I. (40 points)

(1) Write down a system of 3 linear equations in 3 variables whose solution set in $\mathbb{R}^3$ coincides with the position vectors of the points lying on the unique line $L$ which passes through the point $P_0(1, 2, 0)$ and has direction vector $\vec{d} := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(2) Solve the system of linear equations you wrote down in part (i) by the method of row reduction.

One of many possible answers is

\[
\begin{cases}
  x - 2 = 1 \\
  y = 2 \\
  x + y - 2 = 3
\end{cases}
\]
\[(2)\]
\[
\begin{align*}
X - z &= 1 \\
y &= 2 \\
X + y - 2 &= 3
\end{align*}
\]
\[
\Rightarrow \begin{bmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 0 & 2 \\
1 & 1 & -1 & 3
\end{bmatrix}
\Rightarrow \begin{bmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 0 & 2
\end{bmatrix}
\]

Hence, \(z\) is free variable.

Set \(z = t\)

Row 1 gives: \(X - z = 1\)

So \(X = z + 1 = t + 1\)

Row 2 gives: \(y = 2\)

\(X = 1 + t\)
\(y = 2\)
\(z = t\)

\[
\begin{bmatrix}
X \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
1 \\
2 \\
t
\end{bmatrix}
+ t \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}
\]

which is the same as the line \(L\), as expected.
II. (20 points)

(1) Write down the equation of the unique plane $\Pi$ in $\mathbb{R}^3$ which contains the points $P_1(1,0,0)$, $P_2(0,1,0)$ and $P_3(0,0,1)$.

(2) Find the intersection between the plane $\Pi$ and the line which is perpendicular on $\Pi$ and contains the point $O(0,0,0)$.

(1) $\Pi$ is set of points which satisfies $ax + by + cz = d$ for some $a, b, c, d$.

Plug in $P_1: a \cdot 1 + b \cdot 0 + c \cdot 0 = d \Rightarrow a = d$

$P_2: a \cdot 0 + b \cdot 1 + c \cdot 0 = d \Rightarrow b = d$

$P_3: a \cdot 0 + b \cdot 0 + c \cdot 1 = d \Rightarrow c = d$

so eqn for $\Pi$ is $dx + dy + dz = d$.

$d \neq 0$, so can multiply by $d^{-1}$

get $\frac{1}{d}x + y + z = 1$

(2) $L$ is perpendicular on $\Pi$

so direction vector for $L$ = normal vector for $\Pi = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$L$ contains $(0,0,0) \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

is the equation for $L$.\[\]
To find where \( L \) and \( \Pi \) intersect, we suppose a point in \( L \), \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} \)

satisfies \( x + y + z = 1 \)

\[ \Rightarrow (t) + (t) + (t) = 1 \quad \Rightarrow 3t = 1 \quad \Rightarrow t = \frac{1}{3} \]

Hence, \( \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \) is a point which satisfies \( x + y + z = 1 \) and is on the line \( L \), so it is the intersection.
III. (40 points)

(1) Let \( A \in M_{m \times n}(\mathbb{R}) \) and \( B \in M_{n \times r}(\mathbb{R}) \), for some natural numbers \( m, n, r \). Prove that if the rows of \( A \) (viewed as vectors in \( \mathbb{R}^n \)) are linearly dependent, then the columns of \( (B^t \cdot A^t) \) (viewed as vectors in \( \mathbb{R}^r \)) are linearly dependent.

(2) Without using determinants decide whether the rows of the matrix

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
3 & 3 & 4 \\
2 & 2 & 3
\end{bmatrix}
\]

are linearly independent or not. Justify your answer.

(1) Let \( r_i(A) \) denote the \( i \)th row of \( A \). If \( \{r_i(A)\} \) are linearly dependent, then there is some row vector \( \mathbf{1}_0 \in \mathbb{R}^m \), \( \mathbf{1}_0 \neq \mathbf{0} \), with \( \mathbf{1}_0 \left[ r_1(A) \right. \left. r_2(A) \right] = \mathbf{0} \); that is, \( \mathbf{1}_0^t A = \mathbf{0} \).

Now the columns of \( B^t A^t \) are the rows of \( (B^t A^t)^t = A^t B^t = AB \), so it suffices to find a nonzero \( \mathbf{1}_0 \in \mathbb{R}^m \) with \( \mathbf{1}_0^t (AB) = \mathbf{0} \). (because then the rows of \( AB \) are linearly dependent)

But the \( \mathbf{1}_0 = \mathbf{1}_0 \) works, as \( \mathbf{1}_0^t (AB) = (\mathbf{1}_0^t A) B \)

\[
= (0) B = 0.
\]
(2) Row reduce:  
\[
\begin{bmatrix}
1 & 0 & 1 \\
3 & 3 & 4 \\
2 & 2 & 3
\end{bmatrix}
R_2 - 3R_1 
\rightarrow 
\begin{bmatrix}
1 & 0 & 1 \\
0 & 3 & 1 \\
0 & 2 & 1
\end{bmatrix}
R_3 - \frac{2}{3}R_2 
\rightarrow 
\begin{bmatrix}
1 & 0 & 1 \\
0 & 3 & 1 \\
0 & 0 & \frac{1}{3}
\end{bmatrix}
\]

Since this square matrix has echelon form with no nonzero rows, it is invertible. 

\[ \iff \text{its rows are linearly dependent.} \]