2. Let
\[ \alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 6 & 1 & 7 & 8 & 6 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix} \]

Write \( \alpha \), \( \beta \), and \( \alpha \beta \) as
(i) products of disjoint cycles

**Solution:** Notice that \( \alpha(1) = 2, \alpha(2) = 3, \alpha(3) = 4, \alpha(4) = 5 \), and \( \alpha(5) = 1 \), so the cycle \((12345)\) appears in \( \alpha \). Similarly, we find that the other cycle is \((678)\). Thus we can write \( \alpha = (12345)(678) \) (the order doesn’t matter because disjoint cycles commute). Meanwhile, \( \beta = (23847)(56) \), and \( \alpha \beta = (12485736) \)

(ii) products of 2-cycles

**Solution:** To do this, we just need to write each individual cycle as a product of 2-cycles. There is a standard way to do this: \((a_1a_2 \cdots a_n) = (a_na_{n-1}) \cdots (a_2a_1)(a_1a_2)\). Thus from part a, \( \alpha = (45)(35)(25)(15)(78)(68) \), while \( \beta = (47)(87)(37)(27)(56) \). Of course, this means \( \alpha \beta = (45)(35)(25)(15)(78)(68)(47)(87)(37)(27)(56) \).

6. What is the order of each of the following permutations?

(i) \[ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 6 & 3 \end{bmatrix} \]

**Solution:** Call this permutation \( \sigma \). Since disjoint cycles commute, we know that the order of \( \sigma \) is simply the lcm of the orders of its disjoint cycles, and that an \( n \)-cycle has order \( n \). In this case, \( \sigma = (12)(356) \), so the order is \( lcm(2, 3) = 6 \).

(ii) \[ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 2 & 3 & 4 & 5 \end{bmatrix} \]

**Solution:** Now call this permutation \( \sigma \). Then we have \( \sigma = (1753)(264) \), so by the same reasoning as above \( |\sigma| = lcm(4, 3) = 12 \).

24. Suppose that \( H \) is a subgroup of \( S_n \) of odd order. Prove that \( H \) is a subgroup of \( A_n \).

**Solution:** This looks similar to problem 25 of homework 5, although since \( H \) might not be cyclic we cannot use method 2 from the solutions. Instead, we can copy method 1. Suppose \( H \) contains an odd permutation \( \sigma \). Given any other permutation \( \tau \in S_n \),
στ is even if τ is odd and odd if τ is even. This is easy to see: if σ = σ₁⋯σₖ and τ = τ₁⋯τₘ with the σᵢ, τᵢ 2-cycles, the στ = σ₁⋯σₖτ₁⋯τₘ, hence can be written as a product of k + m 2-cycles (as we have seen, this means every representation of στ as a product of 2-cycles has the same parity as this one). Since τ is odd, k is odd, and hence m + k ≡ m + 1 mod 2; ie, multiplying by σ changes the parity of the permutation.

This means that the function f(τ) = στ is a function from H → H (since H is closed under multiplication), that restricts to a bijection (because multiplying by σ⁻¹ is the inverse of f) between the subsets of even and odd elements of H. Therefore, the two have the same size, and the sum of their orders, |H| must be even, a contradiction. Thus, H cannot contain an odd element.

32. Let β = (123)(145). Write β^⁹⁹ in disjoint cycle form.

Solution: It is easy to take powers of individual cycles, so it will be helpful to write β as a product of disjoint cycles. To do this, we just check what β does to each element of {1, 2, 3, 4, 5}:

β(1) = (123)(145)(1)
   = (123)(4)
   = 4
β(2) = (123)(2)
   = 3
β(3) = (123)(3)
   = 1
β(4) = (123)(5)
   = 5
β(5) = (123)(1)
   = 2

Putting it all together, β is just the 5-cycle (14523). This means that

β^⁹⁹ = β^{100}β⁻¹
   = (β^⁵)^{20}β⁻¹
   = eβ⁻¹
   = β⁻¹

However, the inverse of a cycle is also easy to calculate: β⁻¹ = (13254). This is in disjoint cycle form already, so we are done.
48. Let $\alpha$ and $\beta$ belong to $S_n$. Prove that $\beta\alpha\beta^{-1}$ and $\alpha$ are both even or both odd.

**Solution:** Write $\alpha$ as a product of $k$ 2-cycles $\alpha_i$, and $\beta$ as a product of $r$ 2-cycles $\beta_j$. It is easy to check (it follows from the general form of the inverse of a product, and that 2-cycles are their own inverses) that $\beta^{-1} = \beta_r \cdots \beta_1$, so it also is a product of $k$ 2-cycles. Thus $\beta\alpha\beta^{-1}$ can be written as a product $k + 2r$ 2-cycles. $2r$ is certainly even, so $k$ and $k + 2r$ have the same parity, giving the result.

55. Show that a permutation with odd order must be an even permutation.

**Solution:** Let $\sigma$ be such a permutation, so in particular $\sigma^r = e$, with $r$ odd. As usual, if we write $\sigma$ as a product of $k$ 2-cycles. Then $\sigma^r$ will be a product of $kr$ 2-cycles. But $e$ is an even permutation (for example, $e = (12)(12)$) so $kr$ must be even by the well-definedness of the parity of a permutation. Since 2 divides $rk$ but not $r$, the only option is if 2 divides $k$; ie, $k$ is even. Thus $\sigma$ is even.