p}$).

Definition The ***normal form of the equation of a plane*** $\mathcal{P}$ in $\mathbb{R}^3$ is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \quad \text{or} \quad \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$$

where $\mathbf{p}$ is a specific point on $\mathcal{P}$ and $\mathbf{n} \neq \mathbf{0}$ is a normal vector for $\mathcal{P}$.

The ***general form of the equation of*** $\mathcal{P}$ is $ax + by + cz = d$, where $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a normal vector for $\mathcal{P}$.

Note that any scalar multiple of a normal vector for a plane is another normal vector.

Example 1.30 Find the normal and general forms of the equation of the plane that contains the point $P = (6, 0, 1)$ and has normal vector $\mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Solution With $\mathbf{p} = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we have $\mathbf{n} \cdot \mathbf{p} = 1 \cdot 6 + 2 \cdot 0 + 3 \cdot 1 = 9$, so the normal equation $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ becomes the general equation $x + 2y + 3z = 9$.

Geometrically, it is clear that parallel planes have the same normal vector(s). Thus, their general equations have left-hand sides that are multiples of each other. So, for example, $2x + 4y + 6z = 10$ is the general equation of a plane that is parallel to the plane in Example 1.30, since we may rewrite the equation as $x + 2y + 3z = 5$—from which we see that the two planes have the same normal vector $\mathbf{n}$. (Note that the planes do not coincide, since the right-hand sides of their equations are distinct.)

We may also express the equation of a plane in vector or parametric form. To do so, we observe that a plane can also be determined by specifying one of its points P (by the vector $\mathbf{p}$) and *two* direction vectors $\mathbf{u}$ and $\mathbf{v}$ parallel to the plane (but not parallel to each other). As Figure 1.61 shows, given any point X in the plane (located

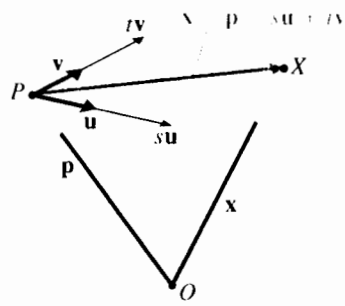


Figure 1.61
$\mathbf{x} - \mathbf{p} = s\mathbf{u} + t\mathbf{v}$

by $\mathbf{x}$), we can always find appropriate multiples $s\mathbf{u}$ and $t\mathbf{v}$ of the direction vectors such that $\mathbf{x} - \mathbf{p} = s\mathbf{u} + t\mathbf{v}$ or $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$. If we write this equation componentwise, we obtain parametric equations for the plane.

Definition The ***vector form of the equation of a plane*** $\mathcal{P}$ in $\mathbb{R}^3$ is

$$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$$

where $\mathbf{p}$ is a point on $\mathcal{P}$ and $\mathbf{u}$ and $\mathbf{v}$ are direction vectors for $\mathcal{P}$ ($\mathbf{u}$ and $\mathbf{v}$ are nonzero and parallel to $\mathcal{P}$, but not parallel to each other).

The equations corresponding to the components of the vector form of the equation are called ***parametric equations*** of $\mathcal{P}$.

Example 1.31 Find vector and parametric equations for the plane in Example 1.30.

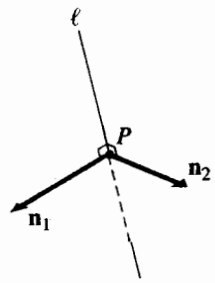


Figure 1.62
Two normals determine a line

Solution We need to find two direction vectors. We have one point $P = (6, 0, 1)$ in the plane; if we can find two other points Q and R in $\mathcal{P}$, then the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ can serve as direction vectors (unless by bad luck they happen to be parallel!). By trial and error, we observe that $Q = (9, 0, 0)$ and $R = (3, 3, 0)$ both satisfy the general equation $x + 2y + 3z = 9$ and so lie in the plane. Then we compute

$$\mathbf{u} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \text{ and } \mathbf{v} = \overrightarrow{PR} = \mathbf{r} - \mathbf{p} = \begin{bmatrix} -3 \\ 3 \\ -1 \end{bmatrix}$$

which, since they are not scalar multiples of each other, will serve as direction vectors. Therefore, we have the vector equation of $\mathcal{P}$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} -3 \\ 3 \\ -1 \end{bmatrix}$$

and the corresponding parametric equations,

$$\begin{aligned} x &= 6 + 3s - 3t \\ y &= 3t \\ z &= 1 - s - t \end{aligned}$$

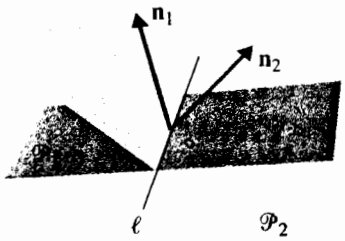


Figure 1.63
The intersection of two planes is a line

[What would have happened had we chosen $R = (0, 0, 3)$?]

Remarks

- A plane is a two-dimensional object, and its equation, in vector or parametric form, requires *two* parameters.
- As Figure 1.59 shows, given a point P and a nonzero vector $\mathbf{n}$ in $\mathbb{R}^3$, there are infinitely many lines through P with $\mathbf{n}$ as a normal vector. However, P and two nonparallel normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ do serve to locate a line ℓ uniquely, since ℓ must then be the line through P that is perpendicular to the plane with equation $\mathbf{x} = \mathbf{p} + s\mathbf{n}_1 + t\mathbf{n}_2$ (Figure 1.62). Thus, a line in $\mathbb{R}^3$ can also be specified by a pair of equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \end{aligned}$$

one corresponding to each normal vector. But since these equations correspond to a pair of nonparallel planes (why nonparallel?), this is just the description of a line as the intersection of two nonparallel planes (Figure 1.63). Algebraically, the line consists of all points (x, y, z) that simultaneously satisfy both equations. We will explore this concept further in Chapter 2 when we discuss the solution of systems of linear equations.

Tables 1.2 and 1.3 summarize the information presented so far about the equations of lines and planes.

Observe once again that a single (general) equation describes a line in $\mathbb{R}^2$ but a plane in $\mathbb{R}^3$. [In higher dimensions, an object (line, plane, etc.) determined by a single equation of this type is usually called a ***hyperplane.***] The relationship among the

Table 1.2 Equations of Lines in $\mathbb{R}^2$

Normal Form	General Form	Vector Form	Parametric Form
$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$	$ax + by = c$	$\mathbf{x} = \mathbf{p} + t\mathbf{d}$	$\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \end{cases}$

Table 1.3 Lines and Planes in $\mathbb{R}^3$

	Normal Form	General Form	Vector Form	Parametric Form
Lines	$\begin{cases} \mathbf{n}_1 \cdot \mathbf{x} = \mathbf{n}_1 \cdot \mathbf{p}_1 \\ \mathbf{n}_2 \cdot \mathbf{x} = \mathbf{n}_2 \cdot \mathbf{p}_2 \end{cases}$	$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$	$\mathbf{x} = \mathbf{p} + t\mathbf{d}$	$\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \\ z = p_3 + td_3 \end{cases}$
Planes	$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$	$ax + by + cz = d$	$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$	$\begin{cases} x = p_1 + su_1 + tv_1 \\ y = p_2 + su_2 + tv_2 \\ z = p_3 + su_3 + tv_3 \end{cases}$

dimension of the object, the number of equations required, and the dimension of the space is given by the "balancing formula":

(*dimension* of the object) + (*number* of general equations) = dimension of the space

The higher the dimension of the object, the fewer equations it needs. For example, a plane in $\mathbb{R}^3$ is two-dimensional, requires one general equation, and lives in a three-dimensional space: $2 + 1 = 3$. A line in $\mathbb{R}^3$ is one-dimensional and so needs $3 - 1 = 2$ equations. Note that the dimension of the object also agrees with the number of parameters in its vector or parametric form. Notions of "dimension" will be clarified in Chapters 3 and 6, but for the time being, these intuitive observations will serve us well.

We can now find the distance from a point to a line or a plane by combining the results of Section 1.2 with the results from this section.

Example 1.32 Find the distance from the point $B = (1, 0, 2)$ to the line ℓ through the point $A = (3, 1, 1)$ with direction vector $\mathbf{d} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Solution As we have already determined, we need to calculate the length of $\overrightarrow{PB}$, where P is the point on ℓ at the foot of the perpendicular from B. If we label $\mathbf{v} = \overrightarrow{AB}$, then $\overrightarrow{AP} = \text{proj}_{\mathbf{d}}(\mathbf{v})$ and $\overrightarrow{PB} = \mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})$ (see Figure 1.64). We do the necessary calculations in several steps.

Step 1: $\mathbf{v} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

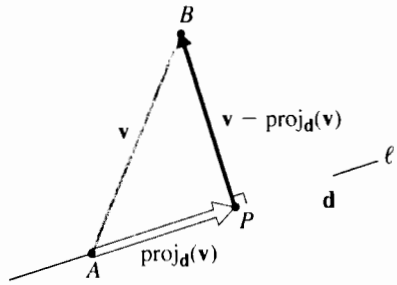


Figure 1.64
$d(B, \ell) = \|\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})\|$

Step 2: The projection of $\mathbf{v}$ onto $\mathbf{d}$ is

$$\begin{aligned}
\text{proj}_{\mathbf{d}}(\mathbf{v}) &= \left(\frac{\mathbf{d}\cdot\mathbf{v}}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d} \\
&= \left(\frac{(-1)\cdot(-2) + 1\cdot(-1) + 0\cdot 1}{(-1)^2 + 1 + 0}\right)\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\
&= \tfrac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}
\end{aligned}$$

Step 3: The vector we want is

$$\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v}) = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

Step 4: The distance $d(B, \ell)$ from B to ℓ is

$$\|\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})\| = \left\| \begin{bmatrix} -\frac{3}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix} \right\|$$

Using Theorem 1.3(b) to simplify the calculation, we have

$$\begin{aligned}
\|\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})\| &= \tfrac{1}{2}\left\| \begin{bmatrix} -3 \\ -3 \\ 2 \end{bmatrix} \right\| \\
&= \tfrac{1}{2}\sqrt{9 + 9 + 4} \\
&= \tfrac{1}{2}\sqrt{22}
\end{aligned}$$

Note

- In terms of our earlier notation, $d(B, \ell) = d(\mathbf{v}, \text{proj}_{\mathbf{d}}(\mathbf{v}))$.

In the case where the line ℓ is in $\mathbb{R}^2$ and its equation has the general form $ax + by = c$, the distance $d(B, \ell)$ from $B = (x_0, y_0)$ is given by the formula

$$d(B, \ell) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}} \tag{3}$$

You are invited to prove this formula in Exercise 39.

Example 1.33 Find the distance from the point $B = (1, 0, 2)$ to the plane $\mathcal{P}$ whose general equation is $x + y - z = 1$.

Solution In this case, we need to calculate the length of $\overrightarrow{PB}$, where P is the point on $\mathcal{P}$ at the foot of the perpendicular from B. As Figure 1.65 shows, if A is any point on $\mathcal{P}$ and we situate the normal vector $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ of $\mathcal{P}$ so that its tail is at A, then we need to find the length of the projection of $\overrightarrow{AB}$ onto $\mathbf{n}$. Again we do the necessary calculations in steps.

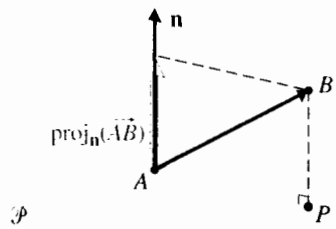


Figure 1.65
$d(B, \mathcal{P}) = \|\text{proj}_{\mathbf{n}}(\overrightarrow{AB})\|$

Step 1: By trial and error, we find any point whose coordinates satisfy the equation $x + y - z = 1$. $A = (1, 0, 0)$ will do.

Step 2: Set

$$\mathbf{v} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Step 3: The projection of $\mathbf{v}$ onto $\mathbf{n}$ is

$$\begin{aligned} \text{proj}_{\mathbf{n}}(\mathbf{v}) &= \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}\right)\mathbf{n} \\ &= \left(\frac{1 \cdot 0 + 1 \cdot 0 - 1 \cdot 2}{1 + 1 + (-1)^2}\right)\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ &= -\tfrac{2}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \end{aligned}$$

Step 4: The distance $d(B, \mathcal{P})$ from B to $\mathcal{P}$ is

$$\begin{aligned}\|\text{proj}_{\mathbf{n}}(\mathbf{v})\| &= \left|-\tfrac{2}{3}\right| \left\| \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\| \\ &= \tfrac{2}{3} \left\| \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\| \\ &= \tfrac{2}{3}\sqrt{3}\end{aligned}$$

In general, the distance $d(B, \mathcal{P})$ from the point $B = (x_0, y_0, z_0)$ to the plane whose general equation is $ax + by + cz = d$ is given by the formula

$$d(B, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} \tag{4}$$

You will be asked to derive this formula in Exercise 40.

Exercises 1.3

In Exercises 1 and 2, write the equation of the line passing through P with normal vector **n** *in (a) normal form and (b) general form.*

1. $P = (0, 0)$, $\mathbf{n} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ **2.** $P = (2, 1)$, $\mathbf{n} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$

In Exercises 3–6, write the equation of the line passing through P with direction vector **d** *in (a) vector form and (b) parametric form.*

3. $P = (1, 0)$, $\mathbf{d} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ **4.** $P = (3, -3)$, $\mathbf{d} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

5. $P = (0, 0, 0)$, $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ **6.** $P = (-3, 1, 2)$, $\mathbf{d} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$

In Exercises 7 and 8, write the equation of the plane passing through P with normal vector **n** *in (a) normal form and (b) general form.*

7. $P = (0, 1, 0)$, $\mathbf{n} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ **8.** $P = (-3, 1, 2)$, $\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$

In Exercises 9 and 10, write the equation of the plane passing through P with direction vectors **u** *and* **v** *in (a) vector form and (b) parametric form.*

9. $P = (0, 0, 0)$, $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

10. $P = (4, -1, 3)$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 11 and 12, give the vector equation of the line passing through P and Q.

11. $P = (1, -2)$, $Q = (3, 0)$

12. $P = (4, -1, 3)$, $Q = (2, 1, 3)$

In Exercises 13 and 14, give the vector equation of the plane passing through P, Q, and R.

13. $P = (1, 1, 1)$, $Q = (4, 0, 2)$, $R = (0, 1, -1)$

14. $P = (1, 0, 0)$, $Q = (0, 1, 0)$, $R = (0, 0, 1)$

15. Find parametric equations and an equation in vector form for the lines in $\mathbb{R}^2$ with the following equations:

(a) $y = 3x - 1$ **(b)** $3x + 2y = 5$

16. Consider the vector equation $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$, where $\mathbf{p}$ and $\mathbf{q}$ correspond to distinct points P and Q in $\mathbb{R}^2$ or $\mathbb{R}^3$.
   (a) Show that this equation describes the line segment $\overline{PQ}$ as t varies from 0 to 1.
   (b) For which value of t is $\mathbf{x}$ the midpoint of $\overline{PQ}$, and what is $\mathbf{x}$ in this case?
   (c) Find the midpoint of $\overline{PQ}$ when $P = (2, -3)$ and $Q = (0, 1)$.
   (d) Find the midpoint of $\overline{PQ}$ when $P = (1, 0, 1)$ and $Q = (4, 1, -2)$.
   (e) Find the two points that divide $\overline{PQ}$ in part (c) into three equal parts.
   (f) Find the two points that divide $\overline{PQ}$ in part (d) into three equal parts.

17. Suggest a "vector proof" of the fact that, in $\mathbb{R}^2$, two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$.

18. The line ℓ passes through the point $P = (1, -1, 1)$ and has direction vector $\mathbf{d} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. For each of the following planes $\mathcal{P}$, determine whether ℓ and $\mathcal{P}$ are parallel, perpendicular, or neither:
   (a) $2x + 3y - z = 1$ (b) $4x - y + 5z = 0$
   (c) $x - y - z = 3$ (d) $4x + 6y - 2z = 0$

19. The plane $\mathcal{P}_1$ has the equation $4x - y + 5z = 2$. For each of the planes $\mathcal{P}$ in Exercise 18, determine whether $\mathcal{P}_1$ and $\mathcal{P}$ are parallel, perpendicular, or neither.

20. Find the vector form of the equation of the line in $\mathbb{R}^2$ that passes through $P = (2, -1)$ and is perpendicular to the line with general equation $2x - 3y = 1$.

21. Find the vector form of the equation of the line in $\mathbb{R}^2$ that passes through $P = (2, -1)$ and is parallel to the line with general equation $2x - 3y = 1$.

22. Find the vector form of the equation of the line in $\mathbb{R}^3$ that passes through $P = (-1, 0, 3)$ and is perpendicular to the plane with general equation $x - 3y + 2z = 5$.

23. Find the vector form of the equation of the line in $\mathbb{R}^3$ that passes through $P = (-1, 0, 3)$ and is parallel to the line with parametric equations

$$\begin{aligned} x &= 1 - t \\ y &= 2 + 3t \\ z &= -2 - t \end{aligned}$$

24. Find the normal form of the equation of the plane that passes through $P = (0, -2, 5)$ and is parallel to the plane with general equation $6x - y + 2z = 3$.

25. A cube has vertices at the eight points (x, y, z), where each of x, y, and z is either 0 or 1. (See Figure 1.34.)
   (a) Find the general equations of the planes that determine the six faces (sides) of the cube.
   (b) Find the general equation of the plane that contains the diagonal from the origin to $(1, 1, 1)$ and is perpendicular to the xy-plane.
   (c) Find the general equation of the plane that contains the side diagonals referred to in Example 1.22.

26. Find the equation of the set of all points that are equidistant from the points $P = (1, 0, -2)$ and $Q = (5, 2, 4)$.

In Exercises 27 and 28, find the distance from the point Q to the line ℓ.

27. $Q = (2, 2)$, ℓ with equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

28. $Q = (0, 1, 0)$, ℓ with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

In Exercises 29 and 30, find the distance from the point Q to the plane $\mathcal{P}$.

29. $Q = (2, 2, 2)$, $\mathcal{P}$ with equation $x + y - z = 0$

30. $Q = (0, 0, 0)$, $\mathcal{P}$ with equation $x - 2y + 2z = 1$

Figure 1.66 suggests a way to use vectors to locate the point R on ℓ that is closest to Q.

31. Find the point R on ℓ that is closest to Q in Exercise 27.

32. Find the point R on ℓ that is closest to Q in Exercise 28.

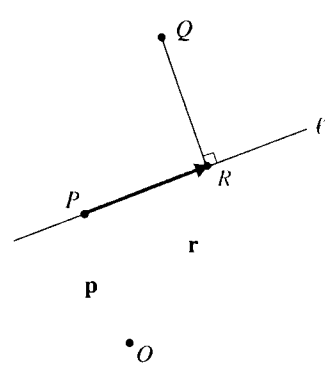


Figure 1.66
$\mathbf{r} = \mathbf{p} + \overrightarrow{PR}$

Figure 1.67 suggests a way to use vectors to locate the point R on $\mathcal{P}$ that is closest to Q.

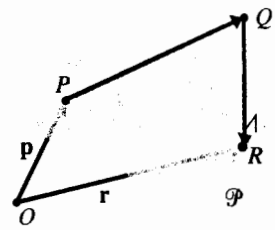


Figure 1.67
$\mathbf{r} = \mathbf{p} + \overrightarrow{PQ} + \overrightarrow{QR}$

33. Find the point R on $\mathcal{P}$ that is closest to Q in Exercise 29.

34. Find the point R on $\mathcal{P}$ that is closest to Q in Exercise 30.

In Exercises 35 and 36, find the distance between the parallel lines.

35. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} + t\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

36. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 37 and 38, find the distance between the parallel planes.

37. $2x + y - 2z = 0$ and $2x + y - 2z = 5$

38. $x + y + z = 1$ and $x + y + z = 3$

39. Prove equation (3) on page 43.

40. Prove equation (4) on page 44.

41. Prove that, in $\mathbb{R}^2$, the distance between parallel lines with equations $\mathbf{n} \cdot \mathbf{x} = c_1$ and $\mathbf{n} \cdot \mathbf{x} = c_2$ is given by $\dfrac{|c_1 - c_2|}{\|\mathbf{n}\|}$.

42. Prove that the distance between parallel planes with equations $\mathbf{n} \cdot \mathbf{x} = d_1$ and $\mathbf{n} \cdot \mathbf{x} = d_2$ is given by $\dfrac{|d_1 - d_2|}{\|\mathbf{n}\|}$.

If two nonparallel planes $\mathcal{P}_1$ and $\mathcal{P}_2$ have normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ and θ is the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$, then we define the angle between $\mathcal{P}_1$ and $\mathcal{P}_2$ to be either θ or $180° - \theta$, whichever is an acute angle. (Figure 1.68)

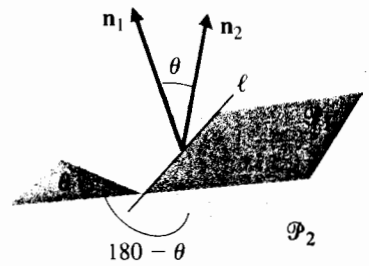


Figure 1.68

In Exercises 43–44, find the acute angle between the planes with the given equations.

43. $x + y + z = 0$ and $2x + y - 2z = 0$

44. $3x - y + 2z = 5$ and $x + 4y - z = 2$

In Exercises 45–46, show that the plane and line with the given equations intersect, and then find the acute angle of intersection between them.

45. The plane given by $x + y + 2z = 0$ and the line given by
$$\begin{aligned} x &= 2 + t \\ y &= 1 - 2t \\ z &= 3 + t \end{aligned}$$

46. The plane given by $4x - y - z = 6$ and the line given by
$$\begin{aligned} x &= t \\ y &= 1 + 2t \\ z &= 2 + 3t \end{aligned}$$

Exercises 47–48 explore one approach to the problem of finding the projection of a vector onto a plane. As Figure 1.69 shows, if $\mathcal{P}$ is a plane through the origin in $\mathbb{R}^3$ with normal vector $\mathbf{n}$, and $\mathbf{v}$ is a vector in $\mathbb{R}^3$, then $\mathbf{p} = \text{proj}_{\mathcal{P}}(\mathbf{v})$ is a vector in $\mathcal{P}$ such that $\mathbf{v} - c\mathbf{n} = \mathbf{p}$ for some scalar c.

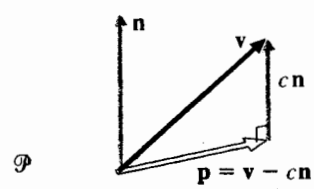


Figure 1.69
Projection onto a plane

47. Using the fact that **n** is orthogonal to every vector in $\mathcal{P}$ (and hence to **p**), solve for c and thereby find an expression for **p** in terms of **v** and **n**.

48. Use the method of Exercise 43 to find the projection of

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

onto the planes with the following equations:

(a) $x + y + z = 0$ **(b)** $3x - y + z = 0$

(c) $x - 2z = 0$ **(d)** $2x - 3y + z = 0$