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Integral and p-adic Refinements of the Abelian Stark Conjecture

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In loving memory of my father, Dumitru D. Popescu (1937–2005) and my mentor, Nicolae Popescu (1937–2010).

Abstract. We give a formulation of the abelian case of Stark's Main Conjecture in terms of determinants of projective modules and briefly show how this formulation leads naturally to its Equivariant Tamagawa Number Conjecture (ETNC) – type integral refinements. We discuss the Rubin-Stark integral refinement of an idempotent p1 iece of Stark's Abelian Main Conjecture. In the process, we give a new formulation of its particular case, the Brumer-Stark conjecture, in terms of annihilators of generalized Arakelov class-groups (first Chow groups.) In this context, we discuss somewhat in detail our recent theorems with Greither, settling refinements of the Brumer-Stark conjecture, under certain hypotheses. We formulate a general Gross–type refinement of the Rubin–Stark conjecture, which has emerged fairly recently from work of Gross, Tate, Tan, Burns, Greither and the author, and interpret it in terms of special values of p–adic L–functions. Finally, we discuss the recent overwhelming evidence in support of the combined Gross-Rubin-Stark conjecture, which is a consequence of independent work of Burns and that of Greither and the author on the ETNC and various Equivariant Main Conjectures in Iwasawa theory, respectively.

Introduction

In the 1970s and early 1980s, Stark (see [**34**], [**35**], [**36**], [**37**]) developed a remarkable Galois–equivariant conjectural link between the values at $s = 0$ of the first non–vanishing derivatives of the S–imprimitive Artin L–functions $L_{K/k,S}(\rho, s)$

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associated to a Galois extension K/k of number fields and a certain $\mathbb{Q}[\mathrm{Gal}(K/k)]$–module invariant of the group U_S of S–units in K. Stark's Main Conjecture should be viewed as a vast Galois–equivariant generalization of the unrefined, *rational version* of Dirichlet's class–number formula

$$\lim_{s\to 0}\frac{1}{s^r}\zeta_k(s)\in\mathbb{Q}^\times\cdot R_k\,,$$

in which the zeta function $\zeta_k(s)$ is replaced by a Galois–equivariant L–function

$$\Theta_{K/k,S}(s)=\sum_{\rho\in\widehat{G}}L_{K/k,S}(\rho,s)\cdot e_{\rho^\vee}\,,$$

with values in the center of the group–ring $\mathcal{Z}(\mathbb{C}[\mathrm{Gal}(K/k)])$, the regulator R_k and the rank r of the group of units in k are replaced by a Galois–equivariant regulator with values in $\mathcal{Z}(\mathbb{C}[\mathrm{Gal}(K/k)])$ and the local rank function of the projective $\mathbb{Q}[\mathrm{Gal}(K/k)]$–module $\mathbb{Q}U_S$, respectively. In [**19**], Gross formulated a p–adic version of Stark's Main Conjecture, bringing into the picture the p–adic L–functions (p–adic interpolations of Artin's L–functions), which had just been constructed in [**1**], [**8**], and [**12**], independently.

In the 1980s, work of Stark, loc.cit., Tate [**42**], and Chinburg [**9**], [**10**], among others, revealed not only the depth and importance of Stark's Main Conjecture for number theory, but also the fact that an *integral refinement* of this statement, in the spirit of the full strength, integral Dirichlet class–number formula

$$\lim_{s\to 0}\frac{1}{s^r}\zeta_k(s)=-\frac{h_k}{w_k}\cdot R_k\,,$$

would have very far reaching applications to major unsolved problems in the field (e.g. Hilbert's 12–th problem, asking for an explicit class-field theory over arbitrary number fields.)

In [**37**], Stark himself formulated such an integral refinement for the particular case of *abelian extensions* K/k and their associated imprimitive L–functions $L_{K/k,S}(\chi,s)$ of *order of vanishing at most* 1 *at* $s=0$, under additional assumptions on S, and proved it in the cases where k is either $\mathbb{Q}$ or imaginary quadratic. The proofs revealed deep connections between the (however limited in scope) integral conjecture and the theory of Gauss sums, cyclotomic units, and elliptic units, respectively. In [**19**] and [**20**], Gross developed a p–adic refinement of Stark's integral conjecture in this context and proved it for imaginary, abelian extensions of $\mathbb{Q}$, as a direct consequence of his joint work with Koblitz [**21**].

Then, Tate's influential book [**42**] was published, casting the conjectures developed so far and their potential applications into a completely new light and extending them to function fields (characteristic p global fields.) An ℓ–adic étale cohomological proof of the integral Stark conjecture for function fields, due to Deligne and Tate, was given in Chpt. V of [**42**]. Almost immediately after, Hayes (see [**22**] and [**23**]) gave explicit proofs of Stark's integral conjecture and its refinement by Gross, in the case of function fields, establishing beautiful connections with the theory of torsion points of rank 1, sign normalized Drinfeld modules.

In the late 1980s and early 1990s, after the extremely successful emergence of Kolyvagin's theory of Euler systems, Rubin (see [**31**]) realized that, if true, Stark's integral conjecture is a rich source of Euler systems. The Euler systems of Gauss sums, cyclotomic units, elliptic units, and torsion points of rank 1 Drinfeld modules are very particular cases of this very general fact. Driven by this realization, Rubin (see [**32**]) generalized Stark's integral conjecture to the case of abelian extensions of number fields K/k and their associated S–imprimitive L–functions of minimal order of vanishing $r \geq 0$, under certain additional assumptions on S. This is the Rubin–Stark conjecture, one of the central themes of this paper. The Rubin-Stark conjecture was extended to function fields, and proved in fairly general cases in that context in [**27**] by the author. As a consequence, a general Gras-type conjecture for function fields was proved in [**26**]. (The cyclotomic Gras conjecture in number fields had been proved earlier by Mazur–Wiles and Kolyvagin.) Recently, a Gross–type refinement of the Rubin–Stark conjecture has emerged out of work of Tate, Gross, Tan, Burns, Greither and the author. This has led to a combined Gross-Rubin-Stark Conjecture, which encapsulates the Rubin-Stark and Gross conjectures and predicts, among other things, a subtle link between special values of derivatives of global and p–adic L–functions.

In [**2**], Bloch and Kato formulated a very general conjecture on special values of L–functions associated to motives with abelian coefficients (in particular, Dirichlet motives, whose L–functions are the object of study of Stark's Abelian Main Conjecture.) Now, this statement is known as the Equivariant Tamagawa Number Conjecture (ETNC.) Burns and Flach (see [**3**] and [**4**]) extended this statement to motives with non-abelian coefficients. They also showed that, in the case of Artin motives, the ETNC can be viewed as an integral refinement of Stark's Main Conjecture in its most general setting. Recently, Burns has shown in [**6**] that the ETNC for Dirichlet motives implies (refined versions of) the combined Gross–Rubin–Stark Conjecture. In [**5**], Burns and Greither (with later contributions by Flach on the 2–primary side) proved the ETNC for Dirichlet motives with base field $\mathbb{Q}$. Recently, Burns (see [**6**] and the references therein) has proved the ETNC for Dirichlet motives over function fields as well. Therefore, the Gross–Rubin–Stark conjecture was proved in those cases.

In recent joint work with Greither (see [**15**], [**16**], [**17**]), the author has proved an Equivariant Main Conjecture in Iwasawa theory for both function fields and number fields (away from the 2–primary part and under a certain μ–invariant hypothesis, in the number field case.) It turns out that this conjecture implies the Gross–Rubin–Stark Conjecture (in full generality, in function fields and in an imprimitive form and away from 2, for CM abelian extensions of totally real number fields.) In particular, it implies refined versions of an imprimitive Brumer–Stark Conjecture and of the full étale Coates–Sinnott Conjecture (discussed in Manfred Kolster's contribution to this volume.) Ultimately, our Equivariant Main Conjecture implies the full ETNC for function fields and its "minus" piece under the action of complex conjugation, for abelian CM extensions of totally real number fields, providing new evidence in support of the ETNC and the combined Gross–Rubin–Stark Conjecture.

This paper is organized as follows. After a brief §1 on various algebraic tools needed throughout, in §2, we give a formulation of the abelian case of Stark's Main Conjecture in terms of determinants of projective modules. In §3, we briefly show how this formulation leads naturally to its ETNC–type integral refinements and discuss in detail the Rubin-Stark integral refinement of an idempotent piece of Stark's Abelian Main Conjecture. In §4, we restrict ourselves to the order of vanishing 1 case of the Rubin-Stark conjecture. In particular, in §4.3, we give a new formulation of the Brumer-Stark conjecture (a special case of the order of vanishing 1 Rubin-Stark conjecture) in terms of annihilators of generalized Arakelov class-groups (first Chow groups.) In this context, we discuss somewhat in detail our recent theorems with Greither, settling refinements of the Brumer-Stark conjecture, under certain hypotheses. In §5, we formulate a Gross–type refinement of the Rubin–Stark conjecture (the so-called combined Gross–Rubin–Stark Conjecture) and interpret it in terms of special values of p–adic L–functions. In §5.5, we discuss the recent overwhelming evidence in support of the combined Gross-Rubin-Stark conjecture, which has emerged from work of Burns and that of Greither and the author on the ETNC and various Equivariant Main Conjectures in Iwasawa theory, respectively.

We conclude this introduction by extending our apologies to several people who have made significant contributions to this area over the years and whose names and specific results will not be mentioned in what follows, due to our lack of time, space and expertise. These are: Noboru Aoki, Samit Dasgupta, Henri Darmon, David Dummit, Caleb Emmons, Anthony Hayward, Radan Kučera, Joongul Lee, Jonathan Sands, Barry Smith, David Solomon, Brett Tangedal.

1. Algebraic preliminaries

In what follows, R will denote a commutative, reduced, Noetherian ring. Typically, R will be a group algebra associated to a finite, commutative group G, with coefficients in $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ or the ring of ℓ–adic integers $\mathbb{Z}_\ell$, for an arbitrary prime number ℓ. If M is an R–module and R' is a commutative R–algebra, then $R'M := R'\otimes_R M$.

1.1. Duals

If M is an R–module, we let $M^* := \mathrm{Hom}_R(M, R)$ denote the R–dual of M. We endow M^* with the usual R–module structure $(r\cdot\phi)(m) = r\phi(m)$, for all $r \in R$, $\phi \in M^*$ and $m \in M$.

Remark 1.1.1. *If M is a $\mathbb{Z}[G]$–module, where G is a finite, commutative group, then there is a canonical $\mathbb{Z}[G]$–module isomorphism*

$$\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}[G]}(M,\mathbb{Z}[G]), \qquad \phi \longrightarrow \psi\,,$$

where $\psi(m) := \sum_{\sigma\in G}\phi(\sigma^{-1}\cdot m)\cdot\sigma$, for all $\phi\in\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ and all $m \in M$. Here, $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ is endowed with the usual $\mathbb{Z}[G]$–module structure given by $(\sigma\cdot\phi)(m) = \phi(\sigma\cdot m)$, for all $\sigma\in G$, $\phi\in\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$, and $m \in M$.

Any $\phi \in M^*$ induces an R–linear morphism

$$\phi^{(r)} : \wedge_R^r M \longrightarrow \wedge_R^{r-1}M, \qquad x_1\wedge\cdots\wedge x_r \longrightarrow \sum_{i=1}^{r}(-1)^{i-1}\phi(x_i)\cdot x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge x_r\,,$$

for all $r \in \mathbb{Z}_{\geq 1}$. If the context is clear and r is fixed, then we will sometimes abusively write ϕ instead of $\phi^{(r)}$. The construction above leads to the more general R–linear morphism

$$\begin{array}{ccc} \wedge_R^i M^*\otimes_R\wedge_R^r M & \longrightarrow & \wedge_R^{r-i}M \\ (\phi_1\wedge\cdots\wedge\phi_i)\otimes(x_1\wedge\cdots\wedge x_r) & \longrightarrow & \phi_i^{(r-i+1)}\circ\cdots\circ\phi_1^{(r)}(x_1\wedge\cdots\wedge x_r), \end{array}$$

for all $r, i \in \mathbb{Z}_{\geq 1}$ with $i \leq r$ and all $\phi_1, \dots, \phi_i \in M^*$ and $x_1, \dots, x_r \in M$,

Remark 1.1.2. *With notations as above, note that if $r = i$, then we have*

$$\phi_r^{(1)} \circ \cdots \circ \phi_1^{(r)}(x_1 \wedge \cdots \wedge x_r) = \det(\phi_i(x_j);\, 1 \leq i, j \leq r)\,,$$

for all $\phi_1, \dots, \phi_r \in M^*$ and all $x_1, \dots, x_r \in M$. This establishes an R–module morphism

$$\wedge_R^r M^* \longrightarrow (\wedge_R^r M)^*, \qquad \phi_1 \wedge \cdots \wedge \phi_r \longrightarrow \phi_r^{(1)} \circ \cdots \circ \phi_1^{(r)}\,.$$

It is very easy to check that if M is a free R–module (or, more generally, a projective R–module), then the morphism above is an isomorphism.

Let $\mathcal{K} := Q(R)$ be the total ring of fractions of R (i.e. the ring of fractions obtained from R by inverting all its non–zero–divisors.) Note that R is a subring of $\mathcal{K}$. Since R is Noetherian and reduced, $\mathcal{K}$ is a finite direct sum of fields. Indeed, if $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ are the minimal prime ideals of R, then it is easy to show that we have a ring isomorphism $\mathcal{K} \simeq \oplus_{i=1}^n Q(R/\mathfrak{p}_i)$, where $Q(R/\mathfrak{p}_i)$ is the field of fractions of the integral domain $R/\mathfrak{p}_i$, for all $i = 1, \dots, n$. Now, let M be a finitely generated R–module. We have a canonical R–module morphism

$$\text{(1)} \qquad \mathrm{Hom}_R(M, R) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(\mathcal{K}M, \mathcal{K})\,, \qquad \phi \longrightarrow \psi\,,$$

where $\psi(x/s) = \phi(x)/s$, for all $x \in M$ and all non–zero–divisors $s \in R$. This map happens to be injective in this case and it induces a canonical isomorphism of $\mathcal{K}$–modules

$$\mathrm{Hom}_R(M, R) \otimes_R \mathcal{K} \simeq \mathrm{Hom}_{\mathcal{K}}(\mathcal{K}M, \mathcal{K})\,.$$

For every $r \in \mathbb{Z}_{\geq 0}$, we have an R–module morphism

$$\begin{aligned} E_r : \wedge_R^r M^* \otimes_R \wedge_{\mathcal{K}}^r \mathcal{K}M &\longrightarrow \mathcal{K} \\ (\phi_1 \wedge \cdots \wedge \phi_r) \otimes (x_1 \wedge \cdots \wedge x_r) &\longrightarrow \psi_r^{(1)} \circ \cdots \circ \psi_1^{(r)}(x_1 \wedge \cdots \wedge x_r) = \det(\psi_i(x_j)) \end{aligned}$$

where $\phi_i \in M^*$, $x_i \in \mathcal{K}M$ and $\phi_i \longrightarrow \psi_i$ via the map (1) above, for all $i = 1, \dots, r$.

Definition 1.1.3. *For R, M and r as above, let $\mathcal{L}_R(M, r)$ be the R–submodule of $\wedge_{\mathcal{K}}^r \mathcal{K}M$ given by*

$$\mathcal{L}_R(M, r) := \{\epsilon \in \wedge_{\mathcal{K}}^r \mathcal{K}M \mid E_r((\phi_1 \wedge \cdots \wedge \phi_r) \otimes \epsilon) \in R\,, \quad \forall\, (\phi_1 \wedge \cdots \wedge \phi_r) \in \wedge_R^r M^*\}.$$

Remark 1.1.4. *Note the following.*

1. *$\mathcal{L}_R(M, 0) = R$, for all R and M as above.*
2. *The canonical $\mathcal{K}$–module isomorphism $\mathcal{K} \otimes_R \wedge_R^r M \simeq \wedge_{\mathcal{K}}^r \mathcal{K}M$ induces an obvious R–linear map $\wedge_R^r M \longrightarrow \wedge_{\mathcal{K}}^r \mathcal{K}M$. This map is not injective, in general. Its kernel consists of those elements in $\wedge_R^r M$ which are annihilated by non–zero–divisors in R. We denote by $\widetilde{\wedge_R^r M}$ its image. We have an obvious double-inclusion of R–modules*
$$\widetilde{\wedge_R^r M} \subseteq \mathcal{L}_R(M, r) \subseteq \wedge_{\mathcal{K}}^r \mathcal{K}M\,.$$
3. *If M is a projective R–module, then we leave it as an exercise for the reader to check that*
$$\mathcal{L}_R(M, r) = \widetilde{\wedge_R^r M} \simeq \wedge_R^r M.$$

Note that since M is R–projective, so is $\wedge_R^r M$. Consequently, the R–linear morphism $\wedge_R^r M \longrightarrow \wedge_{\mathcal{K}}^r \mathcal{K}M$ is injective in this case, which leads to the isomorphism above. Standard properties of exterior powers of direct sums of modules reduce the verification of the equality above to the case where M is R–free, which is straightforward.

(4) *In particular, if $R = \mathbb{Z}[1/|G|][G]$, where G is a finite, abelian group, then R is easily seen to be isomorphic to a finite direct sum of Dedekind domains (rings of integers in cyclotomic fields.) Consequently, any finitely generated $\mathbb{Z}[1/|G|][G]$–module M which has no $\mathbb{Z}$–torsion is $\mathbb{Z}[1/|G|][G]$–projective. Consequently, (3) above applies, for all r.*

Example 1.1.5. Let G be a finite, abelian group and $r \in \mathbb{Z}_{\geq 1}$. Let $R := \mathbb{Z}[G]$ and $M := \mathbb{Z}^r$, viewed as a $\mathbb{Z}[G]$–module with trivial G–action. In this case, it is easily seen that $\mathcal{K} := Q(\mathbb{Z}[G]) = \mathbb{Q}[G]$. Consequently, $\mathcal{K}M = \mathbb{Q}^r$, with trivial G–action. If $\{e_1, \dots, e_r\}$ is a $\mathbb{Z}$–basis for M, then the reader can check that the following equalities and isomorphisms of $\mathbb{Z}[G]$–modules hold

$$\wedge_{\mathcal{K}}^r \mathcal{K}M = \mathbb{Q}e_1 \wedge \cdots \wedge e_r \simeq \mathbb{Q}, \quad \wedge_R^r M \simeq \widetilde{\wedge_R^r M} = \mathbb{Z}e_1 \wedge \cdots \wedge e_r \simeq \mathbb{Z},$$
$$\mathcal{L}_R(M, r) = \tfrac{1}{|G|^{r-1}} \mathbb{Z}e_1 \wedge \cdots \wedge e_r,$$

where the G–actions are trivial everywhere. If $r \geq 2$ and $|G| > 1$, the inclusion $\widetilde{\wedge_R^r M} \subseteq \mathcal{L}_R(M, r)$ is strict. If $r = 1$, the inclusion in question is an equality. Obviously, the module M is not R–projective for any $r \geq 1$ and non–trivial group G (more on this in the next section.)

Example 1.1.6. If G is a finite, abelian group, $R := \mathbb{Z}[G]$, and M is an R–module, then we have

$$\mathcal{L}_R(M, 1) = \widetilde{M}\,,$$

as it easily follows from Remark 1.1.1 above.

1.2. Evaluation maps

Let M denote a finitely generated $\mathbb{Z}[G]$–module, where G is a finite, abelian group. As usual, we let $M^* := \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])$. In this section, we take a closer look at the elements of the lattice $\mathcal{L}_{\mathbb{Z}[G]}(M, r)$, for a fixed $r \in \mathbb{Z}_{\geq 0}$. The properties discussed below will not be used until the very last section of this article.

Let us fix $\varepsilon \in \mathcal{L}_{\mathbb{Z}[G]}(M, r)$. Then, from the definition of $\mathcal{L}_{\mathbb{Z}[G]}(M, r)$, the element ε gives rise to a $\mathbb{Z}$–linear evaluation map

$$\mathrm{ev}_{\varepsilon, \mathbb{Z}} : \wedge_{\mathbb{Z}}^r M^* \longrightarrow \mathbb{Z}[G]\,, \qquad \phi_1 \wedge \cdots \wedge \phi_r \to (\phi_r^{(1)} \circ \cdots \circ \phi_1^{(r)})(\varepsilon)\,,$$

for all $\phi_1, \dots, \phi_r \in M^*$. It is easy to see that the map $\mathrm{ev}_{\varepsilon, \mathbb{Z}}$ factors through a $\mathbb{Z}[G]$–linear map

$$\widetilde{\mathrm{ev}_{\varepsilon, \mathbb{Z}}} : \wedge_{\mathbb{Z}[G]}^r M^* \longrightarrow \mathbb{Z}[G]\,.$$

The main goal of this section is to generalize the evaluation map $\mathrm{ev}_{\varepsilon, \mathbb{Z}}$ to some extent. For that purpose, we let $\mathcal{R}$ denote an arbitrary commutative ring with 1. We view $\mathcal{R}[G]$ as a $\mathbb{Z}[G]$–algebra in the usual way and will make use of the ring isomorphism

$$i_{\mathcal{R}} : \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathcal{R} \simeq \mathcal{R}[G]\,, \qquad (x \otimes a) \to xa\,.$$

We let $M^*_{\mathcal{R}} := \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathcal{R}[G])$, endowed with the usual $\mathcal{R}[G]$–module structure. For the sake of consistency, note that $M^* = M^*_{\mathbb{Z}}$. It is easy to see that the isomorphism in Remark 1.1.1 leads to an isomorphism of $\mathcal{R}[G]$–modules

$$j_{M,\mathcal{R}} : M^* \otimes_{\mathbb{Z}} \mathcal{R} \simeq M^*_{\mathcal{R}}\,, \qquad \phi \otimes a \to \psi\,,$$

where $\psi(x) = \phi(x)a$, for all $\phi \in M^*$, $x \in M$ and $a \in \mathcal{R}$. Consequently, for any such ring $\mathcal{R}$, the element ε gives rise to the more general $\mathcal{R}[G]$–linear evaluation map

$$\begin{array}{rl} \mathrm{ev}_{\varepsilon,\mathcal{R}} : \wedge^r_{\mathcal{R}} M^*_{\mathcal{R}} \xrightarrow[\sim]{\wedge^r j^{-1}_{M,\mathcal{R}}} \wedge^r_{\mathcal{R}}(M^* \otimes_{\mathbb{Z}} \mathcal{R}) \xrightarrow[\sim]{} & (\wedge^r_{\mathbb{Z}} M^*) \otimes_{\mathbb{Z}} \mathcal{R} \\ & \Big\downarrow \mathrm{ev}_{\varepsilon,\mathbb{Z}} \otimes \mathbf{1}_{\mathcal{R}} \\ & \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathcal{R} \xrightarrow[\sim]{i_{\mathcal{R}}} \mathcal{R}[G]\,. \end{array}$$

Above, the only non-labeled isomorphism is a consequence of the canonical $\mathcal{R}$–linear isomorphism $\wedge^r_{\mathcal{R}}(N \otimes_{\mathbb{Z}} \mathcal{R}) \simeq (\wedge^r_{\mathbb{Z}} N) \otimes_{\mathbb{Z}} \mathcal{R}$, for all $\mathbb{Z}$–modules N. Just as in the case $\mathcal{R} = \mathbb{Z}$, it is easy to see that the map $\mathrm{ev}_{\varepsilon,\mathcal{R}}$ factors through an $\mathcal{R}[G]$–linear map

$$\widetilde{\mathrm{ev}_{\varepsilon,\mathcal{R}}} : \wedge^r_{\mathcal{R}[G]} M^*_{\mathcal{R}} \longrightarrow \mathcal{R}[G]\,.$$

Remark 1.2.1. *In order to get a feel for the map* $\mathrm{ev}_{\varepsilon,\mathcal{R}}$*, note that, if* $\psi_1, \dots, \psi_r$ *are elements in* $M^*_{\mathcal{R}}$*, such that* $\psi_i = j_{M,\mathcal{R}}(\phi_i \otimes a_i)$*, with* $\phi_i \in M^*$ *and* $a_i \in \mathcal{R}$*, for all* $i = 1, \dots, r$*, then we have*

$$\mathrm{ev}_{\varepsilon,\mathcal{R}}(\psi_1 \wedge \cdots \wedge \psi_r) = \mathrm{ev}_{\varepsilon,\mathbb{Z}}(\phi_1 \wedge \cdots \wedge \phi_r) \cdot \prod_{i=1}^{r} a_i\,. \tag{2}$$

This equality permits us to make the following useful observation. For that purpose, let us assume further that $\mathcal{R}$ *is a* $\mathbb{N}$*–graded ring*

$$\mathcal{R} = \mathcal{R}^{(0)} \oplus \mathcal{R}^{(1)} \oplus \cdots \oplus \mathcal{R}^{(i)} \oplus \cdots\,,$$

where $\mathcal{R}^{(i)}$ *denotes the homogeneous component of degree* i *of* $\mathcal{R}$ *in the given grading, for all* $i \in \mathbb{N}$*. For a* $\mathbb{Z}[G]$*–module* M *as above, let*

$$M^*_{\mathcal{R}^{(i)}} := \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathcal{R}^{(i)} \otimes_{\mathbb{Z}} \mathbb{Z}[G]),$$

for all $i \in \mathbb{N}$*. Then,* $M^*_{\mathcal{R}}$ *is an* $\mathbb{N}$*–graded* $\mathcal{R}$*–module, with the grading given by*

$$M^*_{\mathcal{R}} = \bigoplus_{i \in \mathbb{N}} M^*_{\mathcal{R}^{(i)}}\,,$$

where $M^*_{\mathcal{R}^{(i)}}$ *is viewed as a* $\mathbb{Z}[G]$*–submodule of* $M^*_{\mathcal{R}}$ *via the injection*

$$\mathcal{R}^{(i)} \otimes_{\mathbb{Z}} \mathbb{Z}[G] \hookrightarrow \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}[G] \simeq \mathcal{R}[G],$$

for all $i \in \mathbb{N}$*. Equality (2), combined with the* $\mathcal{R}$*–linearity of* $\mathrm{ev}_{\varepsilon,\mathcal{R}}$*, leads to*

$$\mathrm{ev}_{\varepsilon,\mathcal{R}}(\psi_1 \wedge \cdots \wedge \psi_r) \in \mathcal{R}^{(r)}[G] := i_{\mathcal{R}}(\mathcal{R}^{(r)} \otimes_{\mathbb{Z}} \mathbb{Z}[G]),$$

for all $\psi_1, \dots, \psi_r \in M^*_{\mathcal{R}^{(1)}}$*. This observation will be very useful in the last section of this article.*

1.3. Projective modules

Recall that a finitely generated R–module P is called projective if there exists an R–module Q, an $n \in \mathbb{Z}_{\geq 0}$ and an R–module isomorphism $P \oplus Q \simeq R^n$. Recall that P is projective if and only if it is locally free. This means that for all $\mathfrak{p} \in \mathrm{Spec}(R)$, there exists a (unique) $r_{\mathfrak{p}} \in \mathbb{Z}_{\geq 0}$ and an isomorphism of $R_{\mathfrak{p}}$–modules $P_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^{r_{\mathfrak{p}}}$, where $P_{\mathfrak{p}}$ and $R_{\mathfrak{p}}$ denote the usual localizations at the prime ideal $\mathfrak{p}$.

Consequently, any finitely generated projective R–module P gives rise to a rank–function

$$\mathrm{rk}_P : \mathrm{Spec}(R) \longrightarrow \mathbb{Z}, \qquad \mathrm{rk}_P(\mathfrak{p}) := \mathrm{rank}_{R_{\mathfrak{p}}}(P_{\mathfrak{p}}) = r_{\mathfrak{p}}\,.$$

It is easy to see that this function is locally constant (i.e. continuous with respect to the Zariski topology on $\mathrm{Spec}(R)$ and the discrete topology on $\mathbb{Z}$.) Indeed, for any prime $\mathfrak{p} \in \mathrm{Spec}(R)$, this function is constant on $\mathrm{Spec}(R/\mathfrak{p}) \subseteq \mathrm{Spec}(R)$. Consequently, if $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are the minimal prime ideals of R, then rk_P is constant on the irreducible components $\mathrm{Spec}(R/\mathfrak{p}_i)$ of $\mathrm{Spec}(R)$ and therefore locally constant on $\mathrm{Spec}(R)$.

Let $R = \oplus_{i=1}^{s} R_i$ be a decomposition of R as a direct sum of rings R_i with connected spectra. Of course, this corresponds to a decomposition $1_R = e_1 + \cdots + e_s$ of the identity element 1_R into a sum of indecomposable idempotents e_i and also to the disjoint union decomposition $\mathrm{Spec}(R) = \cup_{i=1}^{s} \mathrm{Spec}(R_i)$ of $\mathrm{Spec}(R)$ into its connected components. Let P be a finitely generated projective R–module and let r_i be the (constant) value of rk_P restricted to $\mathrm{Spec}(R_i)$, for all $i = 1, \dots, s$. Obviously, we have a direct sum decomposition $P = \oplus_{i=1}^{s} P_i$, where $P_i := e_i \cdot P \simeq P \otimes_R R_i$ is a projective R_i–module of constant rank r_i, for all $i = 1, \dots, s$.

Definition 1.3.1. *1) With notations as above, the determinant–module associated to P (or, simply, the determinant of P) is the projective R–module of constant rank 1 given by*

$$\det_R(P) := \bigoplus_{i=1}^{s} \wedge_{R_i}^{r_i} P_i\,.$$

2) Further, if Q is a projective R–module with the same rank function as P (i.e. $\mathrm{rk}_Q = \mathrm{rk}_P$) and $f \in \mathrm{Hom}_R(P, Q)$, then the determinant–morphism $\tilde{f}$ associated to f is the R–module morphism

$$\tilde{f} : \det_R(P) \longrightarrow \det_R(Q), \qquad \tilde{f} := \oplus_{i=1}^{s} \wedge^{r_i} f_i\,,$$

where f_i is the restriction of f to P_i viewed in $\mathrm{Hom}_{R_i}(P_i, Q_i)$, for all $i = 1, \dots, s$.

3) If $f \in \mathrm{Hom}_R(P, P)$, then the determinant of f is the unique element $\det_R(f) \in R$ satisfying the property that for all primes $\mathfrak{p} \in \mathrm{Spec}(R)$ its image via the canonical map $R \longrightarrow R_{\mathfrak{p}}$ is the usual determinant of the endomorphism $f_{\mathfrak{p}} := 1_{R_{\mathfrak{p}}} \otimes f$ of the free $R_{\mathfrak{p}}$–module $P_{\mathfrak{p}} \simeq R_{\mathfrak{p}} \otimes_R P$.

Remark 1.3.2. *The uniqueness of $\det_R(f)$ in part 3) of the above definition is clear since the diagonal map $R \longrightarrow \prod_{\mathfrak{p}} R_{\mathfrak{p}}$ is injective. Its existence is proved by picking an R–module Q, such that $P \oplus Q \simeq R^n$, for some $n \in \mathbb{Z}_{\geq 1}$, and then defining $\det_R(f)$ as the usual determinant of the endomorphism $f \oplus 1_Q$ of the free R-module $P \oplus Q$.*

Recall that if Q is a projective R–module of constant rank 1, there is an R–module isomorphism

$$\mathrm{ev}_Q : Q \otimes_R Q^* \xrightarrow{\sim} R, \qquad x \otimes f \longrightarrow f(x)\,,$$

called the evaluation map associated to Q. More generally, if P is a finitely generated projective R–module P, then we denote by ev_P the evaluation map associated to $\det_R(P)$

$$\mathrm{ev}_P : \det_R(P) \otimes_R \det_R(P)^* \xrightarrow{\sim} R\,.$$

Example 1.3.3. Let us consider the ring $R := F[G]$, where G is a finite abelian group and F a field of characteristic 0. We denote by $\widehat{G}(F)$ the set of characters of all irreducible linear representations of G defined over F. If F is algebraically closed, then $\widehat{G}(F)$ is the group consisting of the characters of all the 1–dimensional representations of G defined over F. In general, there is a one–to–one correspondence

$$\widehat{G}(\overline{F})/\sim_F \longleftrightarrow \widehat{G}(F)\,, \qquad [\chi] \longrightarrow \psi := \sum_{\tau \in \mathrm{Gal}(F(\chi)/F)} \chi^\tau\,, \tag{3}$$

where $\overline{F}$ is the algebraic closure of F, $G_F := \mathrm{Gal}(\overline{F}/F)$, $\sim_F$ is the equivalence relation given by the natural G_F–action on $\widehat{G}(\overline{F})$ and $[\chi]$ is the equivalence class of a character $\chi \in \widehat{G}(\overline{F})$. There is an F–algebra isomorphism

$$F[G] \xrightarrow{\sim} \bigoplus_{[\chi]} F(\chi)\,, \qquad \sum_{\sigma \in G} a_\sigma \cdot \sigma \longrightarrow (\sum_{\sigma \in G} a_\sigma \cdot \chi(\sigma))_{[\chi]}\,, \tag{4}$$

where $[\chi]$ runs through $\widehat{G}(\overline{F})/\sim_F$ and $F(\chi)$ is the minimal field-extension of F which contains the values of χ. Consequently, there is a one-to-one correspondence

$$\widehat{G}(F) \longleftrightarrow \mathrm{Spec}(F[G])\,, \quad \psi \longrightarrow \mathfrak{p}_\psi := \ker(F[G] \xrightarrow{\chi} F(\chi))\,, \text{ if } [\chi] \longrightarrow \psi \text{ via (3).} \tag{5}$$

Here, $\chi : F[G] \longrightarrow F(\chi)$ is the morphism of F–algebras sending σ to $\chi(\sigma)$, for all $\sigma \in G$. Note that the prime ideal $\mathfrak{p}_\psi$ only depends on the conjugacy class $[\chi]$ of χ. In what follows, e_ψ denotes the usual idempotent associated to $\psi \in \widehat{G}(F)$ in $F[G]$, defined by

$$e_\psi := \frac{1}{|G|} \sum_{\sigma \in G} \psi(\sigma) \cdot \sigma^{-1}.$$

Obviously, if $[\chi] \longrightarrow \psi$ via (3), then we have $e_\psi = \sum_\tau e_{\chi^\tau}$ in $\overline{F}[G]$. Consequently, we have $\mathfrak{p}_\psi = (1 - e_\psi)F[G]$ and $e_\psi F[G] \xrightarrow{\sim} F(\chi)$ via isomorphism (4), for all $\psi \in \widehat{G}(F)$. The decomposition of $R := F[G]$ into direct sum of rings with connected spectra is therefore given by

$$F[G] = \bigoplus_{\psi \in \widehat{G}(F)} e_\psi F[G]\,,$$

where $e_\psi F[G]$ is a field (finite extension of F), for all ψ.

Now, if M is a finitely generated $F[G]$–module, then $M = \oplus_\psi M^\psi$, where

$$M^\psi := e_\psi M$$

is a finite dimensional vector space over the field $e_\psi F[G]$. Consequently, M is always $F[G]$–projective and its rank function is given by

$$\mathrm{rk}_M : \mathrm{Spec}(F[G]) \longrightarrow \mathbb{Z}, \qquad \mathrm{rk}_M(\mathfrak{p}_\psi) = \dim_{e_\psi F[G]}(M^\psi)\,, \quad \forall\, \psi \in \widehat{G}(F).$$

Note that M is $F[G]$–free if and only if rk_M is a constant function. If $h : M \longrightarrow M$ is an $F[G]$–module endomorphism, then $h = \oplus_\psi h^\psi$, where $h^\psi : M^\psi \longrightarrow M^\psi$ is the restriction of h to M^ψ. Consequently, we have the following equality in $F[G]$.

$$\det_{F[G]}(h) = \sum_{\psi \in \widehat{G}(F)} \det_{e_\psi F[G]}(h^\psi)\,.$$

Example 1.3.4. Now, let us consider the ring $R := \mathbb{Z}[G]$. It is well–known that $\mathrm{Spec}(\mathbb{Z}[G])$ is connected. Indeed, otherwise, there would exist a non-trivial idempotent $\alpha = \sum_{\sigma\in G}\alpha_\sigma \cdot \sigma$ in $\mathbb{Z}[G]$. Under the embedding $\mathbb{Z}[G] \hookrightarrow \mathbb{C}[G]$, α is a non–trivial idempotent in $\mathbb{C}[G]$ and therefore $\alpha = \sum_{\chi\in\mathcal{S}} e_\chi$, where χ runs through a non-empty, strict subset $\mathcal{S}$ of $\widehat{G}(\mathbb{C})$. This means that

$$\frac{1}{|G|}\sum_{\chi\in\mathcal{S}}\chi(\sigma) = \alpha_\sigma \in \mathbb{Z}\,, \quad \forall\, \sigma \in G\,.$$

However, since $|\mathcal{S}| < |G| = |\widehat{G}(\mathbb{C})|$ and $|\chi(\sigma)| = 1$, for all χ and σ, we have $|\alpha_\sigma| < 1$, therefore $\alpha_\sigma = 0$, for all σ. Consequently, $\alpha = 0$ and therefore $\mathrm{Spec}(\mathbb{Z}[G])$ is connected.

A well–known criterion of Nakayama (see [**42**], Chpt. II, §4) states that a finitely generated $\mathbb{Z}[G]$–module is projective if and only if it is $\mathbb{Z}$–free and G–cohomologically trivial (i.e. the Tate cohomology groups $\widehat{\mathrm{H}}^i(\mathfrak{g}, M)$ vanish, for all $i \in \mathbb{Z}$ and all subgroups $\mathfrak{g}$ of G.) Assuming that we have such a projective module P, its rank function rk_P is constant, say equal to $r \in \mathbb{Z}_{\geq 0}$, on $\mathrm{Spec}(\mathbb{Z}[G])$. Consequently, the projective $\mathbb{Q}[G]$–module $\mathbb{Q}\otimes_{\mathbb{Z}} P$ has constant rank r and it is therefore free of rank r over $\mathbb{Q}[G]$ (see Example 1.3.3 above for $F := \mathbb{Q}$.) The $\mathbb{Z}[G]$–module P may not be free, of course. However, a well–known theorem of Swan (see [**42**], Chpt. II, §4) states that there exists an ideal $\mathfrak{a}$ of $\mathbb{Z}[G]$ whose index in $\mathbb{Z}[G]$ is finite and coprime to $|G|$ and an isomorphism of $\mathbb{Z}[G]$–modules

$$P \xrightarrow{\sim} \mathfrak{a} \oplus \mathbb{Z}[G]^{r-1}.$$

Note that any such ideal $\mathfrak{a}$ is $\mathbb{Z}[G]$–projective of rank 1. (To prove projectivity, apply Nakayama's criterion stated above !)

1.4. Extension of scalars.

Next, we discuss briefly the behavior of some of the constructions in the previous section under extension of scalars. For that purpose, let $\rho : R \longrightarrow R'$ be a morphism of commutative, reduced, Noetherian rings and let P be a finitely generated projective R–module. We let $P' := R'P := R'\otimes_R P$, which is obviously a finitely generated projective R'–module. There are canonical (and obvious) isomorphisms of R'–modules

$$(6) \qquad R'\otimes_R \det_R(P) \simeq \det_{R'}(P'), \qquad R'\otimes_R \det_R(P)^* \simeq \det_{R'}(P')^*\,.$$

The isomorphisms above induce canonical R–module morphisms

$$\iota : \det_R(P) \longrightarrow \det_{R'}(P'), \qquad \iota^* : \det_R(P)^* \longrightarrow \det_{R'}(P')^*\,.$$

It is easy to check that the evaluation maps commute with extension of scalars, which is to say that the following diagram is commutative.

$$\begin{array}{ccc} \det_R(P)\otimes_R \det_R(P)^* & \xrightarrow{\iota\otimes\iota^*} & \det_{R'}(P')\otimes_{R'}\det_{R'}(P')^* \\ {\scriptstyle \mathrm{ev}_P}\downarrow\wr & & {\scriptstyle \mathrm{ev}_{P'}}\downarrow\wr \\ R & \xrightarrow{\rho} & R' \end{array}$$

An example of extension of scalars which will be of particular interest to us is the case where $R' := eR$, where e is an idempotent in R and $\rho : R \longrightarrow eR$ sends x to ex, for all $x \in R$. In this case, isomorphisms (6) above can be rewritten as

$$e\det_R(P) \simeq \det_{eR}(eP) \qquad e\det_R(P)^* \simeq \det_{eR}(eP)^*\,. \tag{7}$$

One last property we would like to point out is the following. Consider an R'–linear endomorphism $h : P' \longrightarrow P'$. Now, we take the composition of the following R–linear maps

$$\begin{array}{ccc} \det_R(P)\otimes_R \det_R(P)^* & \xrightarrow{\iota\otimes\iota^*} & \det_{R'}(P')\otimes_{R'}\det_{R'}(P')^* & & \\ & & \downarrow{\scriptstyle \tilde h\otimes\mathrm{id}} & & \\ & & \det_{R'}(P')\otimes_{R'}\det_{R'}(P')^* & \xrightarrow[\sim]{\mathrm{ev}_{P'}} & R'. \end{array}$$

It is easy to check (locally !) that we have an equality of R–submodules of R'.

$$\mathrm{Im}(\mathrm{ev}_{P'}\circ(\tilde h\otimes\mathrm{id})\circ(\iota\otimes\iota^*)) = \rho(R)\cdot\det_{R'}(h)\,. \tag{8}$$

In particular, if $R = R'$, $h \in \mathrm{End}_R(P)$, and $\rho = \mathrm{id}_R$, then we have an equality of R–ideals

$$\mathrm{Im}(\mathrm{ev}_P\circ(\tilde h\otimes\mathrm{id})) = R\cdot\det_R(h)\,.$$

2. Stark's Main Conjecture in the abelian case

2.1. Notations

In this section, we develop a formulation of Stark's Main Conjecture (see [**34**], [**35**], [**36**], [**37**]), in the particular case of Artin L–functions associated to 1–dimensional complex representations (i.e. abelian L–functions.) Although closer in spirit to the formulation of Conjecture 5.1 in §5 of [**42**], our formulation will be made in terms of determinants of projective modules. The determinant–approach will make the transition to the Equivariant Tamagawa Number Conjecture–type integral refinements of Stark's Main Conjecture seem much more intuitive and natural.

Throughout this section, K/k will denote an abelian extension of global fields of Galois group $G := \mathrm{Gal}(K/k)$. We will fix a finite, non-empty set S of primes in k, containing the set S_{ram} of primes which ramify in K/k as well as the set S_∞ of infinite primes. (If k is a function field, then S_∞ is empty.) We let S_K denote the set of primes in K which sit above primes in S. The group of S_K–units in K is denoted by U_S and its torsion subgroup is denoted by μ_K. As usual, we let $w_K := |\mu_K|$ denote the cardinality of μ_K. The group of S_K–divisors in K is denoted by Y_S. This consists of formal finite integral linear combinations of primes in S_K.

The subgroup of Y_S consisting of divisors of (formal) degree 0 is denoted by X_S. We have an exact sequence of $\mathbb{Z}[G]$–modules

$$0 \longrightarrow X_S \longrightarrow Y_S \xrightarrow{\text{aug}} \mathbb{Z} \longrightarrow 0\,, \tag{9}$$

where (the augmentation map) "aug" is the $\mathbb{Z}$–linear map sending every $w \in S_K$ to 1 and G acts on $\mathbb{Z}$ trivially.

For every prime v in k we denote by G_v the decomposition group of a prime v (relative to the abelian extension K/k). If v is unramified in K/k, then σ_v denotes the Frobenius automorphism associated to v in G. If we pick a prime $w(v) \in S_K$ sitting above v, for every $v \in S$, then we have an isomorphism of $\mathbb{Z}[G]$–modules

$$\bigoplus_{v\in S} \mathbb{Z}[G/G_v] \xrightarrow{\sim} Y_S\,, \qquad (x_v)_{v\in S} \longrightarrow \sum_{v\in S} x_v \cdot w(v), \tag{10}$$

where $(x_v)_{v\in S}$ is an element in the direct sum on the left.

2.2. The L–functions

For every finite prime v of k, we denote $\mathbf{N}v$ the cardinality of its residue field. If v is unramified in K/k, we let σ_v denote its Frobenius automorphism in G. As usual, the S–imprimitive L–function $L_S(\chi, s)$ associated to a character $\chi \in \widehat{G}(\mathbb{C})$ is defined as the meromorphic extension to the whole complex plane of the holomorphic function given by the infinite product

$$\prod_{v\notin S} (1 - \chi(\sigma_v) \cdot \mathbf{N}v^{-s})^{-1}\,, \qquad \Re(s) > 1\,.$$

This product is known to be uniformly and absolutely convergent on compact subsets of the half–plane $\Re(s) > 1$. If $\chi \neq \mathbf{1}_G$, then the L–function $L_S(\chi, s)$ obtained this way is holomorphic everywhere on the complex plane. If $\chi = \mathbf{1}_G$, then $\zeta_{k,S}(s) := L_S(\mathbf{1}_G, s)$ is holomorphic at all $s \neq 1$ and it has a pole of order 1 at $s = 1$. For every χ as above, we let $r_{\chi,S}$ and $L_S^*(\chi, 0)$ denote the order of vanishing and leading term in the Taylor expansion of $L_S(\chi, s)$ at $s = 0$, respectively. As a consequence of the functional equation of $L_S(\chi, s)$, one can easily show that

$$\begin{aligned} r_{S,\chi} := \mathrm{ord}_{s=0} L_S(\chi, s) &= \dim_{\mathbb{C}}(\mathbb{C}X_S)^{\chi} \\ &= \begin{cases} \mathrm{card}\,\{v \in S \mid \chi(G_v) = \{1\}\}, & \text{if } \chi \neq \mathbf{1}_G; \\ \mathrm{card}\, S - 1, & \text{if } \chi = \mathbf{1}_G. \end{cases} \end{aligned} \tag{11}$$

In the present context, it is more convenient to work with the so-called G–equivariant L–function

$$\Theta_S : \mathbb{C} \longrightarrow \mathbb{C}[G], \qquad \Theta_S(s) := \sum_{\chi\in\widehat{G}(\mathbb{C})} L_S(\chi, s) \cdot e_{\chi^{-1}}.$$

An equivalent and, at times, more convenient way of writing $\Theta_S(s)$ is

$$\Theta_S(s) = \sum_{\sigma\in G} \zeta_S(\sigma, s) \cdot \sigma^{-1}\,, \tag{12}$$

where $\zeta_S(\sigma, s)$ is the S–incomplete partial zeta function associated to σ, for all $\sigma \in G$. The leading term in the Taylor expansion of $\Theta_S(s)$ at $s = 0$ is the following:

$$\Theta_S^*(0) := \sum_{\chi\in\widehat{G}(\mathbb{C})} L_S^*(\chi, 0) \cdot e_{\chi^{-1}} \in \mathbb{C}[G]^{\times}\,.$$

Of course, we could talk about the order of vanishing r_S of $\Theta_S(s)$ at $s=0$. However, r_S is not a simple integer any longer, but an integral–valued function

$$r_S : \operatorname{Spec}\mathbb{C}[G] \longrightarrow \mathbb{Z}, \qquad r_S(\mathfrak{p}_\chi) = r_{S,\chi}, \quad \forall\, \chi \in \widehat{G}(\mathbb{C})\,.$$

Remark 2.2.1. *Equalities (11) show that r_S is in fact the rank function $\mathrm{rk}_{\mathbb{C}X_S}$ associated to the finitely generated projective $\mathbb{C}[G]$–module $\mathbb{C}X_S$. In the next section we will see that this coincides with the rank function $\mathrm{rk}_{\mathbb{C}U_S}$ of $\mathbb{C}U_S$.*

2.3. Stark's Main Conjecture (the abelian case)

In what follows, if w is a (finite or infinite) prime in K, we denote by $|\cdot|_w$ the canonically normalized absolute value associated to w. A classical theorem of Dirichlet gives a canonical $\mathbb{C}[G]$–module isomorphism

$$\lambda_S : \mathbb{C}U_S \xrightarrow{\sim} \mathbb{C}X_S, \qquad \tilde{u} \longrightarrow -\sum_{w \in S_K} \log|u|_w \cdot w\,,$$

where u is an arbitrary element in U_S and $\tilde{u}$ is its image via the canonical morphism $U_S \longrightarrow \mathbb{C}U_S$. Note that although a priori the image of λ_S sits inside $\mathbb{C}Y_S$, the classical product formula shows that this image is in fact contained in $\mathbb{C}X_S$. However, since the two complex representations of G involved in the isomorphism above are in fact obtained from the rational representations $\mathbb{Q}X_S$ and $\mathbb{Q}U_S$ by extension of scalars, the original rational representations have to be isomorphic as well. Let us fix a $\mathbb{Q}[G]$–module isomorphism

$$f : \mathbb{Q}X_S \xrightarrow{\sim} \mathbb{Q}U_S\,.$$

This leads to a $\mathbb{C}[G]$–linear automorphism of the projective $\mathbb{C}[G]$–module $\mathbb{C}X_S$

$$\lambda_S \circ (f \otimes \mathbf{1}_{\mathbb{C}}) : \mathbb{C}X_S \xrightarrow{\sim} \mathbb{C}X_S\,.$$

Definition 2.3.1. *The element $\det_{\mathbb{C}[G]}\lambda_S \circ (f \otimes \mathbf{1}_C) \in \mathbb{C}[G]^\times$ is called the Stark regulator associated to the set of data $(K/k, S, f)$.*

With notations as in Example 1.3.3, the following statement is easily seen to be equivalent to Stark's Main Conjecture (as stated in [**42**], Chpt. I, §5), restricted to the case of abelian extensions of global fields.

Conjecture 2.3.2. *$A_{\mathbb{Q}}(K/k, S, f)$. For every f as above, the following equivalent statements hold.*

(1) $\frac{\det_{\mathbb{C}[G]}(\lambda_S \circ (f \otimes \mathbf{1}_C))}{\Theta_S^*(0)} \in \mathbb{Q}[G]^\times\,.$

(2) *If $A_S(\chi, f) := \frac{\det_{\mathbb{C}}(\lambda_S \circ (f \otimes \mathbf{1}_C))^\chi}{L_S^*(\chi^{-1}, 0)}$, then $A_S(\chi, f)^\tau = A_S(\chi^\tau, f)$, for all characters $\chi \in \widehat{G}(\mathbb{C})$ and all $\tau \in \mathrm{Aut}(\mathbb{C})$.*

(3) *There is an equality of rank 1, free $\mathbb{Q}[G]$–submodules of $\mathbb{C}[G]$*

$$\Theta_S^*(0)\mathbb{Q}[G] = \det_{\mathbb{C}[G]}(\lambda_S \circ (f \otimes \mathbf{1}_{\mathbb{C}}))\mathbb{Q}[G]\,.$$

Remark 2.3.3. *It is not difficult to show that the validity of the conjecture above is independent of S, as long as S contains all the K/k–ramified and all the infinite primes of k. (See [**42**], Proposition 7.3, Chpt. I, §7 for a quick proof of this fact.)*

Our next goal is to arrive at a formulation of the conjecture above which is independent of the map f. The first step consists of applying equality (8) above for $P := \mathbb{Q}X_S$, $R := \mathbb{Q}[G]$, $R' := \mathbb{C}[G]$, the natural inclusion $\rho : \mathbb{Q}[G] \longrightarrow \mathbb{C}[G]$ and $h := \lambda_S \circ (f \otimes \mathbf{1}_C)$. We obtain the following equality of $\mathbb{Q}[G]$–submodules of $\mathbb{C}[G]$.

$$\det_{\mathbb{C}[G]}(\lambda_S \circ (f \otimes \mathbf{1}_{\mathbb{C}}))\mathbb{Q}[G] = \operatorname{Im}(\operatorname{ev}_{\mathbb{C}X_S} \circ (\widetilde{\lambda_S \circ (f \otimes \mathbf{1}_{\mathbb{C}})} \otimes \operatorname{id}) \circ (\iota \otimes \iota^*)).$$

However, since the maps $\widetilde{\lambda}_S$, $\widetilde{f}$, ι and ι^* are injective, the equality above can be rewritten as

$$\det_{\mathbb{C}[G]}(\lambda_S \circ (f \otimes \mathbf{1}_{\mathbb{C}}))\mathbb{Q}[G] = \operatorname{ev}_{\mathbb{C}X_S}(\widetilde{\lambda_S}(\tilde{f}(\det_{\mathbb{Q}[G]}(\mathbb{Q}X_S))) \otimes_{\mathbb{Q}[G]} \det_{\mathbb{Q}[G]}(\mathbb{Q}X_S)^*).$$

Now, since f is an isomorphism, we have $\tilde{f}(\det_{\mathbb{Q}[G]}(\mathbb{Q}X_S)) = \det_{\mathbb{Q}[G]}(\mathbb{Q}U_S)$. Therefore, we arrive at the following equality of $\mathbb{Q}[G]$–modules:

$$\det_{\mathbb{C}[G]}(\lambda_S \circ (f \otimes \mathbf{1}_{\mathbb{C}}))\mathbb{Q}[G] = \operatorname{ev}_{\mathbb{C}X_S}(\widetilde{\lambda_S}(\det_{\mathbb{Q}[G]}(\mathbb{Q}U_S)) \otimes_{\mathbb{Q}[G]} \det_{\mathbb{Q}[G]}(\mathbb{Q}X_S)^*).$$

Consequently, the abelian Main Conjecture of Stark can be restated independently of f as follows.

Conjecture 2.3.4. $A_{\mathbb{Q}}(K/k, S)$. *For K/k and S as above, the following equivalent statements hold.*

(1) *We have an equality of $\mathbb{Q}[G]$–submodules of $\mathbb{C}[G]$:*

$$\Theta_S^*(0)\mathbb{Q}[G] = \operatorname{ev}_{\mathbb{C}X_S}(\widetilde{\lambda_S}(\det_{\mathbb{Q}[G]}(\mathbb{Q}U_S)) \otimes_{\mathbb{Q}[G]} \det_{\mathbb{Q}[G]}(\mathbb{Q}X_S)^*).$$

(2) *If $\mathfrak{X}_S^*$ is a generator for the free, rank one $\mathbb{Q}[G]$–module $\det_{\mathbb{Q}[G]}(\mathbb{Q}X_S)^*$, then there exists a unique $\mathfrak{U}_S \in \det_{\mathbb{Q}[G]}(\mathbb{Q}U_S)$ (necessarily a generator of $\det_{\mathbb{Q}[G]}(\mathbb{Q}U_S)$), such that*

$$\Theta_S^*(0) = \operatorname{ev}_{\mathbb{C}X_S}(\widetilde{\lambda_S}(\mathfrak{U}_S) \otimes \mathfrak{X}_S^*).$$

Remark 2.3.5. *Since λ_S is an isomorphism, we have*

$$\widetilde{\lambda_S}(\det_{\mathbb{Q}[G]}(\mathbb{Q}U_S)) = \det_{\mathbb{Q}[G]}(\lambda_S(\mathbb{Q}U_S)).$$

So, one can replace the equality in statement (1) *of the conjecture above with*

$$\Theta_S^*(0)\mathbb{Q}[G] = \operatorname{ev}_{\mathbb{C}X_S}(\det_{\mathbb{Q}[G]}(\lambda_S(\mathbb{Q}U_S)) \otimes_{\mathbb{Q}[G]} \det_{\mathbb{Q}[G]}(\mathbb{Q}X_S)^*).$$

The right-hand side of the equality above is an invertible $\mathbb{Q}[G]$–submodule of $\mathbb{C}[G]$ which measures the "relative position" of the full rank projective $\mathbb{Q}[G]$–submodules $\lambda_S(\mathbb{Q}U_S)$ and $\mathbb{Q}X_S$ of $\mathbb{C}X_S$. This "relative position" is measured by the $\mathbb{Q}[G]$–submodule generated by the elements of $\mathbb{C}[G]$ obtained by evaluating every element in the determinant of the first module against every element in the dual of the determinant of the second. The conjecture above simply states that the leading term $\Theta_S^*(0)$ in the Taylor expansion at $s = 0$ of the equivariant L–function Θ_S captures this "relative position".

$$\begin{array}{ccc}
\det_{\mathbb{C}[G]}(\mathbb{C}X_S) \otimes_{\mathbb{C}[G]} \det_{\mathbb{C}[G]}(\mathbb{C}X_S)^* & \xrightarrow[\sim]{\operatorname{ev}_{\mathbb{C}X_S}} & \mathbb{C}[G] \\
\uparrow & & \uparrow \\
\det_{\mathbb{Q}[G]}(\lambda_S(\mathbb{Q}U_S)) \otimes_{\mathbb{Q}[G]} \det_{\mathbb{Q}[G]}(\mathbb{Q}X_S)^* & \dashrightarrow & \mathbb{Q}[G]\Theta_S^*(0)
\end{array}$$

Example 2.3.6. Let us consider the case $K = k$. Assume that $|S| = r + 1$, for some $r \in \mathbb{Z}_{\geq 1}$. Let $S := \{v_0, v_1, \dots, v_r\}$ and denote by $\{u_1, u_2, \dots, u_r\}$ a set of fundamental S–units. This means that the images $\widetilde{u_i}$ of u_i via the natural morphism $j : U_S \longrightarrow \mathbb{Q}U_S$ form a $\mathbb{Z}$–basis for

$$\widetilde{U_S} := j(U_S) \simeq U_S/\mu_k,$$

where μ_k denotes the group of roots of unity in k. In this case, we have $\Theta_S(s) = \zeta_{k,S}(s)$, where $\zeta_{k,S}(s)$ is the S–incomplete Dedekind zeta-function associated to the field k. The class number formula gives us the equalities

$$\Theta_S^*(0) = \zeta_{k,S}^*(0) = -\frac{R_{k,S} \cdot h_{k,S}}{w_k}\,.$$

Here, $h_{k,S}$ is the cardinality of the ideal–class group of the ring $O_{k,S}$ of S–integers in k, w_k is the cardinality of μ_k and $R_{k,S}$ is the Dirichlet S–regulator, given by

$$R_{k,S} := |\det(\log |u_i|_{v_j})| = \mathrm{vol}(\mathbb{R}X_S/\lambda_S(\widetilde{U_S}))\,,$$

where $i, j \in \{1, 2, \dots, r\}$ and "vol" denotes the (Lebesgue) volume of the quotient of $\mathbb{R}X_S \simeq \mathbb{R}^{|S|-1}$ by its full rank generating sublattice $\lambda_S(\widetilde{U_S})$. Now, since $\mathbb{Q}U_S$ and $\mathbb{Q}X_S$ are free $\mathbb{Q}$–modules of rank r with bases

$$\{\widetilde{u_1}, \dots, \widetilde{u_r}\}, \quad \{(v_1 - v_0), \dots, (v_r - v_0)\},$$

respectively, $\det_{\mathbb{Q}}(\mathbb{Q}U_S) = \wedge^r_{\mathbb{Q}}\mathbb{Q}U_S$ and $\det_{\mathbb{Q}}(\mathbb{Q}X_S) = \wedge^r_{\mathbb{Q}}\mathbb{Q}X_S$ are free $\mathbb{Q}$–modules of rank 1 with bases

$$\{\widetilde{u_1} \wedge \cdots \wedge \widetilde{u_r}\}, \quad \{(v_1 - v_0) \wedge \cdots \wedge (v_r - v_0)\},$$

respectively, where the exterior products are viewed over $\mathbb{Q}$. We let

$$\{(v_1 - v_0)^*, \dots, (v_r - v_0)^*\}$$

denote the basis of $(\mathbb{Q}X_S)^*$ which is dual to $\{(v_1 - v_0), \dots, (v_r - v_0)\}$. Remark 1.1.2 above establishes an explicit isomorphism of $\mathbb{Q}$–modules

$$\begin{array}{ccc} \wedge^r_{\mathbb{Q}}(\mathbb{Q}X_S)^* & \xrightarrow{\sim} & (\wedge^r_{\mathbb{Q}}\mathbb{Q}X_S)^*\,, \\ (v_1 - v_0)^* \wedge \cdots \wedge (v_r - v_0)^* & \longrightarrow & (v_r - v_0)^{*(1)} \circ \cdots \circ (v_1 - v_0)^{*(r)}\,. \end{array}$$

Consequently, we obtain the following equalities of $\mathbb{Q}$–submodules of $\mathbb{C}$.

$$\begin{aligned} \mathrm{ev}_{\mathbb{C}}(\det_{\mathbb{Q}}(\lambda_S(\mathbb{Q}U_S)) \otimes_{\mathbb{Q}} \det_{\mathbb{Q}}(\mathbb{Q}X_S)^*) &= \mathbb{Q} \cdot \det((v_i - v_0)^*(\lambda_S(u_j))) \\ &= \mathbb{Q} \cdot \det(-\log |u_j|_{v_i}) \\ &= \mathbb{Q} \cdot R_{k,S} \\ &= \mathbb{Q} \cdot \Theta_S^*(0)\,. \end{aligned}$$

This verifies Conjecture $A_{\mathbb{Q}}(K/k, S)$ in the case $K = k$, assuming that $|S| \geq 2$. The interested reader is invited to check that in this case, if we take

$$\mathfrak{X}_S^* := (v_r - v_0)^{*(1)} \circ \cdots \circ (v_1 - v_0)^{*(r)},$$

then we have

$$\mathfrak{U}_S := \pm\frac{h_{k,S}}{w_k} \cdot \widetilde{u_1} \wedge \cdots \wedge \widetilde{u_r},$$

where the notations are as in formulation (2) of Conjecture 2.3.4. We leave the case $K = k$ and $|S| = 1$ as an exercise.

2.4. Idempotents

Let us consider a splitting
$$1 = \sum_{e\in E} e\,,$$
of $1 \in \mathbb{Q}[G]$ into a finite sum of idempotents $e \in \mathbb{Q}[G]$. Equation (7) above shows that Conjecture $A_{\mathbb{Q}}(K/k,S)$ holds true if and only if
$$e\cdot\Theta_S^*(0)\mathbb{Q}[G] = \mathrm{ev}_{\mathbb{C}X_S}(\widetilde{\lambda_S}(\det_{e\mathbb{Q}[G]}(e\mathbb{Q}U_S))\otimes_{e\mathbb{Q}[G]}\det_{e\mathbb{Q}[G]}(e\mathbb{Q}X_S)^*),\quad \forall\, e\in E.$$
We will denote the equality above by $eA_{\mathbb{Q}}(K/k,S)$ and refer to it as the "e–component of Stark's Main conjecture $A_{\mathbb{Q}}(K/k,S)$", for all idempotents $e \in \mathbb{Q}[G]$.

A useful decomposition of 1 as a sum of idempotents in $\mathbb{Q}[G]$ is the following. For all $r \in \mathbb{Z}_{\geq 0}$, let
$$e_{S,r} := \sum_{\chi\in\widehat{G}_{S,r}} e_\chi\,,$$
where $\widehat{G}_{S,r} := \{\chi \in \widehat{G} \mid r_{S,\chi} = r\}$. Then $e_{S,r}$ is an idempotent in $\mathbb{C}[G]$. However, equalities (11) show that $r_{S,\chi} = r_{S,\chi'}$ whenever $\chi \sim_{\mathbb{Q}} \chi'$. Consequently, $e_{S,r} \in \mathbb{Q}[G]$, for all $r \in \mathbb{Z}_{\geq 0}$. This means that every set S as above leads to a canonical decomposition
$$1 = \sum_{r\in\mathbb{Z}_{\geq 0}} e_{S,r}$$
of 1 as a sum of orthogonal idempotents in $\mathbb{Q}[G]$. The advantage of using this decomposition over any other stems from the fact that for all $r \in \mathbb{Z}_{\geq 0}$ the projective $e_{S,r}\cdot\mathbb{Q}[G]$–modules $e_{S,r}\mathbb{Q}U_S$ and $e_{S,r}\mathbb{Q}X_S$ are free of rank r (see Remark 2.2.1.) Consequently, we have equalities
$$\det_{e_{S,r}\mathbb{Q}[G]}(e_{S,r}\mathbb{Q}U_S) = e_{S,r}\wedge^r_{\mathbb{Q}[G]}\mathbb{Q}U_S,\qquad \det_{e_{S,r}\mathbb{Q}[G]}(e_{S,r}\mathbb{Q}X_S) = e_{S,r}\wedge^r_{\mathbb{Q}[G]}\mathbb{Q}X_S\,.$$
At the analytic (L–function) level, we have the following equalities, for all $r \in \mathbb{Z}_{\geq 0}$.
$$e_{S,r}\Theta_S^*(0) = e_{S,r}\cdot\frac{1}{r!}\lim_{s\to 0}\frac{1}{s^r}\Theta_S(s).$$
Now, let $\overline{u} := \{u_1,\dots,u_r\}$ and $\overline{x} := \{x_1,\dots,x_r\}$ be bases of the free $e_{S,r}\cdot\mathbb{Q}[G]$–modules $e_{S,r}\mathbb{Q}U_S$ and $e_{S,r}\mathbb{Q}X_S$, respectively and let $\overline{x}^* := \{x_1^*,\dots,x_r^*\}$ be the basis of $e_{S,r}(\mathbb{Q}X_S)^* = (e_{S,r}\mathbb{Q}X_S)^*$ (the first dual is over $\mathbb{Q}[G]$ and the second over $e_{S,r}\mathbb{Q}[G]$) which is dual to $\overline{x}$. Then, it is obvious that for every $r \in \mathbb{Z}_{\geq 0}$, the $e_{S,r}$–component of Stark's Main Conjecture is in fact the following.

Conjecture 2.4.1. $e_{S,r}A_{\mathbb{Q}}(K/k,S)$. *The following equivalent statements hold.*

(1)
$$\frac{e_{S,r}\cdot\frac{1}{r!}\lim_{s\to 0}\frac{1}{s^r}\Theta_S(s)}{\det(x_i^*(\lambda_S(u_j)))} \in (e_{S,r}\mathbb{Q}[G])^\times.$$

(2) *If* $\mathfrak{X}^*_{S,r} := x_1^*\wedge\cdots\wedge x_r^*$, *there exists a unique* $\alpha \in (e_{S,r}\mathbb{Q}[G])^\times$, *such that*
$$e_{S,r}\cdot\frac{1}{r!}\lim_{s\to 0}\frac{1}{s^r}\Theta_S(s) = \mathrm{ev}_{\mathbb{C}X_S}(\widetilde{\lambda_S}(\mathfrak{U}_{S,r})\otimes\mathfrak{X}^*_{S,r}),$$
where $\mathfrak{U}_{S,r} := \alpha\cdot u_1\wedge\cdots\wedge u_r \in e_{S,r}\wedge^r_{\mathbb{Q}[G]}\mathbb{Q}U_S$.

Above, we have identified x_i^* with an element in $(e_{S,r}\mathbb{C}X_S)^*$. This is the scalar extension $(x_i^* \otimes_{\mathbb{Q}} \mathbf{1}_{\mathbb{C}})$ of x_i^* to $e_{S,r}\mathbb{C}X_S$. Also, we have made the identification

$$\wedge^r_{(e_{S,r}\mathbb{C}G)}(e_{S,r}\mathbb{C}X_S)^* = (\wedge^r_{(e_{S,r}\mathbb{C}G)} e_{S,r}\mathbb{C}X_S)^*,$$

as in Remark 1.1.2. Of course, if $r = 0$, the determinant $\det(x_i^*(\lambda_S(u_j)))$ is equal to 1, by definition. Also, note that the uniqueness in the statement above is an immediate consequence of the fact that the map $\mathrm{ev}_{\mathbb{C}X_S}$ is an isomorphism.

Example 2.4.2. In this example, we examine the $e_{S,0}$–component $e_{S,0}A_{\mathbb{Q}}(K/k, S)$ of Conjecture $A_{\mathbb{Q}}(K/k, S)$. According to the above considerations and equality (12), we have the equivalences

$$e_{S,0}A_{\mathbb{Q}}(K/k, S) \quad \Leftrightarrow \quad \Theta_S(0) \in \mathbb{Q}[G] \quad \Leftrightarrow \quad \zeta_S(\sigma, 0) \in \mathbb{Q}\,, \text{ for all } \sigma \in G\,.$$

In the number field case, the third statement above is a deep, well-known theorem of Klingen and Siegel (see [**33**].) In the function field case, a much sharper form of this statement is true, as we will see in the next example. So, conjecture $e_{S,0}A_{\mathbb{Q}}(K/k, S)$ is true in general.

In fact, both the Klingen-Siegel theorem mentioned above and its function field counterpart state that $\Theta_S(1 - n) \in \mathbb{Q}[G]$, for all $n \in \mathbb{Z}_{\geq 1}$ (see Remark 3.2.4 below for a refinement of these results.)

Example 2.4.3. In this example, we examine conjecture $A_{\mathbb{Q}}(K/k, S)$ in the case of abelian extensions K/k of global fields of characteristic p (i.e. function fields), where p is an arbitrary prime number. Let us assume that the exact field of constants of k is $\mathbb{F}_q$, where q is a power of p. For a prime v in either k or K, we let d_v be the degree of v over $\mathbb{F}_q$. This means that the residue field $\kappa(v)$ of v is $\mathbb{F}_{q^{d_v}}$. Consequently, we have $\mathbf{N}v = q^{d_v}$, for all v. Therefore, the Euler product expansion of $L_S(\chi, s)$ has the form

$$L_S(\chi, s) = \prod_{v \notin S} (1 - \chi(\sigma_v) \cdot (q^{-s})^{d_v})^{-1}, \qquad \Re(s) > 1, \quad \chi \in \widehat{G}.$$

This leads naturally to considering the power series of variable u

$$P_S(u) := \prod_{v \notin S} (1 - \sigma_v^{-1} \cdot u^{d_v})^{-1} \in \mathbb{Q}[G][[u]]\,.$$

Note that the infinite product converges in the u–adic topology of $\mathbb{Q}[G][[u]]$ because for every $n \in \mathbb{N}$ there are only finitely many primes v such that $d_v \leq n$. In fact, as shown in [**42**], Chpt. IV, it turns out that $P_S(u)$ is a rational function in the variable u with coefficients in $\mathbb{Q}[G]$,

$$P_S(u) \in \mathbb{Q}[G](u)\,,$$

with a single pole at $u = q^{-1}$ and that we have an equality

$$\Theta_S(s) = P_S(q^{-s}), \qquad \text{for all } s \in \mathbb{C} \setminus \{1\}\,.$$

The last equality implies that we have

(13)

$$e_{S,r}\Theta_S^*(0) = e_{S,r} \cdot \frac{1}{r!}\frac{\mathrm{d}^r P}{\mathrm{d}u^r}(1) \cdot (-\log q)^r \in (e_{S,r}\mathbb{Q}[G])^\times \cdot (\log q)^r\,, \quad \text{for all } r \in \mathbb{Z}_{\geq 0}\,.$$

For $r = 0$, this simply says that $\Theta_S(0) \in \mathbb{Q}[G]$, which settles the $e_{S,0}$–component of conjecture $A_{\mathbb{Q}}(K/k, S)$, as promised at the end of the last example. Next, let us consider an arbitrary $r \in \mathbb{Z}_{\geq 1}$ and analyze the determinant in Conjecture 2.4.1(1) for two bases $\overline{x}$ and $\overline{u}$. For all $u \in U_S$, we have

$$\lambda_S(\tilde{u}) = -\sum_{w \in S_K} \log |u|_w \cdot w = -(\log q) \cdot (\sum_{w \in S_K} d_w \cdot \mathrm{ord}_w(u) \cdot w) \in (\log q) \cdot X_S \,,$$

where $\mathrm{ord}_w(u) \in \mathbb{Z}$ is the evaluation at u of the canonically normalized valuation associated to w. However, this means that $\lambda_S(\mathbb{Q}U_S) \subseteq (\log q) \cdot \mathbb{Q}X_S$. Therefore, we have

$$x_i^*(\lambda_S(u_j)) \in (\log q) \cdot e_{S,r}\mathbb{Q}[G], \text{ for all } i, j = 1, \dots, r.$$

Consequently, we have $\det(x_i^*(\lambda_S(u_j)) \in (\log q)^r \cdot (e_{S,r}\mathbb{Q}[G])^\times$. If we combine this statement with (13) above, we obtain

$$\frac{e_{S,r}\Theta_S^*(0)}{\det(x_i^*(\lambda_S(u_j))} \in (e_{S,r}\mathbb{Q}[G])^\times \,.$$

This verifies conjecture $e_{S,r}A_{\mathbb{Q}}(K/k, S)$, for all $r \in \mathbb{Z}_{\geq 0}$. Therefore, conjecture $A_{\mathbb{Q}}(K/k, S)$ holds true for characteristic p global fields.

3. Integral refinements of Stark's Main Conjecture

The main goal of this section is to try to put an integral structure on conjecture $A_{\mathbb{Q}}(K/k, S)$. The meaning of the phrase "integral structure" will become clear as the section progresses. For the moment, let us just note that in the case $K = k$ (see Example 2.3.6), the left-hand side of the equality in conjecture $A_{\mathbb{Q}}(K/k, S)$ is

$$\mathbb{Q}\zeta_{k,S}^*(0) = \mathbb{Q}R_{k,S} \,,$$

which only determines the regulator $R_{k,S}$ up to a rational factor, missing completely the important integral invariants $h_{k,S}$ and w_k captured by the class number formula for the global field k. So, roughly speaking, we would like to have a conjectural equality which captures the invertible $\mathbb{Z}[G]$–submodule $\mathbb{Z}[G]\Theta_S^*(0)$ of $\mathbb{C}[G]$, rather than the $\mathbb{Q}[G]$–submodule $\mathbb{Q}[G]\Theta_S^*(0)$. The main difficulty in achieving this goal lies in that, at first sight, there is no canonical invertible $\mathbb{Z}[G]$–submodule $\mathcal{M}_S$,

$$\mathcal{M}_S \subseteq \widetilde{\lambda_S}(\det_{\mathbb{Q}[G]}(\mathbb{Q}U_S)) \otimes_{\mathbb{Q}[G]} \det_{\mathbb{Q}[G]}(\mathbb{Q}X_S)^* \,.$$

If we had a canonical $\mathcal{M}_S$, then a natural conjectural equality would simply state

$$\mathrm{ev}_{\mathbb{C}X_S}(\mathcal{M}_S) = \mathbb{Z}[G]\Theta_S^*(0) \,.$$

The difficulty in finding such a module $\mathcal{M}_S$ stems from the non-triviality of the $\mathbb{Z}[G]$–module structure of U_S and X_S. For example, if these $\mathbb{Z}[G]$–modules were *perfect* (i.e. of finite projective dimension, i.e. G–cohomologically trivial and therefore of projective dimension at most 1), the construction of $\mathcal{M}_S$ would be more or less immediate, via an extension of the "determinant" functor from the category of projective modules to that of modules of finite projective dimension, due to Grothendieck and Mumford-Knudsen (see [**24**].) Although X_S and U_S are rarely perfect (see Example 3.1.3 below), it turns out that, under mild hypotheses on S, the difference between their $\mathbb{Z}[G]$–module structures is controlled by a length 2, *perfect complex* of $\mathbb{Z}[G]$–modules (i.e. a complex which is quasi-isomorphic to a complex of perfect $\mathbb{Z}[G]$–modules.)

The main idea behind the most general integral refinement of $A_{\mathbb{Q}}(K/k, S)$ to date is to set the module $\mathcal{M}_S$ above equal to a canonical isomorphic image of the

Grothendieck-Mumford-Knudsen determinant of this perfect complex. Although it is not our goal to focus on this general integral refinement (deferring that for Burns's contribution to this volume), we will describe it very briefly in the next section. This will permit the reader to see how the integral refinements which will be covered in detail in our lectures fit in the big picture.

3.1. The Equivariant Tamagawa Number Conjecture – a brief introduction

For simplicity, in this section we will work under the hypothesis that S is sufficiently large, so that the ideal class–group of the ring of S–integers $O_{K,S}$ is trivial (i.e. $h_{K,S} = 1$, in the notations of Example 2.3.6.) Note that, since the validity of $A_{\mathbb{Q}}(K/k, S)$ does not depend on S (see Remark 2.3.3), this assumption is technically inconsequential in the context of the "rational" conjecture $A_{\mathbb{Q}}(K/k, S)$. Also, note that Tchebotarev's density theorem allows us to enlarge S by adjoining a finite set of primes which split completely in K/k in order achieve $h_{K,S} = 1$, which turn out to be relevant in the context of the Rubin-Stark integral refinement discussed later in this paper.

Theorem 3.1.1 (Tate, [**41**] and [**42**]). *Under the above hypotheses, there exists a canonical class $c_2 \in \mathrm{Ext}^2_{\mathbb{Z}[G]}(X_S, U_S)$ which has a representative*

$$0 \longrightarrow U_S \longrightarrow P_{-1} \longrightarrow P_0 \longrightarrow X_S \longrightarrow 0\,,$$

where P_-1 and P_0 are finitely generated $\mathbb{Z}[G]$–modules of projective dimension ≤ 1 (i.e. G–cohomologically trivial.)

The element $c_2 := c_2(K/k, S)$ is called the Tate class and it is constructed via a clever gluing process out of local and global fundamental classes (see Chpt. II, §5 of [**42**] and also [**10**].) The four term exact sequence in the theorem is called a Tate sequence. An immediate consequence of the above theorem is the following.

Corollary 3.1.2. *For all $i \in \mathbb{Z}$, we have group isomorphisms*

$$\widehat{\mathrm{H}}^i(G, X_S) \xrightarrow[\sim]{\cup\, c_2} \widehat{\mathrm{H}}^{i+2}(G, U_S)\,,$$

given by the cup product with $c_2 \in \mathrm{Ext}^2_{\mathbb{Z}[G]}(X_S, U_S) \simeq \widehat{\mathrm{H}}^2(G, \mathrm{Hom}_{\mathbb{Z}}(X_S, U_S))$.

Example 3.1.3. The Corollary above allows us to compute the cohomology of the "mysterious" $\mathbb{Z}[G]$–module U_S in terms of that of the more down-to-earth module X_S. This is illustrated in the following example. Let p be an odd prime number. Let $K := \mathbb{Q}(\zeta_p)$, $k := \mathbb{Q}$ and $S := \{p, v_\infty, v_1, \dots, v_r\}$, where v_∞ is the infinite prime of $\mathbb{Q}$ and $v_1, \dots, v_r$ are distinct primes in k, which split completely in K/k and are chosen so that $h_{K,S} = 1$. Exact sequence (9) and isomorphism (10) imply immediately that we have an isomorphism of $\mathbb{Z}[G]$–modules

$$X_S \simeq \mathbb{Z}[G]/(1-j) \bigoplus \mathbb{Z}[G]^r,$$

where $j \in G$ is the complex conjugation automorphism of K, obviously satisfying $G_{v_\infty} = \{1, j\}$. For all $i \in \mathbb{Z}$, we have the following group isomorphisms.

$$\widehat{\mathrm{H}}^{2i}(G_{v_\infty}, X_S) \simeq \widehat{\mathrm{H}}^0(G_{v_\infty}, X_S) \simeq \widehat{\mathrm{H}}^0(G_{v_\infty}, \mathbb{Z}[G/G_{v_\infty}]) \simeq (\mathbb{Z}/2\mathbb{Z})^{|G|/2} \neq 0,$$
$$\widehat{\mathrm{H}}^{2i+1}(G_{v_\infty}, X_S) \simeq \widehat{\mathrm{H}}^1(G_{v_\infty}, X_S) \simeq \widehat{\mathrm{H}}^1(G_{v_\infty}, \mathbb{Z}[G/G_{v_\infty}]) \simeq 0.$$

In the two sequences of isomorphisms above, the first isomorphism is a consequence of the cyclicity of G_{v_∞}, the second is due to the fact that $\mathbb{Z}[G]^r$ is G–cohomologically trivial, and the third is based on the definitions of $\widehat{\mathrm{H}}^0$ and $\widehat{\mathrm{H}}^1$, respectively, and the fact that G_{v_∞} acts trivially on $\mathbb{Z}[G/G_{v_\infty}]$. The isomorphisms above combined with the last Corollary, lead to

$$\widehat{\mathrm{H}}^{2i}(G_{v_\infty}, U_S) \simeq (\mathbb{Z}/2\mathbb{Z})^{|G|/2} \neq 0, \qquad \widehat{\mathrm{H}}^{2i+1}(G_{v_\infty}, U_S) = 0,$$

for all $i \in \mathbb{Z}$. In particular, these calculations show that the $\mathbb{Z}[G]$–modules X_S and U_S are not of finite projective dimension in this case. We leave it as an exercise for the reader to compute the cohomology groups $\widehat{\mathrm{H}}^i(G, U_S)$, for all $i \in \mathbb{Z}$ in this case. Also, the reader might be interested in repeating this exercise for the case $k = \mathbb{Q}$, $K = \mathbb{Q}(\zeta_p)^+$. He will discover that X_S and U_S are of finite projective dimension in that case.

Roughly speaking, the Equivariant Tamagawa Number Conjecture (ETNC) of Bloch-Kato (see [**2**]) states that the elusive module $\mathcal{M}_S$ in the introduction to this section is a specific isomorphic image of the Grothendieck-Mumford-Knudsen determinant $\mathbf{det}(P^\bullet)$ of the perfect complex of $\mathbb{Z}[G]$–modules

$$P^\bullet : \cdots \longrightarrow 0 \longrightarrow P_{-1} \longrightarrow P_0 \longrightarrow \cdots$$

(extended by 0 at levels other than -1 and 0) arising from the Tate sequence above (see [**24**] for the definition of $\mathbf{det}(P^\bullet)$.) More precisely, $\mathcal{M}_S$ is obtained as follows. One considers the following injective morphism of $\mathbb{Z}[G]$–modules:

$$\begin{aligned} \rho_S \ : \ \mathbf{det}(P^\bullet) \overset{\iota}{\hookrightarrow} \mathbb{Q}\mathbf{det}(P^\bullet) \ &\overset{e}{\simeq} \ \mathbf{det}(\mathbb{Q}P^\bullet) \\ &\overset{c}{\simeq} \ \det\nolimits_{\mathbb{Q}[G]}(\mathbb{Q}U_S) \otimes_{\mathbb{Q}[G]} \det\nolimits_{\mathbb{Q}[G]}(\mathbb{Q}X_S)^* \\ &\simeq \ \widetilde{\lambda_S}(\det\nolimits_{\mathbb{Q}[G]}(\mathbb{Q}U_S)) \otimes_{\mathbb{Q}[G]} \det\nolimits_{\mathbb{Q}[G]}(\mathbb{Q}X_S)^*. \end{aligned}$$

Above, the morphism ι sends x to $1_\mathbb{Q} \otimes x$, for all $x \in \mathbf{det}(P^\bullet)$. The isomorphism e is the analogue of (6) for the Grothendieck-Mumford-Knudson determinant functor at the level of perfect complexes. It turns out that **det** also commutes with base-change. The isomorphism c arises as follows: the perfect complex of $\mathbb{Q}[G]$–modules $\mathbb{Q}P^\bullet$ is in fact cohomologically perfect. This means that its cohomology groups $\mathrm{H}^{-1}(\mathbb{Q}P^\bullet) = \mathbb{Q}U_S$ and $\mathrm{H}^0(\mathbb{Q}P^\bullet) = \mathbb{Q}X_S$ are perfect $\mathbb{Q}[G]$–modules (i.e. of finite projective dimension over $\mathbb{Q}[G]$.) Obviously, these cohomology groups are in fact projective $\mathbb{Q}[G]$–modules. If R is a commutative ring and $\mathcal{C}^\bullet$ is a perfect complex of R–modules which is cohomologically perfect, there is a canonical isomorphism of R–modules

$$c_{\mathcal{C}^\bullet} : \mathbf{det}(\mathcal{C}^\bullet) \simeq \left(\bigotimes_{i \in \mathbb{Z}} \mathbf{det}(\mathrm{H}^{2i+1}(\mathcal{C}^\bullet)) \right) \otimes_R \left(\bigotimes_{i \in \mathbb{Z}} \mathbf{det}(\mathrm{H}^{2i}(\mathcal{C}^\bullet))^* \right).$$

The isomorphism c above is the canonical $c_{\mathbb{Q}P^\bullet}$. Also, note that since $\mathbb{Q}U_S$ and $\mathbb{Q}X_S$ are projective $\mathbb{Q}[G]$–modules, their Grothendieck-Mumford-Knudson determinants **det** coincide with their classical determinants $\det_{\mathbb{Q}[G]}$ defined earlier in these lectures. Finally, the last isomorphism above is $\widetilde{\lambda_S} \otimes \mathrm{id}$. Now, we define

$$\mathcal{M}_S := \mathrm{Im}(\rho_S) = \rho_S(\mathbf{det}(P^\bullet)).$$

This is an invertible $\mathbb{Z}[G]$–submodule of $\det_{\mathbb{C}[G]}(\mathbb{C}U_S) \otimes_{\mathbb{C}[G]} \det_{\mathbb{C}[G]}(\mathbb{C}X_S)^*$ which turns out to be independent on the Tate sequence representing the Tate canonical class c_2.

Conjecture 3.1.4 (ETNC, Bloch-Kato [**2**])**.** *Under the above hypotheses, we have the following equality of invertible $\mathbb{Z}[G]$–submodules of $\mathbb{C}[G]$.*

$$\mathrm{ev}_{\mathbb{C}X_S}(\mathcal{M}_S) = \mathbb{Z}[G]\Theta_S^*(0)\,.$$

As mentioned above, we will not insist upon commenting further on this conjecture in these lectures. For further details, we refer the reader to Burns's contribution to this volume. At this point, it suffices to say that all refinements (both integral and p–adic) of $A_{\mathbb{Q}}(K/k,S)$ which will be discussed in detail in these lectures are consequences of the ETNC, as shown in [**6**]. (See also §5.5, below.)

3.2. The integral conjecture of Rubin-Stark

In this section, we will discuss a conjecture stated by Rubin in [**32**], which generalizes an earlier integral conjecture of Stark [**37**]. It would be misleading to say that Rubin's conjecture is an integral refinement of $A_{\mathbb{Q}}(K/k,S)$. As it will become very clear by the end of this section, Rubin's conjecture is in fact an integral refinement of an idempotent piece $e_{S,r}A_{\mathbb{Q}}(K/k,S)$ of Stark's Main Conjecture $A_{\mathbb{Q}}(K/k,S)$, under some additional hypotheses on S and for an appropriately chosen integer r. (See the notations and definitions of section 2.4.)

3.2.1. **Additional hypotheses on S.** Let us fix an integer $r \geq 0$. Let us suppose that the set S contains r distinct primes $v_1,\dots,v_r$ which split completely in K/k and that $\mathrm{card}(S) \geq r+1$. Let us pick a prime $v_0 \in S \setminus \{v_1,\dots,v_r\}$. Equalities (11) show that $e_{S,r'} = 0$, for all $r' \leq r$. Rubin's conjecture aims for an integral refinement of $e_{S,r}A_{\mathbb{Q}}(K/k,S)$ (i.e. the first possibly non–trivial idempotent piece of $A_{\mathbb{Q}}(K/k,S)$), under the above additional hypotheses on the set S.

Our first goal is to give the rationale behind the new hypotheses on S as well as the choice of the particular idempotent $e_{S,r} \in \mathbb{Q}[G]$. For that purpose, we use the notations introduced in §2.4. Let $S_0 := S \setminus \{v_1,\dots,v_r\}$. Let $W = (w_0, w_1, \dots, w_r)$ be an ordered r–tuple of primes in K, such that w_i sits above v_i, for all i. Note that since v_i splits completely in K/k, we have $\mathrm{Ann}_{\mathbb{Z}[G]}(w_i - w_0) = \mathrm{Ann}_{\mathbb{Z}[G]}(w_i) = 0$, for all $i = 1,\dots,r$ (see isomorphism (10).) Consequently, X_S has a natural direct sum decomposition

$$X_S = X_{S_0} \oplus \left(\oplus_{i=1}^r \mathbb{Z}[G](w_i - w_0)\right),$$

where the second direct summand is a free $\mathbb{Z}[G]$–module of rank r and basis $\{w_1 - w_0, \dots, w_r - w_0\}$. The above decomposition is a direct consequence of the exact sequence of $\mathbb{Z}[G]$–modules

$$0 \longrightarrow X_{S_0} \longrightarrow X_S \xrightarrow{\ \pi\ } \oplus_{i=1}^r \mathbb{Z}[G]w_i \longrightarrow 0\,,$$

where π is the forgetful map sending $\sum_{w\in S_K} a_w \cdot w$ to $\sum_{w\in S_K\setminus S_{0,K}} a_w \cdot w$. The map π has an obvious section sending w_i to $(w_i - w_0)$, for all $i = 1,\dots,r$, which splits the exact sequence and leads to the above decomposition. Since $e_{S,r}\mathbb{Q}X_S$ is a free $e_{S,r}\mathbb{Q}[G]$–module of rank r, we have

$$e_{S,r}\mathbb{Q}X_S = \oplus_{i=1}^r e_{S,r}\mathbb{Q}[G](w_i - w_0)\,,$$

and a natural $e_{S,r}\mathbb{Q}[G]$–basis $\overline{x} = \{x_1, \dots, x_r\}$ for $e_{S,r}\mathbb{Q}X_S$, given by

$$x_i := e_{S,r}(w_i - w_0)\,, \qquad \text{for all } i = 1, \dots, r\,.$$

The free (full rank r) $e_{S,r}\mathbb{Z}[G]$–submodule of $e_{S,r}\mathbb{Q}X_S$ of basis $\overline{x}$ puts a natural integral structure on $e_{S,r}\mathbb{Q}X_S$. This leads to a natural integral structure on $\det_{e_{S,r}\mathbb{Q}[G]}(e_{S,r}\mathbb{Q}X_S)^*$ given by its free (full rank, i.e. rank one) submodule $e_{S,r}\mathbb{Z}[G](x_1^* \wedge \dots \wedge x_r^*)$, where the exterior powers are viewed over $e_{S,r}\mathbb{Z}[G]$. (See the notations and identifications made in §2.4.)

The conclusion is that the additional conditions imposed upon S and the choice of the particular idempotent $e_{S,r}$ give rise to a canonical generator

$$\mathfrak{X}_{S,r}^* := x_1^* \wedge \dots \wedge x_r^*$$

of the free, rank one $e_{S,r}\mathbb{Q}[G]$–module $\det_{e_{S,r}\mathbb{Q}[G]}(e_{S,r}\mathbb{Q}X_S)^*$. Also, it is important to note that evaluating against $\mathfrak{X}_{S,r}^*$ is particularly simple. Indeed, if we let

$$R_W : e_{S,r} \wedge_{\mathbb{C}[G]}^r \mathbb{C}U_S \xrightarrow{\sim} e_{S,r}\mathbb{C}[G], \qquad R_W(\mathfrak{U}) := \mathrm{ev}_{\mathbb{C}X_S}(\widetilde{\lambda_S}(\mathfrak{U}) \otimes \mathfrak{X}_{S,r}^*)\,,$$

then for $\mathfrak{U} = e_{S,r}\widetilde{u}_1 \wedge \dots \wedge \widetilde{u_r}$, with $u_1, \dots, u_r \in U_S$, we have

$$R_W(\mathfrak{U}) = e_{S,r} \cdot \det(-\sum_{\sigma \in G} \log |u_i^\sigma|_{w_j} \cdot \sigma^{-1})\,.$$

This is implied by the equality $x_j^*(\lambda_S(u_i)) = (-\sum_{\sigma \in G} \log |u_i^\sigma|_{w_j} \cdot \sigma^{-1}) \cdot e_{S,r}$, for all $i, j = 1, \dots, r$, which follows from the definition of λ_S and from the fact that v_j is completely split in K/k.

Definition 3.2.1. *The map R_W defined above is called the Rubin-Stark regulator.*

Remark 3.2.2. *It is easy to see that for two different choices of ordered r–tuples W and W' corresponding to possibly different orderings of the set $\{v_1, \dots, v_r\}$, we have $\mathcal{R}_W = \pm\sigma \cdot \mathcal{R}_{W'}$, where $\sigma \in G$. Since $\pm\sigma \in \mathbb{Z}[G]^\times$, the truth of the conjectures below is independent of these choices.*

Note that, under the current hypotheses, we have

$$e_{S,r}\Theta_S^*(0) = \frac{1}{r!}\lim_{s\to 0}\frac{1}{s^r}\Theta_S(s).$$

Following Rubin, we will denote this special value by $\Theta_S^{(r)}(0)$. Under the current hypotheses, conjecture $e_{S,r}A_{\mathbb{Q}}(K/k, S)$ can be restated as follows.

Conjecture 3.2.3. $A_{\mathbb{Q}}(K/k, S, r)$. *For all K/k, S, and r satisfying the above hypotheses, there exists a unique element $\varepsilon_S \in e_{S,r} \wedge_{\mathbb{Q}[G]}^r \mathbb{Q}U_S$, such that*

$$R_W(\varepsilon_S) = \Theta_S^{(r)}(0)\,.$$

Note that the uniqueness in the conjecture holds true and is an immediate consequence of the fact that $\mathrm{ev}_{\mathbb{C}X_S}$ is an isomorphism which renders the Rubin-Stark regulator R_W bijective. The equality in the conjecture above can be rewritten as

$$\mathrm{ev}_{\mathbb{C}X_S}(\widetilde{\lambda_S}(\mathbb{Z}[G]\varepsilon_S) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]\mathfrak{X}_S^*) = \mathbb{Z}[G]\Theta_S^{(r)}(0)\,.$$

This may lure the reader into believing that we have arrived this way at an integral refinement of $A_{\mathbb{Q}}(K/k, S, r)$. However, this is deceiving: although the $\mathbb{Z}[G]$–module $\mathbb{Z}[G]\mathfrak{X}_S^*$ is indeed canonical, that is not the case with $\mathbb{Z}[G]\varepsilon_S$, as we have no integral information on the conjectural element ε_S. The only information we have on ε_S is rational in nature. Namely, if ε_S indeed exists, then we have an equality

$$\mathbb{Q}[G]\varepsilon_S = e_{S,r} \cdot \wedge^r_{\mathbb{Q}[G]} \mathbb{Q}U_S.$$

As we will see in the next few subsections, Rubin's Conjecture does not quite identify the module $\mathbb{Z}[G]\varepsilon_S$ with a canonical $\mathbb{Z}[G]$–submodule of $e_{S,r} \cdot \wedge^r_{\mathbb{Q}[G]} \mathbb{Q}U_S$. Instead, it specifies a canonical $\mathbb{Z}[G]$–submodule of $e_{S,r} \cdot \wedge^r_{\mathbb{Q}[G]} \mathbb{Q}U_S$ in which a certain "multiple" of $\mathbb{Z}[G]\varepsilon_S$ sits with a finite index.

3.2.2. **The additional set of primes T.** To the set of data $(K/k, S, r)$ satisfying the hypotheses in the previous section, we add a new finite, non-empty set of primes T of k, which is disjoint from S and satisfies the following condition:

$$\{\zeta \in \mu_K \mid \zeta \equiv 1 \mod w, \quad \forall w \in T_K\} = \{1\}\,. \tag{14}$$

Note that, if $\mathrm{char}(k) = p > 0$, then $T \neq \emptyset$ implies the condition above. If $\mathrm{char}(k) = 0$, then the condition above is satisfied if, for example, T contains two primes of distinct residual characteristics or a prime of residual characteristic larger than $w_K = |\mu_K|$. We let

$$U_{S,T} := \{u \in U_S \mid u \equiv 1 \mod w, \quad \forall w \in T_K\}\,.$$

The new group of units is a $\mathbb{Z}[G]$–submodule of U_S with no $\mathbb{Z}$–torsion (see (14)). There is an obvious exact sequence of $\mathbb{Z}[G]$–modules (see [**32**]):

(15)

$$1 \longrightarrow U_{S,T} \longrightarrow U_S \longrightarrow \Delta_T := \bigoplus_{w \in T_K} \kappa(w)^\times \longrightarrow A_{S,T} \longrightarrow A_S \longrightarrow 1\,,$$

where A_S is the ideal class-group of $O_{K,S}$ and $A_{S,T}$ is a T–modified version of A_S, defined by

$$A_{S,T} := \frac{\{\text{fractional ideals of } O_{K,S}, \text{ coprime to } T\}}{\{fO_{K,S} \mid f \in K^\times, \quad f \equiv 1 \mod w, \quad \forall w \in T_K\}}\,.$$

The Artin reciprocity map associated to K establishes a group isomorphism between $A_{S,T}$ and the Galois group of the maximal abelian extension of K which is completely split at primes in S, is unramified away from T, and at most tamely ramified at all primes in T. We define the T–modified S–imprimitive ζ–function associated to K by

$$\zeta_{K,S,T}(s) := \prod_{w \in T_K} (1 - \mathrm{N}w^{1-s}) \cdot \zeta_{K,S}(s), \qquad \forall s \in \mathbb{C}.$$

This way, we obtain a holomorphic function. Note the simple pole at $s = 1$ of $\zeta_{K,S}$ is annihilated by the zero at $s = 1$ of the product

$$\delta_{K,T}(s) := \prod_{w \in T_K} (1 - \mathrm{N}w^{1-s})\,.$$

If we define the (S,T)–regulator of K by $R_{K,S,T} := \mathrm{vol}(\mathbb{R}X_S/\lambda_S(U_{S,T}))$ and let $h_{K,S} := |A_S|$ and $h_{K,S,T} := |A_{S,T}|$, then the exact sequence (15) gives us an equality

$$R_{K,S,T} = R_{K,S} \cdot [\widetilde{U_S} : U_{S,T}] = |\Delta_T| \cdot w_K^{-1} \cdot h_{K,S,T}^{-1} \cdot h_{K,S}.$$

If we combine this equality with the S–class number formula (see Example 2.3.6) and the observation that $\delta_{K,T}(0) = \pm|\Delta_T|$, we obtain the (S,T)–class number formula

$$\zeta^*_{K,S,T}(0) = \delta_{K,T}(0) \cdot \zeta^*_{K,S}(0) = \pm h_{K,S,T} \cdot R_{K,S,T}\,, \tag{16}$$

where "+" arises if and only if $\delta_{K,T}(0) < 0$. These constructions can be extended to the G–equivariant L–function. We define the T–modified version of Θ_S by setting

$$\Theta_{S,T}(s) := \prod_{v\in T}(1 - \mathrm{N}v^{1-s}\cdot\sigma_v^{-1})\cdot\Theta_S(s)\,.$$

This is a $\mathbb{C}[G]$–valued, holomorphic function. The simple pole at $s=1$ of $\Theta_S(s)$, supported at the trivial character idempotent $e_{\mathbf{1}_G}$, is annihilated by the zero at $s=1$ of the $\mathbb{C}[G]$–valued, holomorphic

$$\delta_{K/k,T}(s) := \prod_{v\in T}(1 - \mathrm{N}v^{1-s}\cdot\sigma_v^{-1})\,,$$

supported at the same idempotent.

Remark 3.2.4. *Above, we noticed that $\delta_{K,T}(0) = \pm|\Delta_T|$. A better way of writing this equality is $\mathrm{Fit}_{\mathbb{Z}}(\Delta_T) = \delta_{K,T}(0)\mathbb{Z}$. It turns out that the more general $\delta_{K/k,T}(0)$ has a similar interpretation. Indeed, we have isomorphisms of $\mathbb{Z}[G]$–modules*

$$\begin{aligned}\Delta_T \simeq \bigoplus_{v\in T}\kappa(w(v))^{\times}\otimes_{\mathbb{Z}[G_v]}\mathbb{Z}[G] \quad &\simeq \quad \bigoplus_{v\in T}\mathbb{Z}[G_v]/(1-\mathrm{N}v\cdot\sigma_v^{-1})\otimes_{\mathbb{Z}[G_v]}\mathbb{Z}[G]\\ &\simeq \quad \bigoplus_{v\in T}\mathbb{Z}[G]/(1-\mathrm{N}v\cdot\sigma_v^{-1})\,,\end{aligned}$$

*where $w(v)$ is a fixed prime in K sitting above v, for each $v\in T$ (see [**28**].) Consequently, we have*

$$\delta_{K/k,T}(0)\cdot\mathbb{Z}[G] = \prod_{v\in T}(1-\mathrm{N}v\cdot\sigma_v^{-1})\cdot\mathbb{Z}[G] = \mathrm{Fit}_{\mathbb{Z}[G]}(\Delta_T).$$

In particular, since $\mu_K\subseteq\Delta_T$ (see property (14)), we have $\delta_{K/k,T}(0)\in\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$. In fact, the following equality of $\mathbb{Z}[G]$–ideals holds

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K) = \langle\delta_{K/k,T}(0)\mid T\rangle_{\mathbb{Z}[G]}\,,$$

*where the ideal on the right is generated by $\delta_{K/k,T}(0)$, for all sets T satisfying the above properties, provided that S is fixed (see [**42**], Lemme 1.1, Chpt. IV.) Next, we observe that $\mu_K = (\mathbb{Q}/\mathbb{Z})'(1)^{G_K}$, where "(1)" denotes the Tate twist with respect to the first power of the cyclotomic character $c_k : G_k \longrightarrow \mathrm{Aut}(\mu_{\overline{k}})\subseteq\widehat{\mathbb{Z}}^{\times}$, G_L denotes the absolute Galois group of the global field L and*

$$(\mathbb{Q}/\mathbb{Z})' = \begin{cases}\mathbb{Q}/\mathbb{Z}, & \text{if } char(k)=0;\\ \mathbb{Q}/\mathbb{Z}[1/p], & \text{if } char(k)=p>0.\end{cases}$$

We remind the reader that if $n\in\mathbb{Z}$ and $\mathcal{M}$ is a torsion $\mathbb{Z}$–module endowed with a left G_k–action $m\longrightarrow {}^{\sigma}m$, for all $\sigma\in G_k$ and $m\in\mathcal{M}$, then the n–th Tate twist $\mathcal{M}(n)$ of $\mathcal{M}$ is, by definition, the module $\mathcal{M}$ with a new G_k–action given by

$$\sigma * m := c_k(\sigma)^n\cdot{}^{\sigma}m.$$

It turns out that if we assume in addition that the sets T contain at least two primes of distinct residual characteristic (if $\mathrm{char}(k)>0$ this assumption is unnecessary), then we have an equality

$$\mathrm{Ann}_{\mathbb{Z}[G]}\left((\mathbb{Q}/\mathbb{Z})'(n)^{G_K}\right) = \langle\delta_{K/k,T}(1-n)\mid T\rangle_{\mathbb{Z}[G]}\,,$$

for all $n \in \mathbb{Z}_{\geq 1}$ *(see Lemma 2.3 in* [**11**].*) The reader familiar with étale cohomology of arithmetic schemes, may want to rethink the above equality in terms of the well known* $\mathbb{Z}[G]$*–module isomorphisms*

$$\mathrm{H}^1_{\mathrm{et}}(O_{K,S}, \mathbb{Z}(n))_{\mathrm{tors}} \simeq (\mathbb{Q}/\mathbb{Z})'(n)^{G_K}, \quad \textit{for all } n \in \mathbb{Z}_{\geq 1}.$$

An even more remarkable property of the elements $\delta_{K/k,T}(1-n)$ *considered above is the following theorem due to Deligne-Ribet* [**12**], *if* $\mathrm{char}(k) = 0$ *and to Deligne-Tate (*[**42**], *Chpt. V), if* $\mathrm{char}(k) > 0$. *This gives an integral refinement of the Klingen-Siegel Theorem and its function field counterpart mentioned in Example 2.4.2 above.*

Theorem 3.2.5. *For all* $n \in \mathbb{Z}_{\geq 1}$ *and all sets* T *as above, we have*

$$\Theta_{S,T}(1-n) = \delta_{K/k,T}(1-n) \cdot \Theta_S(1-n) \in \mathbb{Z}[G],$$

provided that T *contains at least two primes of different residual chars., if* $n > 1$ *and* $\mathrm{char}(k) = 0$.

Now, we reformulate conjecture $A_{\mathbb{Q}}(K/k, S, r)$ in terms of the T–modified equivariant L–function $\Theta_{S,T}$. This is a simple task for two reasons. First, since $\delta_{K/k,T}(0) \in \mathbb{Q}[G]^\times$, we have

$$\Theta^{(r)}_{S,T}(0) = \delta_{K/k,T}(0) \cdot \Theta^{(r)}_S(0).$$

Second, since $U_S/U_{S,T}$ is finite (see exact sequence (15)), we have $\mathbb{Q}U_S = \mathbb{Q}U_{S,T}$. Consequently, conjecture $A_{\mathbb{Q}}(K/k, S, r)$ is equivalent to the following.

Conjecture 3.2.6. $A_{\mathbb{Q}}(K/k, S, T, r)$. *For all* K/k, S, T *and* r *satisfying the above hypotheses, there exists a unique element* $\varepsilon_{S,T} \in e_{S,r} \wedge^r_{\mathbb{Q}[G]} \mathbb{Q}U_{S,T}$, *such that*

$$R_W(\varepsilon_{S,T}) = \Theta^{(r)}_{S,T}(0).$$

Of course, the conjectural ε_S and $\varepsilon_{S,T}$ are linked by $\varepsilon_{S,T} = \delta_{K/k,T}(0) \cdot \varepsilon_S$. Since $\delta_{K/k,T}(0) \in \mathbb{Q}[G]^\times$, we have an equality of $\mathbb{Q}[G]$–modules $\mathbb{Q}[G]\varepsilon_S = \mathbb{Q}[G]\varepsilon_{S,T}$. However, we only have an inclusion

$$\mathbb{Z}[G]\varepsilon_{S,T} \subseteq \mathbb{Z}[G]\varepsilon_S.$$

The index $[\,\mathbb{Z}[G]\varepsilon_S : \mathbb{Z}[G]\varepsilon_{S,T}\,]$ is finite and supported at primes dividing μ_K (a direct consequence of Remark 3.2.4.) As we will see in the next section, Rubin's Conjecture is an integral refinement of the conjecture above, in the sense that it specifies a canonical $\mathbb{Z}[G]$–submodule of $e_{S,r} \cdot \wedge^r_{\mathbb{Q}[G]} \mathbb{Q}U_{S,T}$ which contains $\mathbb{Z}[G]\varepsilon_{S,T}$ with a finite index.

3.2.3. **The Rubin-Stark lattice.** In this section, we assume that the set of data $(K/k, S, T, r)$ satisfies the hypotheses listed in the previous section. In the following, we use the notations of Definition 1.1.3 in the "Algebraic Preliminaries" section.

Definition 3.2.7. *The Rubin-Stark lattice* $\Lambda_{S,T}$ *is given by*

$$\Lambda_{S,T} := \mathcal{L}_{\mathbb{Z}[G]}(U_{S,T},\, r) \cap e_{S,r} \cdot \wedge^r_{\mathbb{Q}[G]} \mathbb{Q}U_{S,T}.$$

More explicitly, $\Lambda_{S,T}$ *consists of all* $\epsilon \in e_{S,r} \cdot \wedge^r_{\mathbb{Q}[G]} \mathbb{Q}U_{S,T}$, *such that*

$$(\phi_r^{(1)} \circ \cdots \circ \phi_1^{(r)})(\epsilon) \in \mathbb{Z}[G],$$

for all $\phi_1, \ldots, \phi_r \in U^*_{S,T}$, *where* $U^*_{S,T} := \mathrm{Hom}_{\mathbb{Z}[G]}(U_{S,T}, \mathbb{Z}[G])$.

Remark 3.2.8. *If M is a $\mathbb{Z}[G]$–module, then we let*

$$M_{S,r} := \widetilde{M} \cap e_{S,r}\mathbb{Q}M = \{x \in \widetilde{M} \mid (1 - e_{S,r})x = 0 \text{ in } \mathbb{Q}M\}.$$

The following are direct consequences of Remark 1.1.4 and Example 1.1.6.

(1) *If $r = 0$, then $\Lambda_{S,T} = \mathbb{Z}[G]_{S,0}$*
(2) *If $r = 1$, then $\Lambda_{S,T} = (U_{S,T})_{S,1}$. Note that since $U_{S,T}$ has no non-trivial $\mathbb{Z}$–torsion, we have an equality $U_{S,T} = \widetilde{U_{S,T}}$.*
(3) *For $r \geq 1$, we have a double inclusion*

$$(\wedge^r_{\mathbb{Z}[G]} U_{S,T})_{S,r} \subseteq \Lambda_{S,T} \subseteq \mathbb{Z}[1/|G|](\wedge^r_{\mathbb{Z}[G]} U_{S,T})_{S,r}.$$

3.2.4. **The Rubin-Stark conjecture.** The integral refinement proposed by Rubin in [**32**] for conjecture $A_{\mathbb{Q}}(K/k, S, T, r)$ is the following.

Conjecture 3.2.9. *$A_{\mathbb{Z}}(K/k, S, T, r)$ (Rubin-Stark). For all K/k, S, T and r satisfying the above hypotheses, there exists a unique element $\varepsilon_{S,T} \in \Lambda_{S,T}$, such that*

$$R_W(\varepsilon_{S,T}) = \Theta^{(r)}_{S,T}(0)\,.$$

In the case $r = 1$, the above statement is easily seen to be equivalent to the integral conjecture formulated by Stark himself in [**37**]. The key ingredients in establishing this equivalence are Proposition 1.2 in [**42**], Chpt. IV and Remark 3.2.4 above regarding the various properties of $\delta_{K/k,T}(0)$. From now on, we will refer to the unique element $\varepsilon_{S,T} \in \mathbb{C}(\wedge^r_{\mathbb{Z}[G]} U_{S,T})_{S,r}$ satisfying the regulator equality in the conjecture above as *the Rubin-Stark element* for the data $(K/k, S, T, r)$.

Example 3.2.10. Let us assume that $r = 0$. Then, $\Lambda_{S,T} = \mathbb{Z}[G]_{S,0}$ and the Rubin-Stark regulator is the inclusion $e_{S,0}\mathbb{Q}[G] \subseteq e_{S,0}\mathbb{C}[G]$. Consequently, the Rubin-Stark conjecture states that

$$\Theta_{S,T}(0) \in \mathbb{Z}[G]_{S,0}\,.$$

Theorem 3.2.5 above combined with the obvious equality $(1 - e_{S,0})\Theta_{S,T}(0) = 0$ implies that the conjecture holds true in this case.

Example 3.2.11. (compare to Example 2.3.6.) Let us consider the case where S contains at least $r+1$ distinct primes $v_0, v_1, \dots, v_r$ which split completely in K/k. Unless $S = \{v_0, \dots, v_r\}$, we have $e_{S,r} = 0$ and the conjecture is trivially true in that case. So, let us assume that $S = \{v_0, \dots, v_r\}$. Then, $e_{S,r} = e_{\mathbf{1_G}}$ and therefore we have

$$\Theta^{(r)}_{S,T}(0) = \zeta^{(r)}_{k,S,T}(0) \cdot e_{\mathbf{1_G}} = \zeta^{*}_{k,S,T}(0) \cdot e_{\mathbf{1_G}}\,.$$

Let $u_1, \dots, u_r$ be a $\mathbb{Z}$–basis for the group of (S,T)–units $U_{k,S,T}$ of k. Since

$$\sum_{\sigma \in G} \log|u_i^{\sigma}|_{w_j} \cdot \sigma^{-1} = \log|u_i|_{v_j} \cdot |G| e_{\mathbf{1_G}},$$

for all $i, j = 1, \dots, n$, the (S,T)–class number formula (16) implies that in this case the Rubin-Stark element $\varepsilon_{S,T}$ is given by

$$\varepsilon_{S,T} = \pm \frac{h_{k,S,T}}{|G|^r} u_1 \wedge \cdots \wedge u_r\,,$$

where the exterior products are viewed over $\mathbb{Z}[G]$. The question that needs to be answered is whether $\varepsilon_{S,T}$ belongs to the Rubin-Stark lattice $\Lambda_{S,T}$ or not. Let $\phi_1, \ldots, \phi_r \in U_{S,T}^*$. Since u_i belongs to $U_{k,S,T} \subseteq (U_{S,T})^G$, we have

$$\phi_j(u_i) \in \mathbb{Z}[G]^G = \mathbb{Z} \cdot |G| e_{\mathbf{1_G}}\,,$$

for all i, j as above. Consequently, we have

$$(\phi_r^{(1)} \circ \cdots \circ \phi_1^{(r)})(\varepsilon_{S,T}) \in \mathbb{Z} h_{k,S,T} e_{\mathbf{1_G}} \subseteq \frac{h_{k,S,T}}{|G|} \mathbb{Z}[G]\,.$$

Now, since K/k is unramified everywhere and completely split above S, the class-field theoretical interpretation of the (S,T)–class group $A_{k,S,T}$ (see §3.2.2 above) implies that $|G| \mid h_{k,S,T}$. This shows that, indeed, $\varepsilon_{S,T} \in \Lambda_{S,T}$ in this case. Note that, in particular, this settles the Rubin-Stark conjecture in the case $K = k$.

Example 3.2.12. A question which arises naturally in the context of the Rubin-Stark conjecture is what prevents the Rubin-Stark element $\varepsilon_{S,T}$ from lying in the somewhat more natural guess for a Rubin-Stark lattice given by $(\wedge_{\mathbb{Z}[G]}^r U_{S,T})_{S,r}$. Of course, if $r = 1$, then $\Lambda_{S,T} = (U_{S,T})_{S,1}$. For r large, Rubin gave number field examples where $\varepsilon_{S,T} \notin (\wedge_{\mathbb{Z}[G]}^r U_{S,T})_{S,r}$ (see [**32**].) In [**29**] and [**30**], such examples were constructed in the case $r = 2$, for both function fields and number fields. All these examples arise in situations where $U_{S,T}$ is not $\mathbb{Z}[G]$–projective (i.e. not G–cohomologically trivial.) When the $\mathbb{Z}[G]$–module $U_{S,T}$ is projective, then $\mathcal{L}_{\mathbb{Z}[G]}(U_{S,T}, r) = \widetilde{\wedge_{\mathbb{Z}[G]}^r U_{S,T}}$ (see Remark 1.1.4(3)) and therefore

$$\Lambda_{S,T} = (\wedge_{\mathbb{Z}[G]}^r U_{S,T})_{S,r}.$$

Next, we describe a somewhat systematic way of producing examples where $U_{S,T}$ is a projective $\mathbb{Z}[G]$–module.

Let k be a number field with non-trivial class-number h_k and let $S_0 := S_\infty$ be the set of infinite primes in k. Let p be a prime number which divides h_k and T be a finite set of non-archimedean primes in k, containing at least two primes of distinct residual characteristics. Let K/k be an abelian extension of degree p which is unramified everywhere. (The existence of K/k is a consequence of the assumption $p \mid h_k$.) Let $G := \mathrm{Gal}(K/k)$. Tchebotarev's density theorem permits us to construct a finite set of primes S, which contains S_0 and is disjoint from T, such that all but one prime v_0 in $S \setminus S_0$ splits completely in K/k, the prime v_0 is inert in K/k, and $h_{K,S,T} = 1$. We examine conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$, where $r = |S| - 1$. Note that every prime in $S \setminus \{v_0\}$ splits completely in K/k. Since v_0 is inert in K/k, we have $e_{S,r} = 1$ in this case. Since $h_{K,S,T} = 1$, we have an exact sequence of $\mathbb{Z}[G]$–modules (see (15)):

$$1 \longrightarrow U_{S,T} \longrightarrow U_S \longrightarrow \Delta_T \longrightarrow 1.$$

Further, since $\delta_{T,K/k}(0) := \prod_{v \in T}(1 - \mathrm{N}v \cdot \sigma_v^{-1})$ is a not a zero-divisor in $\mathbb{Z}[G]$ (check this !), Remark 3.2.4 leads to an exact sequence of $\mathbb{Z}[G]$–modules

$$0 \longrightarrow \bigoplus_{v \in T} \mathbb{Z}[G] \xrightarrow{\bigoplus_{v\in T} j_v} \bigoplus_{v \in T} \mathbb{Z}[G] \longrightarrow \Delta_T \longrightarrow 0,$$

where j_v is the multiplication by $(1 - \mathrm{N}v \cdot \sigma_v^{-1})$ map. This implies that the $\mathbb{Z}[G]$–module Δ_T is G–cohomologically trivial. Consequently, U_S and $U_{S,T}$ have isomorphic G–cohomology groups. On the other hand, since $h_{K,S,T} = 1$, we have $h_{K,S} = 1$

(note that $h_{K,S} \mid h_{K,S,T}$.) Consequently, Corollary 3.1.2 combined with the fact that G is cyclic (and therefore its Tate cohomology is periodic of period 2) implies that U_S and X_S have isomorphic G–cohomology groups. However, it is easy to check that in this case we have isomorphisms of $\mathbb{Z}[G]$–modules

$$Y_S \simeq \mathbb{Z} \oplus \mathbb{Z}[G]^r, \qquad X_S \simeq \mathbb{Z}[G]^r,$$

where G acts trivially in $\mathbb{Z}$. Consequently, X_S is G–cohomologically trivial. Therefore, $U_{S,T}$ is G–cohomologically trivial and therefore $\mathbb{Z}[G]$–projective. As discussed above, this implies that

$$\Lambda_{S,T} = \widetilde{\wedge^r_{\mathbb{Z}[G]} U_{S,T}} = \wedge^r_{\mathbb{Z}[G]} U_{S,T}.$$

The $\mathbb{Z}[G]$–projectivity of $U_{S,T}$ is not necessary in order to have an equality $\Lambda_{S,T} = (\wedge^r_{\mathbb{Z}[G]} U_{S,T})_{S,r}$. In order to see this, let us examine further the example above. However, instead of assuming that v_0 is inert, assume that v_0 splits completely. Therefore, S consists of $(r+1)$ distinct primes which split completely, as in Example 3.2.11. Obviously, in order to show that $\Lambda_{S,T} = (\wedge^r_{\mathbb{Z}[G]} U_{S,T})_{S,r}$, it suffices to show that $\mathbb{Z}_{(p)}\Lambda_{S,T} = \mathbb{Z}_{(p)}(\wedge^r_{\mathbb{Z}[G]} U_{S,T})_{S,r}$, where $\mathbb{Z}_{(p)}$ denotes the usual localization of $\mathbb{Z}$ at its prime ideal $p\mathbb{Z}$ (see part (3) of Remark 3.2.8.) We have an exact sequence of $\mathbb{Z}[G]$–modules

$$0 \longrightarrow X_S \longrightarrow Y_S \simeq \mathbb{Z}[G]^{r+1} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

which implies that

$$\widehat{H}^0(G, X_S) \simeq \widehat{H}^1(G, \mathbb{Z}) = 0$$
$$\widehat{H}^1(G, X_S) \simeq \widehat{H}^0(G, \mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}.$$

Of course, the same isomorphisms hold true for the G–cohomology groups of $U_{S,T}$ (see above.) Since $U_{S,T}$ is not G–cohomologically trivial, $U_{S,T}$ is not $\mathbb{Z}[G]$–projective in this case. Now, since G is of prime order p and $U_{S,T}$ is $\mathbb{Z}$-free, Theorem 4.19 in [**39**] implies that we have an isomorphism

$$\mathbb{Z}_{(p)} U_{S,T} \simeq \mathbb{Z}_{(p)} I_G \oplus \mathbb{Z}_{(p)}[G]^r$$

in the category of $\mathbb{Z}_{(p)}[G]$–modules, where I_G is the augmentation ideal of $\mathbb{Z}[G]$, generated by $(\sigma - 1)$, for any generator σ of G. (In applying Theorem 4.19 loc. cit., note that $\mathbb{Z}_{(p)}[G]$ is a local ring of maximal ideal generated by p and $(\sigma - 1)$.) Pick units $U_0, U_1, \ldots, U_r$ in $U_{S,T}$, such that

$$\mathbb{Z}_{(p)} U_{S,T} \simeq \oplus_{i=0}^r \mathbb{Z}_{(p)}[G] U_i, \qquad \mathbb{Z}_{(p)}[G] U_0 \simeq \mathbb{Z}_{(p)} I_G, \quad \mathbb{Z}_{(p)}[G] U_i \simeq \mathbb{Z}_{(p)}[G],$$

for all $i = 1, \ldots, r$. Since $e_{S,r} = e_{\mathbf{1_G}}$ and $e_{\mathbf{1_G}} I_G = 0$, we have an equality

$$e_{S,r} \cdot (\mathbb{Q} \wedge^r_{\mathbb{Z}[G]} U_{S,T}) = e_{\mathbf{1_G}} \cdot \mathbb{Q}(U_1 \wedge \cdots \wedge U_r)\,,$$

where the exterior products are viewed over $\mathbb{Z}[G]$. Now, it is very easy to see that

$$\mathbb{Z}_{(p)}\Lambda_{S,T} = \mathbb{Z}_{(p)}(\wedge^r_{\mathbb{Z}[G]} U_{S,T})_{S,r} = |G| e_{\mathbf{1_G}} \cdot \mathbb{Z}_{(p)}(U_1 \wedge \cdots \wedge U_r)\,.$$

(To prove this, consider the particular elements $\phi_i \in \mathbb{Z}_{(p)} U^*_{S,T}$, satisfying $\phi_i(U_j) = \delta_{i,j}$, for all values $i = 1, \ldots, n$ and $j = 0, 1, \ldots, n$.) In order to reconcile the above description of $\mathbb{Z}_{(p)}\Lambda_{S,T}$ with the concrete formula for $\varepsilon_{S,T}$ obtained in Example 3.2.11, observe that since $\widehat{H}^0(G, U_{S,T}) = 0$, we have

$$U_{k,S,T} = (U_{K,S,T})^G = N_G \cdot U_{K,S,T}, \quad \mathbb{Z}_{(p)} U_{k,S,T} = \oplus_{i=1}^r \mathbb{Z}_{(p)}(N_G U_i)\,,$$

where $N_G := |G|e_{\mathbf{1_G}}$ is the usual norm element in $\mathbb{Z}[G]$. Note that $N_G U_0 = 1$. Consequently, the $\mathbb{Z}$–basis $\{u_1, \dots, u_r\}$ for $U_{k,S,T}$ considered in Example 3.2.11 can be taken to satisfy $u_i^{\alpha_i} = N_G U_i^{\beta_i}$, for some $\alpha_i, \beta_i \in \mathbb{Z} \setminus p\mathbb{Z}$, for all $i = 1, \dots, r$. Let $\alpha := \prod_i \alpha_i$ and $\beta := \prod_i \beta_i$. Since $N_G^r = |G|^{r-1} N_G$, we have the following

$$\begin{aligned}\varepsilon_{S,T} = \pm \frac{\beta}{\alpha} \cdot \frac{h_{k,S,T}}{|G|^r}(N_G U_1 \wedge \cdots \wedge N_G U_r) \quad &= \pm \tfrac{\beta}{\alpha} \cdot \tfrac{h_{k,S,T}}{|G|} N_G \cdot (U_1 \wedge \cdots \wedge U_r) \\ &\in |G| e_{\mathbf{1_G}} \cdot \mathbb{Z}_{(p)}(U_1 \wedge \cdots \wedge U_r)\,.\end{aligned}$$

Indeed, this reconciles the two calculations.

3.2.5. **Functorial behavior.** In this section, we will describe very briefly several functorial properties of conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$: dependence on the top field K; dependence on S (and implicitly on r); dependence on T. All these properties are consequences of the existence and uniqueness of the Rubin-Stark element $\varepsilon_{S,T}$ in the space $e_{S,r}(\mathbb{C} \wedge^r_{\mathbb{Z}[G]} U_{S,T})$. Below, we assume that the set of data $(K/k, S, T, r)$ satisfies the hypotheses in the Rubin-Stark conjecture.

First, let K' be an intermediate field, $k \subseteq K' \subseteq K$. Then, it is obvious that $(K'/k, S, T, r)$ also satisfies the hypotheses in the Rubin-Stark conjecture.

Proposition 3.2.13. *If conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$ holds, then so does conjecture $A_{\mathbb{Z}}(K'/k, S, T, r)$.*

Proof. (Sketch; see [**32**] for details.) Let $G' := G(K'/k)$ and $H := G(K/K')$. We let $\mathcal{R}_{W'}$ be the Rubin-Stark regulator for $(K'/k, S, T, r)$ and the r–tuple $W' := (w_1', \dots, w_r')$, where w_i' is the prime in K' sitting below w_i, for all $i = 1, \dots, r$. We let $\varepsilon_{S,T}'$ and $\Lambda_{S,T}'$ denote the corresponding Rubin-Stark element and lattice, respectively. If $U_{S,T}' := U_{K',S,T} = (U_{S,T})^H$, then $N_H : U_{S,T} \to U_{S,T}'$ denotes the usual norm map. It is easy to check that the $\mathbb{C}[G]$–module morphism

$$\wedge^r N_H : e_{S,r} \wedge^r_{\mathbb{C}[G]} \mathbb{C}U_{S,T} \twoheadrightarrow e_{S,r} \wedge^r_{\mathbb{C}[G']} \mathbb{C}U_{S,T}'\,,$$

obtained by first extending N_H by $\mathbb{C}$–linearity to $\mathbb{C}U_S$ and then taking exterior powers, maps $\varepsilon_{S,T}$ to $\varepsilon_{S,T}'$ and also maps $\Lambda_{S,T}$ into $\Lambda_{S,T}'$. □

Next, let us assume that $S' = S \cup \{v_{r+1}, \dots, v_{r'}\}$, where $r' > r$ and $v_{r+1}, \dots, v_{r'}$ are distinct primes in k, which do not belong to $S \cup T$ and split completely in K/k. Then, it is obvious that $(K/k, S', T, r')$ also satisfy the hypotheses of the Rubin-Stark conjecture.

Proposition 3.2.14. *If conjecture $A_{\mathbb{Z}}(K/k, S', T, r')$ holds, then so does conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$.*

Proof. (Sketch; see [**32**] for details.) First, it is obvious that $e_{S,r} = e_{S',r'}$. We extend the ordered r-tuple $W = (w_1, \cdots, w_r)$ to an ordered r'–tuple $W' = (w_1, \dots, w_r, w_{r+1}, \dots, w_{r'})$ of primes w_i in K sitting above v_i and denote by $\varepsilon_{S',T}$ the Rubin-Stark element in $e_{S,r} \wedge^{r'}_{\mathbb{C}[G]} \mathbb{C}U_{S',T}$ corresponding to the regulator $R_{W'}$ associated to W'. For all $i = r+1, \dots, r'$, we define $w_i^* \in (U_{S',T})^*$ by

$$w_i^*(u) = \sum_{\sigma \in G} \mathrm{ord}_{w_i}(u^\sigma) \cdot \sigma^{-1}, \qquad \text{for all } u \in U_{S',T}\,,$$

and extend these by $\mathbb{C}$–linearity to elements in $\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}U_{S',T}, \mathbb{C}[G])$. The extensions will be abusively denoted by w_i^* as well. It is not difficult to see that the $\mathbb{C}[G]$–module morphism

$$(w_{r'}^*)^{(r+1)} \circ \cdots \circ (w_{r+1}^*)^{(r')} : e_{S,r} \wedge_{\mathbb{C}[G]}^{r'} \mathbb{C}U_{S',T} \longrightarrow e_{S,r} \wedge_{\mathbb{C}[G]}^{r} \mathbb{C}U_{S',T}$$

is an isomorphism onto $e_{S,r} \wedge_{\mathbb{C}[G]}^{r} \mathbb{C}U_{S,T}$, which maps $\varepsilon_{S',T}$ to $\varepsilon_{S,T}$ and also maps the lattice $\Lambda_{S',T}$ into the lattice $\Lambda_{S,T}$. Note that since $\mathbb{C}U_{S,T}$ is a direct summand of $\mathbb{C}U_{S',T}$ in the category of $\mathbb{C}[G]$–modules, $e_{S,r} \wedge_{\mathbb{C}[G]}^{r} \mathbb{C}U_{S,T}$ is a direct summand of $e_{S,r} \wedge_{\mathbb{C}[G]}^{r} \mathbb{C}U_{S',T}$. □

Remark 3.2.15. *As remarked before, Burns has shown in* [**6**] *that the ETNC (Conjecture 3.1.4 above) implies the Rubin-Stark Conjecture. In fact, one starts with data $(K/k, S, T, r)$ satisfying the hypotheses in conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$, then one uses Tchebotarev's density theorem to construct a set S' as in Proposition 3.2.14 above and satisfying the additional hypothesis that the S'–ideal class group of K is trivial (i.e. $h_{K,S'} = 1$) and then one proves that the ETNC for data $(K/k, S')$ implies conjecture $\mathbb{A}_{\mathbb{Z}}(K/k, S', T, r')$. Finally, one uses Proposition 3.2.14 above to derive conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$.*

Next, let S' be any finite set of primes in k, such that $S \subseteq S'$ and $S' \cap T = \emptyset$. Then, the set of data $(K/k, S', T, r)$ satisfies the hypotheses in the Rubin-Stark conjecture as well.

Proposition 3.2.16. *If conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$ holds, then so does conjecture $A_{\mathbb{Z}}(K/k, S', T, r)$.*

Proof. Since $\Theta_{S',T}^{(r)}(0) = \prod_{v \in S' \setminus S}(1 - \sigma_v^{-1}) \cdot \Theta_{S,T}^{(r)}(0)$, we have

$$e_{S',r}\mathbb{C}[G] = \prod_{v \in S' \setminus S} (1 - \sigma_v^{-1}) \cdot e_{S,r}\mathbb{C}[G]$$

(check this equality !) and

$$\varepsilon_{S',T} = (\prod_{v \in S' \setminus S} (1 - \sigma_v^{-1})) \cdot \varepsilon_{S,T}\,.$$

Since $U_{S',T}/U_{S,T}$ has no $\mathbb{Z}$–torsion (check this !), Remark 1.1.1 implies that the inclusion of $\mathbb{Z}[G]$–modules $U_{S,T} \hookrightarrow U_{S',T}$ induces a surjective $\mathbb{Z}[G]$–module morphism $(U_{S',T})^* \twoheadrightarrow (U_{S',T})^*$. This implies right away that we have an inclusion

$$(\prod_{v \in S' \setminus S} (1 - \sigma_v^{-1})) \cdot \Lambda_{S,T} \subseteq \Lambda_{S',T}.$$

Combined with the last displayed equality, this concludes the proof. Note that, tacitly, we have used the fact that since $\mathbb{C}U_{S,T}$ is a direct summand of $\mathbb{C}U_{S',T}$, $e_{S,r} \wedge_{\mathbb{C}[G]}^{r} \mathbb{C}U_{S,T}$ is a direct summand of $e_{S,r} \wedge_{\mathbb{C}[G]}^{r} \mathbb{C}U_{S',T}$ as well (all in the category of $\mathbb{C}[G]$–modules.) □

Finally, let us assume that T' is a finite set of primes in k, such that $T \subseteq T'$ and $S \cap T' = \emptyset$. Then, the set of data $(K/k, S, T', r)$ satisfies the hypotheses in the Rubin-Stark conjecture as well.

Proposition 3.2.17. *If conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$ holds, then so does conjecture $A_{\mathbb{Z}}(K/k, S, T', r)$.*

Proof. See Proposition 5.3.1 of [**28**]. □

4. The Rubin-Stark conjecture in the case $r = 1$

4.1. General considerations

In this section, we take a closer look at the predictions of conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$ in the important particular case when $r = 1$. As we mentioned earlier, in this case, the conjecture had already been formulated by Stark himself for number fields in [**37**] and extended by Tate to function fields in [**42**]. Although Stark's and Tate's formulations only depend on the set S and do not involve the set T, it is not difficult to prove that for a given S as above the Stark-Tate conjecture for the data $(K/k, S, 1)$ is true if and only if the conjectures $A_{\mathbb{Z}}(K/k, S, T, 1)$ are true, for all T, such that $(K/k, S, T, 1)$ satisfy the above hypotheses (see Remark 4.1.4 below.) The case $r = 1$ of the Rubin-Stark conjecture reads as follows.

Conjecture 4.1.1. $A_{\mathbb{Z}}(K/k, S, T, 1)$ *(Stark, Tate) If the set of data* $(K/k, S, T, 1)$ *satisfies the hypotheses above and* w_1 *is a prime in* K *sitting above the distinguished prime* v_1 *in* S *which splits completely in* K/k*, then there exists a unique* $\varepsilon_{S,T} \in (U_{S,T})_{S,1}$*, such that*

$$\Theta'_{S,T}(0) = -\sum_{\sigma \in G} \log |\varepsilon_{S,T}^{\sigma^{-1}}|_{w_1} \cdot \sigma \,.$$

Let $v_0 \in S$, with $v_0 \neq v_1$ and let us fix a prime w_0 in K sitting above v_0. Note that the $e_{S,1}\mathbb{R}[G]$–module $e_{S,1}(\mathbb{R}X_S)$ is free of rank one of basis $e_{S,1}(w_1 - w_0)$. This observation, combined with the obvious equality $e_{S,1}\mathbb{R}[G] = \Theta'_{S,T}(0)\mathbb{R}[G]$ (by the definition of $e_{S,1}$) and the injectivity of the $\mathbb{Z}[G]$–module morphism $\lambda_S : U_{S,T} \hookrightarrow \mathbb{R}X_S \subseteq \mathbb{R}Y_S$ leads right away to the following alternative description of $(U_{S,T})_{S,1}$.

$$(U_{S,T})_{S,1} = \{u \in U_{S,T} \mid \lambda_S(u) \in \Theta'_{S,T}(0)\mathbb{R}[G](w_1 - w_0)\}. \tag{17}$$

In view of the above remarks, the following is an equivalent formulation of conjecture $A_{\mathbb{Z}}(K/k, S, T, 1)$.

Conjecture 4.1.2 (Stark, Tate)**.** *Under the previous assumptions and notations, the following equivalent statements hold.*

(1) *There exists* $\varepsilon_{S,T} \in U_{S,T}$*, such that*

$$\Theta'_{S,T}(0)(w_1 - w_0) = \lambda_S(\varepsilon_{S,T})\,.$$

(2) *There exists* $\varepsilon_{S,T} \in K_T^{\times}$*, such that*

$$\Theta'_{S,T}(0)(w_1 - w_0) = -\sum_{w} \log |\varepsilon_{S,T}|_w \cdot w\,,$$

where the sum is taken over all primes w *in* K *and*

$$K_T^{\times} := \{x \in K^{\times} \mid x \equiv 1 \mod v,\ \forall v \in T\}\,.$$

Remark 4.1.3. *Let* $U_{S,T}^{(v_1)} := \{u \in U_{S,T} \mid -\sum_{\sigma\in G} \log |\varepsilon_{S,T}^{\sigma^{-1}}|_{w_1} \cdot \sigma \in \mathbb{R}[G]\Theta'_{S,T}(0)\}$. *Since* $e_{S,1}w_0 = 0$*, if* $S| > 2$ *and* $e_{S,1}w_0 = e_{\mathbf{1_G}}w_0$*, if* $|S| = 2$ *(by the definition of* $e_{S,1}$ *and the fact that* $e_{S,1}w_1 \neq 0$*), equality (17) leads to the following additional description of* $(U_{S,T})_1$*.*

$$(U_{S,T})_{S,1} = U_{S,T}^{(v_1)} \cap \mathcal{A}_{S,T}^{(v_1)} \tag{18}$$

where $\mathcal{A}_{S,T}^{(v_1)}$ *is a group of so-called anti-units defined by*

$$\mathcal{A}_{S,T}^{(v_1)} := \begin{cases} \{u \in U_{S,T} \mid |u|_w = 1, \text{ for all } w \in S_K, w \nmid v_1\}, & \text{if } |S| > 2; \\ \{u \in U_{S,T} \mid |u|_w = |u|_{w'}, \text{ for all } w, w' \in S_K, w, w' \nmid v_1\}, & \text{if } |S| = 2. \end{cases}$$

Remark 4.1.4. *The unique* (S,T)*–unit* $\varepsilon_{S,T}$ *which would render the above conjecture true is called the* T*–modified Stark unit associated to* $(K/k, S, T, 1)$*. Let us fix* S *and vary* T *in a finite family* $\mathcal{T}$*, such that* $(K/k, S, T, 1)$ *satisfies the above hypotheses for all* $T \in \mathcal{T}$ *and the set* $\{\delta_{K/k,T}(0) \mid T \in \mathcal{T}\}$ *generates the* $\mathbb{Z}[G]$*–ideal* $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ *(see Remark 3.2.4). Let us pick elements* $\alpha_T \in \mathbb{Z}[G]$*, such that*

$$w_K = \sum_{T \in \mathcal{T}} \alpha_T \cdot \delta_{K/k,T}(0)\,.$$

Then, since $\Theta'_{S,T}(0) = \delta_{K/k,T}(0)\Theta'_S(0)$*, for all* $T \in \mathcal{T}$*, we have*

$$\Theta'_S(0) = -\frac{1}{w_K} \sum_{\sigma \in G} \log |\varepsilon_S^{\sigma^{-1}}|_{w_1} \cdot \sigma\,, \qquad \text{for } \varepsilon_S := \prod_{T \in \mathcal{T}} \varepsilon_{S,T}^{\alpha_T}\,,$$

and ε_S *is the unique element in* $(\widetilde{U_S})_{S,1}$ *satisfying the first equality above. The (conjectural) element* ε_S *is called the Stark unit associated to the data* $(K/k, S, 1)$*. Note that* ε_S *denotes in fact a class of* S*–units in* $\widetilde{U_S} = U_S/\mu_K$*. Also, note that the uniqueness of* ε_S *and* $\varepsilon_{S,T}$ *(which are consequences of the injectivity of the Rubin-Stark regulator) give us equalities in* U_S

$$\varepsilon_S^{\delta_{K/k,T}(0)} = \varepsilon_{S,T}^{w_K}, \qquad \text{for all } T \in \mathcal{T}\,.$$

Note that the left-hand side of the above equalities does not depend on the representative we choose for ε_S *in* U_S*. The above equalities combined with Proposition 1.2, Chpt. IV of* [**42**] *show that the field extension* $K(\varepsilon_S^{1/w_K})/k$ *generated by any* w_K*–root of any representative in* U_S *of the conjectural Stark unit* ε_S *is an abelian extension. This abelian condition is built into the original ("*T*-less") Stark–Tate formulation of the above conjecture.*

At this point, it is natural to discuss the conjecture above separately in the case where the split prime v_1 is an archimedean (infinite) prime (which will force k to be a number field) and a non-archimedean prime (in which case k can be either a number field or a function field), respectively.

4.2. Archimedean v_1

Under the hypothesis that the distinguished prime v_1 is archimedean, the conjecture is known to hold in two major particular cases, both due to Stark: the case where $k = \mathbb{Q}$, which we will treat somewhat in detail below, and the case where k is a quadratic imaginary field for which we send the reader to [**42**], Chpt. IV, Proposition 3.9 or [**37**].

Let us assume that $k = \mathbb{Q}$. Since the (unique) infinite prime of k is assumed to split completely in K/k, the top field K has to be a real, abelian extension of $\mathbb{Q}$. Class-field theory forces K to be a subfield of some real cyclotomic field $\mathbb{Q}(\zeta_m)^+ := \mathbb{Q}(\zeta_m + \zeta_m^{-1})$, where ζ_m is root of unity of order m and m can be chosen to be any multiple of the conductor of K/k. Propositions 3.2.13, 3.2.16 and 3.2.17 above permit us to reduce the proof of conjecture $\mathbb{A}_{\mathbb{Z}}(K/\mathbb{Q}, S, T, 1)$ to the situation where $K = \mathbb{Q}(\zeta_m)^+$, $S = S_m := \{v_1 = \infty\} \cup \{p\mathbb{Z} \mid p \text{ prime in } \mathbb{Z},\, p \mid m\}$ and

$T = T_q := \{q\mathbb{Z}\}$, where q is a prime in $\mathbb{Z}$ which does not divide $2m$. (Note that $\mu_K = \{\pm 1\}$ and $w_K = 2$ in this case. Also, note that for such prime q, there are no non-trivial roots of unity in the full cyclotomic field $\mathbb{Q}(\zeta_m)$ which are congruent to 1 modulo q.) Let us recall the classical formulas

$$L'_S(\chi, s) = -\frac{1}{2}\sum_{\sigma\in G}\log|\epsilon_m^{\sigma}|_{w_1}\cdot\chi(\sigma)\,, \qquad \text{for all } \chi\in \mathrm{Gal}(K/\mathbb{Q})\,,$$

where $\epsilon_m := (1-\zeta_m)(1-\zeta_m^{-1})$ is obviously an element of U_S and w_1 is the prime sitting over v_1 in K corresponding to the embedding of $\mathbb{Q}(\zeta_m)^+$ into $\mathbb{C}$ sending ζ_m to $\exp(2\pi i/m)$. (For a quick proof, combine Theorem 4.9 in [**43**] with the functional equation for $L_S(\chi, s)$.) We combine the above character-dependent equalities to obtain the following (equivalent) G–equivariant equality.

$$\Theta'_S(0) = -\frac{1}{2}\sum_{\sigma\in G}\log|\epsilon_m^{\sigma^{-1}}|_{w_1}\cdot\sigma.$$

Now, since the prime q is not ramified in the full cyclotomic extension $\mathbb{Q}(\zeta_m)/\mathbb{Q}$, we can afford to abuse the notation a bit and denote by σ_q its Frobenius automorphism in $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$. We leave it as an (easy) exercise for the reader to check that

$$(1-\zeta_m)^{(1-q\sigma_q^{-1})}\in\mathbb{Q}(\zeta_m)^+\,, \qquad \{(1-\zeta_m)^{(1-q\sigma_q^{-1})}\}^2 = \epsilon_m^{(1-q\sigma_q^{-1})}.$$

Now, if we let $\varepsilon_{S,T} := (1-\zeta_m)^{(1-q\sigma_q^{-1})}$, the last two displayed equalities combined give:

$$\Theta'_{S,T}(0) = (1-q\sigma_q^{-1})\cdot\Theta'_S(0) = -\sum_{\sigma\in G}\log|\varepsilon_{S,T}^{\sigma^{-1}}|_{w_1}\cdot\sigma.$$

Consequently, if we show that $\varepsilon_{S,T}\in(U_{S,T})_{S,1}$, the equality above verifies the Rubin-Stark conjecture $\mathbb{A}_{\mathbb{Z}}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q}, S = S_m, T = T_q, 1)$. Now, since $(1-\zeta_m)$ is an S–unit in $\mathbb{Q}(\zeta_m)$,

$$\varepsilon_{S,T} := (1-\zeta_m)^{(1-q\sigma_q^{-1})}$$

is an (S,T)–unit in $\mathbb{Q}(\zeta_m)$ (use the exact sequence (15) and Remark 3.2.4 for top field $\mathbb{Q}(\zeta_m)$ and sets S_m and T_q). Therefore, $\varepsilon_{S,T}$ is an (S,T)–unit in $\mathbb{Q}(\zeta_m)^+$. In order to show that $\varepsilon_{S,T}$ sits in the subspace $(U_{S,T})_{S,1}$ of $U_{S,T}$, one uses the equality (18) combined with the well-known fact that $(1-\zeta_m)$ is a unit in $\mathbb{Q}(\zeta_m)$, if m is not a power of a prime, and $(1-\zeta_m)$ is a generator of the unique and totally ramified prime $\mathfrak{P}$ sitting above p in $\mathbb{Q}(\zeta_m)$, if m is a power of the prime p. The conclusion of the above considerations is the following.

Theorem 4.2.1 (Stark)**.** *Conjecture $A_{\mathbb{Z}}(K/k, S, T, 1)$ is true if $k = \mathbb{Q}$ and K is any real abelian extension of $\mathbb{Q}$.*

Remark 4.2.2. *Assume that the set of data $(K/\mathbb{Q}, S, T, 1)$ satisfies the usual hypotheses and that K is a any real abelian extension of $\mathbb{Q}$. With notations as above, we can always find a natural number m divisible by the conductor of $K/\mathbb{Q}$, such that $S = S_m$. Then, the calculations preceding the Theorem above combined with Remark 4.1.4 and Propositions 3.2.13, 3.2.16 and 3.2.17 show that the Rubin-Stark unit associated to the data $(K/\mathbb{Q}, S, T, 1)$ is given by*

$$\varepsilon_{S,T} := \mathbf{N}_{\mathbb{Q}(\zeta_m)^+/K}((1-\zeta_m)^{\delta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},T}(0)})\,,$$

whereas the Stark unit associated to the data $(K/\mathbb{Q}, S, 1)$ is given by

$$\varepsilon_S := \mathbf{N}_{\mathbb{Q}(\zeta_m)^+/K}((1-\zeta_m)(1-\zeta_m^{-1})) = \mathbf{N}_{\mathbb{Q}(\zeta_m)/K}(1-\zeta_m).$$

It should be noted that the last equality shows that the Stark units corresponding to real abelian extensions of $\mathbb{Q}$ *are norms (down from full cyclotomic fields) of* m*–torsion points* $(1-\zeta_m)$ *of the multiplicative formal group* $\widehat{G_m}(X,Y) = (X+1)(Y+1)-1$. *Saying that these* m*–torsion points, for varying* m*, generate the maximal abelian extension* $\mathbb{Q}^{\mathrm{ab}}$ *of* $\mathbb{Q}$ *(a restatement of the celebrated Kronecker-Weber Theorem) is equivalent to saying that the Stark units* $\epsilon_m = (1-\zeta_m)(1-\zeta_m^{-1})$ *for the full real cyclotomic extensions* $\mathbb{Q}(\zeta_m)^+/\mathbb{Q}$ *generate the maximal real abelian extension* $\mathbb{Q}^{\mathrm{ab},+}$ *of* $\mathbb{Q}$*. This is not an accident, as Stark himself noticed in* [**36**]*: in general, there is a known and surprising link between Hilbert's 12–th problem for a totally real number field* k *(explicit generation of abelian extensions of* k*) and the* $r = 1$ *case of the Rubin-Stark conjecture for abelian extensions* K/k *in which only one of the real primes of* k *splits completely in* K *(see* [**42**]*, Chpt. IV, Proposition 3.8 and the comments which follow.) Similar, but more general consequences can be derived by assuming the Rubin-Stark conjecture for arbitrary* r*. We will not pursue these ideas here.*

As mentioned above, Stark also proved the following (see [**37**] and [**42**], Chpt. IV, Prop. 3.9.)

Theorem 4.2.3 (Stark)**.** *If* k *is an imaginary quadratic field and* $(K/k, S, T, 1)$ *satisfies the usual hypotheses, then conjecture* $A_{\mathbb{Z}}(K/k, S, T, 1)$ *holds true.*

We will not give the proof of the theorem above. However, it is worth mentioning that in this case the Stark unit corresponding to $(K/k, S, 1)$ is the norm down to K of an elliptic unit in the ray–class field of k of conductor equal to that of K/k. So, the role of the cyclotomic units $\epsilon_m = (1-\zeta_m)(1-\zeta_m^{-1})$ is played by elliptic units in this case. Moreover, the elliptic units in question are constructed out of torsion points of an elliptic curve with CM by O_k. The role of the formal group $\widehat{G_m}$ (or, more precisely that of the multiplicative group G_m) is taken by these CM elliptic curves. The analogue of the Kronecker-Weber Theorem in this case is the Main Theorem of Complex Multiplication which gives a precise recipe for the construction of the maximal abelian extension k^{ab} of k out of the coordinates of the torsion points and the j–invariant of any such elliptic curve with CM by O_k.

4.3. Non-archimedean v_1 – the Brumer-Stark conjecture

In this section, we take a closer look at conjecture $A_{\mathbb{Z}}(K/k, S, T, 1)$ in the situation where the distinguished split prime $v_1 \in S$ is a finite prime. This is the classical Brumer-Stark conjecture (formulated by Stark and strengthening an older conjecture of Brumer in the case of number fields, and extended by Tate to the case of function fields.) Our goal in this section is twofold: *First,* we give a new interpretation of the Brumer-Stark conjecture for arbitrary global fields in terms of the annihilation by the special L–value $\Theta_{S_0,T}(0)$ of certain Arakelov class-groups (Chow groups) associated to the top field K, where $S_0 := S \setminus \{v_1\}$; *Second,* we discuss the overwhelming evidence in support of the Brumer-Stark conjecture in terms of this interpretation, some very recent, some going back to the early 1980s and before.

We start by fixing two finite, disjoint sets of primes $\mathcal{S}$ and $\mathcal{T}$ in K, such that $\mathcal{S}$ contains the archimedean (infinite) primes of K. Note that $\mathcal{T}$ may be empty and, if K is a function field, $\mathcal{S}$ may be empty as well. We will define an Arakelov class-group $CH^1_{\mathcal{S}}(K)^0_{\mathcal{T}}$, generalizing the classical Arakelov class-group $CH^1(K)^0$ of a number field K (see [**25**]) and also generalizing the Picard group of degree zero divisors $\mathrm{Pic}^0(K)$ of a function field K. First, we define the group of Arakelov $(\mathcal{S},\mathcal{T})$–divisors of K by

$$\mathrm{Div}_{\mathcal{S}}(K)_{\mathcal{T}} := \left(\bigoplus_{w\notin\mathcal{S}\cup\mathcal{T}}\mathbb{Z}w\right)\oplus\left(\bigoplus_{w\in\mathcal{S}}\mathbb{R}w\right),$$

where the direct sums are taken with respect to all the primes w of K and the first direct summand is the free $\mathbb{Z}$–module generated by all the primes in K which are outside of $\mathcal{S}\cup\mathcal{T}$ and the second direct summand is the $\mathbb{R}$–vector space of basis $\mathcal{S}$. We define the degree map

$$\deg_{K,\mathcal{S}} : \mathrm{Div}_{\mathcal{S}}(K)_{\mathcal{T}} \longrightarrow \mathbb{R}$$

as the unique map which is $\mathbb{Z}$–linear on the first summand and $\mathbb{R}$–linear on the second and satisfies

$$\deg_{K,\mathcal{S}}(w) = \begin{cases} \log|Nw|, & \text{if } w\notin\mathcal{S}\cup\mathcal{T}\ ;\\ 1, & \text{if } w\in\mathcal{S}.\end{cases}$$

We let $\mathrm{Div}_{\mathcal{S}}(K)^0_{\mathcal{T}}$ denote the kernel of $\deg_{K,\mathcal{S}}$ (i.e. the group of degree 0 Arakelov $(\mathcal{S},\mathcal{T})$–divisors.) Further, we let $K^\times_{\mathcal{T}}$ denote the subgroup of $K^\times$ consisting of all $x\in K^\times$, such that $x\equiv 1 \mod w$ (meaning $\mathrm{ord}_w(x-1)>0$), for all $w\in\mathcal{T}$. The product formula leads to a divisor morphism

$$\mathrm{div}_{K,\mathcal{S}} : K^\times_{\mathcal{T}} \longrightarrow \mathrm{Div}_{\mathcal{S}}(K)^0_{\mathcal{T}}, \qquad x\to\sum_{w\notin\mathcal{S}\cup\mathcal{T}}\mathrm{ord}_w(x)\cdot w+\sum_{w\in\mathcal{S}}(-\log|x|_w)\cdot w\,,$$

where $\mathrm{ord}_w(\cdot)$ and $|\cdot|_w$ denote the canonically normalized valuation and metric associated to w, respectively. Note that, for all $x\in K^\times_{\mathcal{T}}$, we have an equality

(19)
$$-\sum_w \log|x|_w\cdot w = \sum_{w\notin\mathcal{S}\cup\mathcal{T}}\deg_{K,\mathcal{S}}(w)\mathrm{ord}_w(x)\cdot w+\sum_{w\in\mathcal{S}}\deg_{K,\mathcal{S}}(w)(-\log|x|_w)\cdot w,$$

where the left sum is taken over all primes w of K. The Arakelov $(\mathcal{S},\mathcal{T})$–class group is given by

$$CH^1_{\mathcal{S}}(K)^0_{\mathcal{T}} := \frac{\mathrm{Div}_{\mathcal{S}}(K)^0_{\mathcal{T}}}{\mathrm{div}_{K,\mathcal{S}}(K^\times_{\mathcal{T}})}\,.$$

Note that, if K is a number field and $\mathcal{S}=S_\infty$ and $\mathcal{T}=\emptyset$, then $CH^1_{\mathcal{S}}(K)^0_{\mathcal{T}}$ is the classical Arakelov class-group $CH^1(K)^0$ defined in [**25**], for example. Also, if K is a function field and $\mathcal{S}=\mathcal{T}=\emptyset$, then $CH^1_{\mathcal{S}}(K)^0_{\mathcal{T}}$ is the Picard group $\mathrm{Pic}^0(K)$ of divisors of K of degree 0 (which is the function field analogue of $CH^1(K)^0$.) There is an obvious natural exact sequence of groups

$$0\longrightarrow\frac{\mathbb{R}X_{\mathcal{S}}}{\mathrm{div}_{K,\mathcal{S}}(U_{\mathcal{S},\mathcal{T}})}\xrightarrow{\ \iota\ }CH^1_{\mathcal{S}}(K)^0_{\mathcal{T}}\xrightarrow{\ \pi\ }Cl(O_{K,\mathcal{S}})_{\mathcal{T}}\longrightarrow 0\,, \tag{20}$$

where $Cl(O_{K,\mathcal{S}})_{\mathcal{T}}$ is the $(\mathcal{S},\mathcal{T})$–ideal class group $A_{K,\mathcal{S},\mathcal{T}}$ defined just below (15), if $\mathcal{S}\neq\emptyset$ and

$$Cl(O_{K,\mathcal{S}})_{\mathcal{T}} = CH^1_{\emptyset}(K)^0_{\mathcal{T}} =: \mathrm{Pic}^0(K)_{\mathcal{T}},$$

if $\mathcal{S} = \emptyset$ (and therefore K is a function field.) The map ι is induced by the natural inclusion $\mathbb{R}X_{\mathcal{S}} \subseteq \mathrm{Div}_{\mathcal{S}}(K)^0_{\mathcal{T}}$ and the map π sends the class of a divisor $D = \sum_{w \notin \mathcal{T}} a_w \cdot w$ in $\mathrm{Div}_{\mathcal{S}}(K)^0_{\mathcal{T}}$ to the class of $D' := \sum_{w \notin \mathcal{T} \cup \mathcal{S}} a_w \cdot w$ in $Cl(O_{K,\mathcal{S}})_{\mathcal{T}}$.

Let us note that the map $\mathrm{div}_{K,\mathcal{S}}$ restricted to the group of $(\mathcal{S},\mathcal{T})$–units $U_{\mathcal{S},\mathcal{T}}$ of K coincides with the map $\lambda_{\mathcal{S}}$ of the previous sections. Consequently, if $\mathcal{S},\mathcal{T} \neq \emptyset$, then $CH^1_{\mathcal{S}}(K)^0_{\mathcal{T}}$ is a compact group (in the obvious topology) whose volume (with respect to the Lebesgue and counting measures on the infinite and finite sides, respectively) satisfies the remarkable equality

$$\mathrm{vol}(CH^1_{\mathcal{S}}(K)^0_{\mathcal{T}}) = \mathrm{vol}\left(\frac{\mathbb{R}X_{\mathcal{S}}}{\lambda_{\mathcal{S}}(U_{\mathcal{S},\mathcal{T}})}\right) \cdot \mid Cl(O_{K,\mathcal{S}})_{\mathcal{T}} \mid = R_{K,\mathcal{S},\mathcal{T}} \cdot h_{K,\mathcal{S},\mathcal{T}} = \pm\zeta^*_{K,\mathcal{S},\mathcal{T}}(0)\,.$$

Obviously, if K/k is a Galois extension of Galois group G and the sets $\mathcal{S}$ and $\mathcal{T}$ are G–invariant, then all the groups defined above are naturally endowed with $\mathbb{Z}[G]$–module structures and all the above morphisms and exact sequences can be viewed in the category of $\mathbb{Z}[G]$–modules.

Now, we return to conjecture $A_{\mathbb{Z}}(K/k, S, T, 1)$. As promised, we will assume that the distinguished prime $v_1 \in S$ which splits completely in K/k is a finite prime and let $S_0 := S \setminus \{v_1\}$. As before, for simplicity, we will also use S_0 and T to denote the sets of primes in K sitting above primes in S_0 and T, respectively. All the calculations that follow will happen at the level of K, so no confusion will ensue. Now, from the definitions, since $w_1 \notin S_0 \cup T$, we have

$$\Theta'_{S,T}(0) = \log|Nw_1| \cdot \Theta_{S_0,T}(0) = \deg_{K,S_0}(w_1) \cdot \Theta_{S_0,T}(0)\,.$$

The last equality combined with (19) leads to the following equivalent formulation of Conjecture 4.1.2, for fixed sets S_0 and T and arbitrary finite split prime $v \notin S_0 \cup T$.

Conjecture 4.3.1. *$BrSt(K/k, S_0, T)$. Let K/k be an abelian extension of global fields. Let S_0 be a finite, non-empty set of primes in k, containing all the infinite primes and all the primes which ramify in K/k. Let T be a finite set of primes in k, disjoint from S_0, such that $K^\times_T$ has no torsion. Then, for every prime v of k which is not in $S_0 \cup T$ and splits completely in K/k and every prime w in K sitting above v, there exists a unique $\varepsilon_w \in K^\times_T$, such that*

$$\Theta_{S_0,T}(0) \cdot (w - \deg_{K,S_0}(w) \cdot w_0) = \mathrm{div}_{K,S_0}(\varepsilon_w)\,,$$

for any (some) prime w_0 in K sitting above any (some) prime $v_0 \in S_0$.

The statement above is the Brumer-Stark conjecture for data $(K/k, S_0, T)$ and will be denoted $BrSt(K/k, S_0, T)$ in what follows. Although the original formulation due to Stark and Tate did not depend on a set T, Remark 4.1.4 above makes the equivalence between the two formulations clear.

Proposition 4.3.2. *The conjecture $BrSt(K/k, S_0, T)$ is equivalent to*

$$\Theta_{S_0,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[G]} CH^1_{S_0}(K)^0_T\,.$$

Proof. Let v, w, v_0 and w_0 be as in the statement of $BrSt(K/k, S_0, T)$. Then,

$$(w - \deg_{K,S_0}(w) \cdot w_0) \in \mathrm{Div}_{S_0}(K)^0_T.$$

So, if $\Theta_{S_0,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[G]} CH^1_{S_0}(K)^0_T$, then we have

$$\Theta_{S_0,T}(0)(w - \deg_{K,S_0}(w) \cdot w_0) = \mathrm{div}_{K,S_0}(\varepsilon_w)\,,$$

for some $\varepsilon_w \in K_T^{\times}$, by the definition of $CH^1_{S_0}(K)^0_T$. This shows that conjecture $BrSt(K/k, S_0, T)$ holds.

Now, assume that $BrSt(K/k, S_0, T)$ holds. Let us fix v_0 and w_0 as in its statement. Let $D \in \mathrm{Div}_{S_0}(K)^0_T$. The class-field theoretical interpretation of $Cl(O_{K,S_0})_T$ (see the paragraphs below (15)) combined with Tchebotarev's density theorem and exact sequence (20) for $\mathcal{S} = S_0$ and $\mathcal{T} = T$ show that there exists a prime w in K sitting above a prime v in k, such that $v \notin S_0 \cup T$, v splits completely in K/k and

$$D = (w - \deg_{K,S_0}(w) \cdot w_0) + \mathfrak{X}_0 \,,$$

for some $\mathfrak{X}_0 \in \mathbb{R}X_{S_0}$. Note that in the exact sequence (20) the image via π of the class of the degree 0 divisor $(w - \deg_{K,S_0}(w) \cdot w_0)$ in $CH^1_{S_0}(K)^0_T$ is the class of the prime w in $Cl(O_{K,S_0})_T$. Obviously, we have $\Theta_{S_0,T}(0) \cdot \mathbb{R}X_{S_0} = 0$. This is so because for every $\chi \in \widehat{G}(\mathbb{C})$, if $\chi(\Theta_{S_0,T}(0)) \neq 0$, then

$$\dim_{\mathbb{C}}(\mathbb{C}X_{S_0})^{\chi} = \mathrm{ord}_{s=0} L_{S_0,T}(\chi, s) = 0$$

(see (11) above.) Consequently, we have

$$\Theta_{S_0,T}(0) \cdot D = \Theta_{S_0,T}(0) \cdot (w - \deg_{K,S_0}(w) \cdot w_0) = \deg_{K,S_0}(\varepsilon_w)\,,$$

for some $\varepsilon_w \in K_T^{\times}$. Therefore, $\Theta_{S_0,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[G]} CH^1_{S_0}(K)^0_T$, as required. □

Corollary 4.3.3 (top change)**.** *Assume that $(K/k, S_0, T)$ satisfy the above hypotheses. Let K' be a global field, such that $K \subseteq K'$ and K'/k is abelian of Galois group G'. Assume that $(K'/k, S_0, T)$ satisfy the above hypotheses as well. Then*

$$BrSt(K'/k, S_0, T) \Rightarrow BrSt(K/k, S_0, T)\,.$$

Proof. Assume that $BrSt(K'/k, S_0, T)$ holds. Let $\widetilde{\Theta}_{S_0,T}$ be the equivariant L–function associated to $(K'/k, S_0, T)$. The inflation property of Artin L–functions implies that $\widetilde{\Theta}_{S_0,T}(0)$ maps to $\Theta_{S_0,T}(0)$ under the natural projection $\mathbb{Z}[G'] \twoheadrightarrow \mathbb{Z}[G]$. Let $D \in \mathrm{Div}_{S_0}(K)^0_T$. Let j be the natural injective $\mathbb{Z}[G]$–linear morphism $j : \mathrm{Div}_{S_0}(K)^0_T \to \mathrm{Div}_{S_0}(K')^0_T$. Under our working hypotheses, we have

$$j(\Theta_{S_0,T}(0) \cdot D) = \widetilde{\Theta}_{S_0,T}(0) \cdot j(D) = \mathrm{div}_{K',S_0}(\varepsilon')\,,$$

for some $\varepsilon' \in {K'_T}^{\times}$. However, since $j(\Theta_{S_0,T}(0) \cdot D) \in (\mathrm{Div}_{S_0}(K')^0_T)^{G(K'/K)}$, this implies that for every $\sigma \in G(K'/K)$, we have $\mathrm{div}_{K',S_0}({\varepsilon'}^{\sigma}/\varepsilon') = 0$. Since div_{K',S_0} is injective on ${K'_T}^{\times}$ (recall that ${K'_T}^{\times}$ has no $\mathbb{Z}$–torsion), this implies that $\varepsilon' \in K_T^{\times}$. Since j is injective, this implies that

$$\Theta_{S_0,T}(0) \cdot D = \mathrm{div}_{K,S_0}(\varepsilon')\,,$$

which concludes the proof of the Corollary. □

Remark 4.3.4. *The proof of the above proposition can be used to show that if K'/K is a Galois extension of global fields and S_0 and T are finite, disjoint, sets of primes in K, such that $S_\infty \subseteq S_0$ and ${K'_T}^{\times}$ has no torsion, then the natural map*

$$CH^1_{S_0}(K)^0_T \longrightarrow CH^1_{S_0}(K')^0_T$$

is an injective group morphism. In addition, if $S_{ram}(K'/K) \subseteq S_0$, then have

$$CH^1_{S_0}(K)^0_T \simeq (CH^1_{S_0}(K')^0_T)^{G(K'/K)}$$

via the above injective group morphism. This last isomorphism is a consequence of the equality $\widehat{H}^1(G(K'/K), {K'_T}^{\times}) = 0$, which can be checked without difficulty.

Next, we focus on **the number field case** of the Brumer-Stark conjecture. Note that in this case the conjecture is trivially true unless the base field k is totally real and the top field K is totally complex. Indeed, if this condition is not satisfied, then there is an infinite prime in S_0 which splits completely in K/k. Consequently, the conjecture is a particular case of Example 3.2.11 above.

Proposition 4.3.5. *Assume that k is a number field.*

(1) *Then the Brumer-Stark conjecture $BrSt(K/k, S_0, T)$ is equivalent to*

$$\Theta_{S_0,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[G]} CH^1(K)_T^0 \,,$$

where $CH^1(K)_T^0 := CH^1_{S_\infty}(K)_T^0$.

(2) *In particular, conjecture $BrSt(K/k, S_0, T)$ implies that*

$$\Theta_{S_0,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[G]} Cl(O_K)_T \,, \quad \Theta_{S_0,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[G]} Cl(O_K),$$

where $Cl(O_K)_T := Cl(O_{K,S_\infty})_T = A_{K,S_\infty,T}$.

Proof. In order to show (1), in the proof of the previous proposition pick v_0 to be an infinite prime (which is allowed, since $S_\infty \subseteq S_0$) and use the exact sequence (20) for $\mathcal{S} := S_\infty$ and $\mathcal{T} = T$.

Part (2) is a direct consequence of (1) combined with the surjective $\mathbb{Z}[G]$–module morphisms

$$CH^1(K)_T^0 \twoheadrightarrow Cl(O_K)_T \twoheadrightarrow Cl(O_K).$$

The first surjection above is the map π in exact sequence (20) for $\mathcal{S} := S_\infty$ and $\mathcal{T} := T$ and the second comes from exact sequence (15) with $S = S_\infty$. $\square$

Now, let us assume that K is a CM field and k is a totally real field. In this case, the decomposition groups in G of all the infinite primes of k are equal and of order 2. Let us denote by j the (common) generator of these groups. (j is the so-called complex conjugation k–algebra endomorphism of K.) For any $\mathbb{Z}[G]$–module M, one can split the $\mathbb{Z}[1/2][G]$–module $M[1/2] := M \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ into a direct sum $\pm$–eigenspaces with respect to the action of j in the usual way:

$$M[1/2] := M^- \oplus M^+, \quad M^- := \frac{1}{2}(1-j) \cdot M[1/2], \quad M^+ := \frac{1}{2}(1+j) \cdot M[1/2].$$

Proposition 4.3.6. *Assume that K is CM and k is totally real. Then, the following equivalence holds.*

$$\Theta_{S_0,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[G]}(CH^1(K)_T^0[1/2]) \Leftrightarrow \Theta_{S_0,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[G]} Cl(O_K)_T^- \,.$$

Proof. First, under our working hypotheses, S_∞ (the set of infinite primes in k) has to contain more than one element. Indeed, if that were not the case, then $k = \mathbb{Q}$ and K/k is unramified away from S_∞, which means that $K = \mathbb{Q}$ and therefore K would not be CM. This observation, combined with (11), implies that $\Theta_{S_0,T}(0) \in \mathbb{Z}[G]^-$. Consequently, we have an equality

$$\Theta_{S_0,T}(0) \cdot (CH^1(K)_T^0[1/2]) = \Theta_{S_0,T}(0) \cdot (CH^1(K)_T^0)^- \,.$$

Now, if we take "–"–eigenspaces (which is an exact functor) in exact sequence (20) with $\mathcal{S} = S_\infty$ and $\mathcal{T} = T$ and observe that $(\mathbb{R}X_{S_\infty})^- = 0$, we obtain an equality

$$(CH^1(K)_T^0)^- = Cl(O_K)_T^- \,,$$

which concludes the proof of the Proposition. $\square$

Proposition 4.3.7. *Assume that K is totally complex, k is totally real and let $\widetilde{K}$ denote the maximal CM subfield of K. Then, we have an equivalence*

$$\Theta_{S_0,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[G]}(CH^1(K)^0_T[1/2]) \Leftrightarrow \widetilde{\Theta}_{S_0,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[\widetilde{G}]} Cl(O_{\widetilde{K}})^-_T\,,$$

where $\widetilde{G} := G(\widetilde{K}/k)$ and $\widetilde{\Theta}_{S_0,T}$ is the equivariant L–function associated to the data $(\widetilde{K}/k, S_0, T)$.

Proof. Obviously, $\widetilde{K} = K^{\mathfrak{g}}$ (the maximal subfield of K which is fixed by $\mathfrak{g}$), where $\mathfrak{g}$ is the subgroup of G generated by the set $\{\sigma_v \cdot \sigma_{v'} \mid v, v' \in S_\infty\}$. Here, σ_v denotes the Frobenius (complex conjugation) morphism associated to v in G, for all $v \in S_\infty$. Since $\mathfrak{g}$ is a finite (elementary) 2–group, it is easy to see that the natural morphisms $CH^1(\widetilde{K})^0_T \to CH^1(K)^0_T$ and $Cl(O_{\widetilde{K}})_T \to Cl(O_K)_T$ induce isomorphisms of $\mathbb{Z}[\widetilde{G}]$–modules

$$(21) \quad CH^1(\widetilde{K})^0_T[1/2] \simeq CH^1(K)^0_T[1/2]^{\,\mathfrak{g}}\,, \qquad Cl(O_{\widetilde{K}})_T[1/2] \simeq Cl(O_K)_T[1/2]^{\,\mathfrak{g}}.$$

(The second isomorphism above is well known, and it is a consequence of the surjectivity of the norm map $N_{\mathfrak{g}} : Cl(O_K)_T[1/2] \to Cl(O_{\widetilde{K}})_T[1/2]$, which can be obtained from the class-field theoretical interpretation of $Cl(O_K)_T[1/2]$ and $Cl(O_{\widetilde{K}})_T[1/2]$ and the fact that the order of $\mathfrak{g}$ is a power of 2. The first isomorphism is a consequence of the second, exact sequence (20) for $\mathcal{S} = S_\infty$ and $\mathcal{T} = T$ and the obvious equalities $\mathbb{R}X_{S_\infty}[1/2]^{\,\mathfrak{g}} = \mathbb{R}\widetilde{X}_{S_\infty}[1/2]$, $U^{\mathfrak{g}}_{S_\infty,T} = \widetilde{U}_{S_\infty,T}$ and $H^1(\mathfrak{g}, \mathbb{R}X_{S_\infty}/\mathrm{div}_{K,S_\infty}(U_{S_\infty,T})[1/2]) = 0$. Here, $\widetilde{*}$ refers to the usual objects at the $\widetilde{K}$–level.)

Next, we observe that $CH^1(K)^0_T[1/2]^{\,\mathfrak{g}} = e_{\mathbf{1}_{\mathfrak{g}}} \cdot CH^1(K)^0_T[1/2]$. Consequently,

$$(22) \qquad CH^1(K)^0_T[1/2] = CH^1(K)^0_T[1/2]^{\,\mathfrak{g}} \oplus (1 - e_{\mathbf{1}_{\mathfrak{g}}}) \cdot CH^1(K)^0_T[1/2]\,,$$

where $e_{\mathbf{1}_{\mathfrak{g}}} := 1/|\mathfrak{g}| \sum_{\sigma\in\mathfrak{g}} \sigma$ is the usual idempotent associated to the trivial character of $\mathfrak{g}$. Also, let us note that if $\chi \in \widehat{G}(\mathbb{C})$ is non-trivial when restricted to $\mathfrak{g}$, then $\chi(\sigma_v) = 1$, for some $v \in S_\infty$. (Otherwise $\chi(\sigma_v) = -1$, for all $v \in S_\infty$ and therefore χ is trivial on the generators $\sigma_v \cdot \sigma_{v'}$ of $\mathfrak{g}$.) Consequently, we have $\chi(\Theta_{S_0,T}) = 0$, for any such character χ (see (11).) Consequently, we have

$$(1 - e_{\mathbf{1}_{\mathfrak{g}}})\Theta_{S_0,T}(0) = 0\,.$$

Now, combine the last equality with (22) and the first isomorphism in (21) to get

$$\Theta_{S_0,T}(0)\cdot CH^1(K)^0_T[1/2] = \Theta_{S_0,T}(0)\cdot CH^1(K)^0_T[1/2]^{\,\mathfrak{g}} = \widetilde{\Theta}_{S_0,T}(0)\cdot CH^1(\widetilde{K})^0_T[1/2]\,.$$

(Note that the natural surjection $\mathbb{Z}[G] \twoheadrightarrow \mathbb{Z}[\widetilde{G}]$ maps $\Theta_{S_0,T}(0)$ to $\widetilde{\Theta}_{S_0,T}(0)$ by the inflation property of Artin L–functions.) Now, we combine the last equality with Proposition 4.3.6 applied to $\widetilde{K}/k$ to conclude the proof. □

Remark 4.3.8. *If M is a $\mathbb{Z}[G]$–module, we let*

$$M[1/2] := M \otimes_{\mathbb{Z}} \mathbb{Z}[1/2], \qquad M_{(\ell)} := M \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)},$$

where $\mathbb{Z}_{(\ell)}$ denotes the localization of $\mathbb{Z}$ at its maximal ideal $\ell\mathbb{Z}$, for any prime number ℓ. We have obvious injective (diagonal) morphisms of $\mathbb{Z}[G]$–modules

$$M \hookrightarrow M[1/2] \bigoplus M_{(2)}, \qquad M[1/2] \hookrightarrow \bigoplus_{\ell\neq 2} M_{(\ell)}\,.$$

This leads to the following equalities of $\mathbb{Z}[G]$–ideals

$$\mathrm{Ann}_{\mathbb{Z}[G]}(M) = \mathrm{Ann}_{\mathbb{Z}[G]}(M[1/2]) \bigcap \mathrm{Ann}_{\mathbb{Z}[G]}(M_{(2)}),$$
$$\mathrm{Ann}_{\mathbb{Z}[G]}(M[1/2]) = \bigcap_{\ell \neq 2} \mathrm{Ann}_{\mathbb{Z}[G]}(M_{(\ell)})\,.$$

We apply this observation to the $\mathbb{Z}[G]$–module $M := CH^1(K)_T$ and combine it with Propositions 4.3.5 and 4.3.7 to conclude that if K and k are number fields (necessarily totally complex and totally real, respectively), then the Brumer-Stark conjecture $BrSt(K/k, S_0, T)$ is equivalent to

$$\Theta_{S_0,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[G]}(CH^1(K)_T)_{(2)} \ \textit{and}\ \widetilde{\Theta}_{S_0,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[\widetilde{G}]}(Cl(O_{\widetilde{K}})_T^-)_{(\ell)},\ \forall\, \ell \neq 2\,,$$

where the notations are as in Proposition 4.3.7. For a given prime ℓ (which can be 2), we refer to the statement above which corresponds to ℓ as "the ℓ–primary component of $BrSt(K/k, S_0, T)$."

The following theorem settles a refinement of the ℓ–primary component of the Brumer-Stark conjecture for number fields, for any prime $\ell \neq 2$, provided that the Iwasawa μ–invariant associated to ℓ and K vanishes and that the set S_0 contains the set S_ℓ of primes in k which divide ℓ.

Theorem 4.3.9 (Greither-Popescu [**16**])**.** *Let K/k be an abelian extension of number fields of Galois group G, with K CM and k totally real. Let S_0 be a finite set of primes in k, such that $S_\infty \cup S_{\mathrm{ram}}(K/k) \subseteq S_0$. Let $\ell \neq 2$ be a prime number. Assume that the Iwasawa μ–invariant $\mu_{K,\ell}$ associated to K and ℓ vanishes. Then, we have*

$$\Theta_{S_0 \cup S_\ell, T}(0) \in \mathrm{Fitt}_{\mathbb{Z}_{(\ell)}[G]}\left((Cl(O_K)_T^-)_{(\ell)}^\vee\right) \subseteq \mathrm{Ann}_{\mathbb{Z}_{(\ell)}[G]}(Cl(O_K)_T^-)_{(\ell)}\,.$$

for any finite set T of primes in k, disjoint from $S_0 \cup S_\ell$ and such that $K(\mu_\ell)_T^\times$ has no $\mathbb{Z}$–torsion.

We need to explain some notations in the theorem above. If M is a finite $\mathbb{Z}[G]$–module, $M_{(\ell)}^\vee := \mathrm{Hom}_{\mathbb{Z}_{(\ell)}}(M_{(\ell)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is the usual Pontrjagin dual of $M_{(\ell)}$, viewed as a $\mathbb{Z}_{(\ell)}[G]$–module with the co-variant G–action given by ${}^\sigma f(x) := f(\sigma \cdot x)$, for all $f \in M_{(\ell)}^\vee$, $x \in M_{(\ell)}$ and $\sigma \in G$. As usual, Fitt stands for the first Fitting ideal, which is a smaller, refined version of the annihilator of a finitely generated module over an arbitrary commutative ring. The inclusion of $\mathbb{Z}_{(\ell)}[G]$–ideals in the statement above is clear from the definitions and it is strict inclusion, in general.

The theorem above is a consequence of an equivariant Main Conjecture in Iwasawa theory proved in [**16**] and refining the Main Conjecture proved in [**44**]. The arguments involve the Iwasawa cyclotomic tower $K(\mu_{\ell^\infty})/K$, which is ramified at all primes above ℓ, explaining the presence of S_ℓ in the statement of the theorem. The assumption $\mu_{K,\ell} = 0$ (needed for technical reasons in our proof of the equivariant Main Conjecture) is a classical conjecture of Iwasawa, known to hold, for example, for all fields K which are abelian over $\mathbb{Q}$ and all primes ℓ, due to Ferrero-Washington and Sinnott. Of course, for abelian extensions of $\mathbb{Q}$, the following classical theorem of Stickelberger, dated 1890 (see [**38**]), settles the full Brumer-Stark conjecture in an explicit (constructive) way.

Theorem 4.3.10 (Stickelberger [**38**])**.** *Let $K := \mathbb{Q}(\mu_m)$, where $m \geq 3$ and m is either odd or divisible by 4. Let $S_0 := \{p \mid p \text{ prime}, p|m\} \cup \{\infty\}$. Let $\mathfrak{P}$ be a prime in K, which sits above a rational prime $p \notin S_0$. Let $\chi_{\mathfrak{P}} : (O_K/\mathfrak{P})^\times \to \mu_m$ be* *the unique morphism satisfying* $\chi_{\mathfrak{P}}(x) \equiv x^{(N\mathfrak{P}-1)/m} \mod \mathfrak{P}$*, for all* $x \in O_K$*. Let* $\psi : O_K/\mathfrak{P} \to \mu_p$ *be any nontrivial, additive character. Let*

$$\mathcal{G}(\chi_{\mathfrak{P}}, \psi) := \sum_{x \in O_K/\mathfrak{P}} \chi_{\mathfrak{P}}(x)\psi(x) \in \mathbb{Q}(\mu_{pm}), \quad g(\chi_{\mathfrak{P}}, \psi) := \frac{\mathcal{G}(\chi_{\mathfrak{P}}, \psi)}{\sqrt{\pm N\mathfrak{P}}}\,,$$

with the sign chosen so that $g(\chi_{\mathfrak{P}}, \psi) \in \mathbb{Q}(\mu_{4pm})$*. Then, the following are satisfied.*

1. *The element* $g(\mathfrak{P}) := g(\chi_{\mathfrak{P}}, \psi)^{w_K}$ *belongs to* K *and is independent of the choice of* ψ*. Here,* w_K *denotes the number of roots of unity in* K*.*
2. *We have* $|g(\mathfrak{P})|_v = 1$*, for all infinite primes* v *of* K*.*
3. *The* O_K*–ideal* $\mathfrak{P}^{w_K \Theta_{S_0}(0)}$ *is principal, generated by* $g(\mathfrak{P})$*.*

The elements $\mathcal{G}(\chi_{\mathfrak{P}}, \psi)$ and $g(\mathfrak{P})$ defined above are particular cases of Gauss and Jacobi sums, respectively. Statements (1) and (2) are well known properties of Gauss sums. Statement (3) is a consequence of the prime power decomposition of $\mathcal{G}(\chi_{\mathfrak{P}}, \psi)^m$, which is due to Stickelberger. The reader can consult §3 of [**11**] for very lucid proofs of all these facts.

Corollary 4.3.11. *Let* $K/\mathbb{Q}$ *be an abelian extension and assume that* $(K/\mathbb{Q}, S_0, T)$ *satisfy the hypotheses of the Brumer-Stark conjecture. Then,* $BrSt(K/\mathbb{Q}, S_0, T)$ *holds true.*

PROOF. First of all, Proposition 3.2.16, combined with Corollary 4.3.3 and the Kronecker-Weber Theorem permit us to reduce the proof to the case where K and S_0 are as in Stickelberger's Theorem. Let $p \notin S_0 \cup T$ be a prime which splits completely in $K/\mathbb{Q}$ and let $\mathfrak{P}$ be a prime in K dividing p. Since $g(\mathfrak{P})^{1/w_K} = g(\chi_{\mathfrak{P}}, \psi)$ belongs to $\mathbb{Q}(\mu_{4pm})$, which is an abelian extension of $\mathbb{Q}$, Proposition 1.2 in Chpt. IV of [**42**] combined with our Remark 3.2.4 implies that

$$g(\mathfrak{P})^{\delta_T(0)} \in K_T^\times\,.$$

Now, (2) and (3) in Stickelberger's Theorem imply that

$$\mathrm{div}_{K,S_0}(g(\mathfrak{P})^{\delta_T(0)}) = \Theta_{S_0,T}(0) \cdot \mathfrak{P}.$$

However, since $|S_0| > 1$, we have $\Theta_{S_0,T}(0) \cdot \mathfrak{P} = \Theta_{S_0,T}(0)(\mathfrak{P} - \deg_{K,S_0}(\mathfrak{P}) \cdot \infty)$, where ∞ denotes (abusively) a prime in K which sits above the unique infinite prime in $\mathbb{Q}$ (also denoted ∞.) Consequently, the last displayed equality settles conjecture $BrSt(K/\mathbb{Q}, S_0, T)$. (Set $w := \mathfrak{P}$, $v_0 := \infty$ and $\varepsilon_w := g(\mathfrak{P})^{\delta_T(0)}$ in the statement of Conjecture 4.3.1.) □

Finally, we discuss **the function field case** of the Brumer-Stark conjecture. As mentioned before, the conjecture was formulated by Tate in this case (see [**42**], Chpt. V.) and proved independently and with very different methods by Deligne-Tate in loc. cit. and Hayes in [**22**].

Theorem 4.3.12 (Deligne-Tate [**42**], Hayes [**22**])**.** *The Brumer-Stark conjecture* $BrSt(K/k, S_0, T)$ *holds true if* k *is a function field.*

Hayes's proof [**22**] is based on a wonderful explicit construction of the Brumer-Stark (Rubin-Stark) element $\varepsilon_{S,T}$ out of an $\mathfrak{f}$–torsion point of a sign-normalized, rank one Drinfeld module ρ_{v_1} with "CM" by the ring $O_{k,\{v_1\}}$ (the subring of k consisting of elements which are integral away from v_1.) Here, v_1 is an arbitrary prime in k which is not in S_0 and splits completely in K/k, $S := S_0 \cup \{v_1\}$ and $\mathfrak{f}$ is the conductor of K/k (viewed as an ideal of $O_{k,\{v_1\}}$.) Hayes's proof is the perfect function field analogue of the proof of Theorem 4.2.1 outlined above. In Hayes's proof, the role of the formal multiplicative group $\widehat{G_m}$ (which has "CM" by $\mathbb{Z}$, the ring of elements in $\mathbb{Q}$ which are integral away from the split prime $v_1 := \infty$) is taken by the Drinfeld module ρ_{v_1} (with "CM" by $O_{k,\{v_1\}}$.) In fact, the analogy runs much deeper: just the way the $\mathbb{Z}$–torsion points of $\widehat{G_m}$ generate the maximal abelian extension $\mathbb{Q}^{\mathrm{ab},+}$ of $\mathbb{Q}$ in which ∞ splits completely, the $O_{k,\{v_1\}}$–torsion points of ρ_{v_1} generate the maximal abelian extension of k in which v_1 splits completely, provided that the Dedekind domain $O_{k,\{v_1\}}$ has class-number 1. If the class-number of $O_{k,\{v_1\}}$ is not 1, then the situation is a bit more complicated (but still fully understood), as one has to generate the Hilbert-class field of $O_{k,\{v_1\}}$ first. This is in fact a rough description of explicit (abelian) class-field theory over an arbitrary function field k, due independently to Drinfeld and Hayes.

In light of our interpretation of the Brumer-Stark conjecture in terms of Arakelov class-groups, the Deligne-Tate proof can be described roughly as follows. *First,* one constructs a Picard 1–motive $\mathcal{M}_{S_0,T}$ defined over $\mathbb{F}_q$ and depending on the data $(K/k, S_0, T, \mathbb{F}_q)$, where $\mathbb{F}_q$ is the exact field of constants of k. *Second,* one uses the Lefschetz trace formula for the ℓ–adic étale cohomology of smooth, projective curves defined over $\mathbb{F}_q$, for ℓ prime, $\ell \neq p := \mathrm{char}(\mathbb{F}_q)$, to show that

$$\Theta_{S_0,T}(s) = P(q^{-s}), \quad P(t) := \det_{\mathbb{Q}_\ell[G]}(1 - \gamma_q \cdot t \mid T_\ell(\mathcal{M}_{S_0,T}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell).$$

Note that $P(t)$ is a slight variation on the G–equivariant characteristic polynomial of the q–power arithmetic Frobenius morphism γ_q acting on the ℓ–adic Tate module $T_\ell(\mathcal{M}_{S_0,T})$ of $\mathcal{M}_{S_0,T}$, for all primes ℓ, $\ell \neq p$. The polynomial P is independent on the prime $\ell \neq p$ and it turns out that $P \in \mathbb{Z}[G][t]$. For a brief description of the *third step,* we make the simplifying assumption that $\mathbb{F}_q$ is the exact field of constants of K as well. For every $n \in \mathbb{Z}_{\geq 1}$, let $K_n := K \cdot \mathbb{F}_{q^n}$, where the field compositum is viewed inside a fixed separable closure $\overline{k}$ of k. Note that $(K_n/k, S_0, T)$ satisfy the hypotheses of the Brumer-Stark conjecture, for all such n. (Note that K_n/K is unramified and $T \neq \emptyset$.) We let $K_\infty := K \cdot \overline{\mathbb{F}_q} = \cup_n K_n$ and define

$$CH^1_\emptyset(K_\infty)^0_T = \varinjlim_n CH^1_\emptyset(K_n)^0_T\,, \tag{23}$$

where the injective limit is taken with respect to the natural injections

$$CH^1_\emptyset(K_n)^0_T \hookrightarrow CH^1_\emptyset(K_m)^0_T,$$

for all n, m, with $n \mid m$ (see Remark 4.3.4.) Now, it turns out that there exists a semiabelian variety J_T defined over $\mathbb{F}_q$, whose group of $\overline{\mathbb{F}_q}$-rational points satisfies

$$J_T(\overline{\mathbb{F}_q}) = CH^1_\emptyset(K_\infty)^0_T, \qquad J_T(\overline{\mathbb{F}_q})^{\gamma_q^n = 1} = CH^1_\emptyset(K_n)^0_T, \text{ for all } n\,.$$

In fact, J_T is an extension of the Jacobian J of a smooth, projective model of K by a torus which depends on T. There are $\mathbb{Z}_\ell[G]$–linear injective morphisms

$$T_\ell(J_T) \hookrightarrow T_\ell(\mathcal{M}_{S_0,T})\,,$$

preserving the γ_q–action, for all primes ℓ. The second step and the above inclusion imply that

$$P(\gamma_q^{-1}) \cdot T_\ell(\mathcal{M}_{S_0,T}) = 0, \quad P(\gamma_q^{-1}) \cdot T_\ell(J_T) = 0, \quad \text{for all } \ell \neq p\,,$$

where $P(\gamma_q)$ is viewed inside $\mathbb{Z}[G][\gamma_q^{-1}]$, which maps naturally to the rings $\mathrm{End}(J_T)$ and $\mathrm{End}(\mathcal{M}_{S_0,T})$. Now, it is very easy to show that if an endomorphism of a semiabelian variety annihilates its ℓ-adic Tate module, for some prime ℓ different from the characteristic of its field of definition, then the endomorphism in question is trivial. Consequently, we have

$$P(\gamma_q^{-1}) \cdot CH^1_\emptyset(K_\infty)^0_T = 0\,, \quad P(\gamma_q^{-1}) \cdot CH^1_\emptyset(K_n)^0_T = 0, \text{ for all } n\,. \tag{24}$$

Now, we note that γ_q is the canonical topological generator of $\Gamma := \mathrm{Gal}(K_\infty/K)$, which is isomorphic (as a topological group) to the profinite completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$. Under the assumption that $\mathbb{F}_q$ is the exact field of constants of both K and k, we have an obvious isomorphism $G(K_\infty/k) \simeq G \times \Gamma$. Consequently, we can view $P(\gamma_q^{-1})$ as an element of $\mathbb{Z}[G(K_\infty/k)]$. It turns out that, under the natural projections $\mathbb{Z}[G(K_\infty/k)] \twoheadrightarrow \mathbb{Z}[G(K_n/k)]$, the element $P(\gamma_q^{-1})$ maps to $\Theta^{(n)}_{S_0,T}(0)$, where $\Theta^{(n)}_{S_0,T}(s)$ is the equivariant L–function associated to $(K_n/k, S_0, T)$, for all $n \in \mathbb{Z}_{\geq 1}$. We combine this fact with (23) and (24) to conclude that

$$\Theta^{(n)}_{S_0,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[G(K_n/k)]} CH^1_\emptyset(K_n)^0_T\,, \qquad \text{for all } n \in \mathbb{Z}_{\geq 1}\,.$$

Now, the proof of $BrSt(K_n/k, S_0, T)$, for all $n \in \mathbb{Z}_{\geq 1}$, is a consequence of the following, which is the function field analogue of Proposition 4.3.5.

Proposition 4.3.13. *The following statements are equivalent.*

(1) $BrSt(K_n/k, S_0, T)$ *holds true, for all* $n \in \mathbb{Z}_{\geq 1}$.
(2) $\Theta^{(n)}_{S_0,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[G(K_n/k)]} CH^1_\emptyset(K_n)^0_T$, *for all* $n \in \mathbb{Z}_{\geq 1}$.

Proof. The implication (1) $\Rightarrow$ (2) is an immediate consequence of Proposition 4.3.2 and the canonical injective morphisms of $\mathbb{Z}[G(K_n/k)]$–modules

$$j_n : CH^1_\emptyset(K_n)^0_T \hookrightarrow CH^1_{S_0}(K_n)^0_T\,,$$

sending the class in $CH^1_\emptyset(K_n)^0_T$ of the divisor $D := (\sum_{w \notin T} x_w \cdot w) \in \mathrm{Div}_\emptyset(K_n)^0_T$ to the class in $CH^1_{S_0}(K_n)^0_T$ of the divisor

$$D' := (\sum_{w \notin T \cup S_0} x_w \cdot w + \sum_{w \in S_0} (x_w \cdot \log |Nw|) \cdot w) \in \mathrm{Div}_{S_0}(K_n)^0_T.$$

Here, the sums are taken with respect to all the primes w in K_n and, as usual, we abuse notation and let S_0 and T denote the sets of primes in K_n sitting above prime in S_0 and T, respectively. A more appropriate notation would be $S_0(K_n)$ and $T(K_n)$, respectively (see below.)

The implication (2) $\Rightarrow$ (1) is proved as follows. First, we leave it to the reader to check that the cokernel of the maps $\pi_n \circ j_n$ in the diagrams

$$\begin{array}{ccccccccc} & & & & CH^1_\emptyset(K_n)^0_T & & & & \\ & & & & \downarrow{\scriptstyle j_n} & \searrow^{\pi_n \circ j_n} & & & \\ 0 & \longrightarrow & \dfrac{\mathbb{R}X_{S_0}}{\mathrm{div}_{K_n,S_0}(U_{S_0,T})} & \xrightarrow{\iota_n} & CH^1_{S_0}(K_n)^0_T & \xrightarrow{\pi_n} & Cl(O_{K_n,S_0})_T & \longrightarrow & 0\,, \end{array}$$

satisfies $\mathrm{coker}(\pi_n \circ j_n) \simeq \mathbb{Z}/d_{S_0,n}\mathbb{Z}$, for all n, where

$$d_{S_0,n} := \gcd\left([\kappa(w) : \mathbb{F}_{q^n}] \mid w \in S_0(K_n)\right).$$

Here, $\kappa(w)$ denotes the residue field of the prime w. This observation implies right away that for n "sufficiently large" (meaning n divisible by $[\kappa(w) : \mathbb{F}_q]$, for all $w \in S_0(K)$), we have $d_{S_0,n} = 1$ and therefore $(\pi_n \circ j_n)$ is surjective. Assume that (2) in the Proposition holds and let us fix an $m \in \mathbb{Z}_{\geq 1}$. We will show that $BrSt(K_m/k, S_0, T)$ holds. Let n be sufficiently large in the above sense and such that $m|n$. Since $(\pi_n \circ j_n)$ is surjective, we have an equality

$$\mathrm{Im}(\iota_{\mathrm{n}}) + \mathrm{Im}(j_n) = CH^1_{S_0}(K_n)^0_T\,.$$

Part (2) of the Proposition combined with $\Theta^{(n)}_{S_0,T}(0) \cdot \mathbb{R}X_{S_0} = 0$ implies that

$$\Theta^{(n)}_{S_0,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[G(K_n/k)]} CH^1_{S_0}(K_n)^0_T\,.$$

Now, Proposition 4.3.2 combined with Proposition 4.3.3 imply that conjecture $BrSt(K_m/k, S_0, T)$ holds true. □

We conclude this section by stating the function field analogue of Theorem 4.3.9 above, which refines the above results of Deligne and Tate and, implicitly, leads to a refinement of the Brumer-Stark conjecture for function fields.

Theorem 4.3.14 (Greither-Popescu [**15**])**.** *The following hold.*

(1) *The $\mathbb{Z}_\ell[[G(K_\infty/k)]]$–module $T_\ell(\mathcal{M}_{S_0,T})$ has projective dimension 1 and*

$$\mathrm{Fitt}_{\mathbb{Z}_\ell[[G(K_\infty/k)]]}(T_\ell(\mathcal{M}_{S_0,T})) = (P(\gamma_q^{-1})),$$

for all prime numbers ℓ.

(2) *Let $T_\ell(J_T)^* := \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(J_T), \mathbb{Z}_\ell)$ endowed with the co-variant $\mathbb{G}(K_\infty/k)$–action. Then, for all prime numbers ℓ we have*

$$P(\gamma_q^{-1}) \in \mathrm{Fitt}_{\mathbb{Z}_\ell[[G(K_\infty/k)]]}(T_\ell(J_T)^*) \subseteq \mathrm{Ann}_{\mathbb{Z}_\ell[[G(K_\infty/k)]]}(T_\ell(J_T)^*).$$

(3) *For all $n \in \mathbb{Z}_{\geq 1}$, we have*

$$\Theta^{(n)}_{S_0,T}(0) \in \mathrm{Fitt}_{\mathbb{Z}[G(K_n/k)]}(CH^1_\emptyset(K_n)^{0\ \vee}_T) \subseteq \mathrm{Ann}_{\mathbb{Z}[G(K_n/k)]} CH^1_\emptyset(K_n)^0_T\,,$$

where, for any finite $\mathbb{Z}[G(K_n/k)]$–module M, we let $M^\vee := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ and endow it with the co-variant $G(K_n/k)$–action.

Remark 4.3.15. (1) *Part (1) in Theorem 4.3.14 is the function field analogue of the Equivariant Main Conjecture mentioned (but not stated, for technical reasons) in the paragraphs after Theorem 4.3.9.*

(2) *Parts (2) and (3) of Theorem 4.3.14 are more or less immediate consequences of part (1).*

(3) *Note that, in [**15**], $CH^1_\emptyset(K)^0_T$ is denoted by $\mathrm{Pic}^0(K)_T$. This notation is justified by the fact that, indeed, $CH^1_\emptyset(K)^0_T$ is a T–modified version of the classical Picard group $\mathrm{Pic}^0(K)$. In fact, there is a canonical short exact sequence of groups linking the two:*

$$0 \longrightarrow \frac{\bigoplus_{w \in T(K)} \kappa(w)^\times}{\mathbb{F}_q^\times} \longrightarrow CH^1_\emptyset(K)^0_T \longrightarrow \mathrm{Pic}^0(K) \longrightarrow 0.$$

(4) *In recent work [**17**], we have shown that part (1) of Theorem 4.3.14 implies the Equivariant Tamagawa Number Conjecture (Conjecture 3.1.4) for all abelian extensions K/k of function fields. (In this case, the ETNC was also proved by Burns with different methods.)*

(5) *In fact, the Deligne-Tate 1–motive constructions and the techniques involved in the proof of Theorem 4.3.14 were the main inspiration for our proof of the Equivariant Main Conjecture for number fields and, implicitly, the proof of Theorem 4.3.9 above. In the number field case, we construct Iwasawa modules which behave like the ℓ–adic Tate modules $T_\ell(\mathcal{M}_{S_0,T})$ of the Picard 1–motive $\mathcal{M}_{S_0,T}$. Then, we prove the analogue of Theorem 4.3.14(1) for these Iwasawa modules , with $P(\gamma_q^{-1})$ replaced by an appropriate G–equivariant ℓ–adic L–function. The ℓ–adic L–function in question is described somewhat in detail in the next section.*

5. Gross–type (p–adic) refinements of the Rubin-Stark conjecture

With notations as in the previous sections, assume that the data $(K/k, S, T, r)$ satisfies the hypotheses in the Rubin-Stark conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$. In [**19**] and [**20**], Gross proposed a refinement of the Rubin–Stark conjecture in the case where the order of vanishing is $r = 1$. In the number field case, if the distinguished split prime $v_1 \in S$ is finite (call it $\mathfrak{p}$ and assume it sits over the rational prime p), then Gross's conjecture predicts the $\mathfrak{P}$–adic expansion of the Rubin-Stark S–unit $\varepsilon_{S,T} \in U_{S,T}$ at a prime $\mathfrak{P}$ in $S(K)$ sitting above the split prime $\mathfrak{p}$ in terms of values of derivatives of p–adic L–functions. In the function field case, Gross's conjecture was proved by Hayes in [**23**]. In the case where $k = \mathbb{Q}$ and K is imaginary, the conjecture was proved by Gross in [**19**] (see also [**21**] and [**13**].)

In the early 1990s, Gross and Tate (see [**18**]) expressed interest in formulating and hinted at a Gross–type refinement for the Rubin–Stark conjecture for arbitrary orders of vanishing r. Tan formulated and partly proved such a refinement for function fields (see [**40**].) In this section, we describe a general Gross–type refinement of the Rubin-Stark Conjecture which has emerged recently out of work of Burns, the present author, and their collaborators. In the number field case, we interpret a particular case of this conjecture in terms of special values of derivatives of p–adic L–functions. Finally, we discuss some recent results supporting this conjecture.

5.1. The set-up

As above, let us assume that the data $(K/k, S, T, r)$ satisfies the hypotheses in the Rubin-Stark conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$. We let $\varepsilon := \varepsilon_{S,T}$ denote the corresponding Rubin-Stark element. Further, assume that L/k is an abelian extension, *not necessarily finite,* such that $k \subseteq K \subseteq L$, $S_{\rm ram}(L/k) \subseteq S$ and $L_T^\times$ has no $\mathbb{Z}$–torsion. (Recall that $L_T^\times$ denotes the subgroup of $L^\times$ consisting of all elements congruent to 1 module every prime in T.) We let $\Gamma := \mathrm{Gal}(L/K)$ and $\mathcal{G} := \mathrm{Gal}(L/k)$ and $G := \mathrm{Gal}(K/k)$.

The main idea behind the conjecture we are about to describe is as follows. We assume that conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$ holds, meaning that $\varepsilon \in \Lambda_{S,T} \subseteq \mathcal{L}_{\mathbb{Z}[G]}(U_{S,T}, r)$, where $\Lambda_{S,T}$ is Rubin's lattice. Since $\varepsilon \in \mathcal{L}_{\mathbb{Z}[G]}(U_{S,T}, r)$, the constructions in §1.2 permit us to associate evaluation maps $\mathrm{ev}_{\varepsilon,\mathcal{R}}$ with values in $\mathcal{R}[G]$, to arbitrary unital commutative rings $\mathcal{R}$. The conjecture aims at interpreting the value $\Theta_{L/k,S,T}(0)$ of the equivariant L–function associated to $(L/k, S, T)$ in terms of a value of the evaluation map $\mathrm{ev}_{\varepsilon,\mathcal{R}(\Gamma)}$ associated to a certain graded ring $\mathcal{R}(\Gamma)$, which depends on Γ and will be described below.

5.2. The relevant graded rings $\mathcal{R}(\Gamma)$ and $\mathcal{R}_\Gamma$

With notations as above, we let $\mathbb{Z}[[\mathcal{G}]]$ and $\mathbb{Z}[[\Gamma]]$ denote the integral profinite group rings associated to the profinite (possibly infinite) groups $\mathcal{G}$ and Γ, respectively. If Γ and $\mathcal{G}$ are finite, then these are the usual group rings $\mathbb{Z}[\mathcal{G}]$ and $\mathbb{Z}[\Gamma]$. Otherwise,

$$\mathbb{Z}[[\Gamma]] := \varprojlim_{L'} \mathbb{Z}[G(L'/K)]\,,$$

where L' runs over the finite extensions L'/K, with $K \subseteq L' \subseteq L$ and the projective limit is taken with respect to the usual group-ring projections induced by Galois restriction at the level of Galois groups. We have a similar description of $\mathbb{Z}[[\mathcal{G}]]$.

Definition 5.2.1.

(1) *We let $I(\Gamma)$ denote the augmentation ideal of $\mathbb{Z}[[\Gamma]]$, given by*

$$I(\Gamma) := \varprojlim_{L'} I(G(L'/K))\,,$$

where $I(G(L'/K))$ is the augmentation ideal of the group ring $\mathbb{Z}[G(L'/K)]$, i.e. the kernel of the $\mathbb{Z}$–linear map $\mathrm{aug}_{\mathrm{L}'} : \mathbb{Z}[G(L'/K)] \longrightarrow \mathbb{Z}$ satisfying the property that $\mathrm{aug}_{L'}(\sigma) = 1$, for all $\sigma \in G(L'/K)$ and all L' as above.

(2) *We let I_Γ denote the Γ–relative augmentation ideal of $\mathbb{Z}[[\mathcal{G}]]$, given by*

$$I_\Gamma := \varprojlim_{L'} I_{G(L'/K)}\,,$$

where $I_{G(L'/K)}$ is the kernel of the natural projection $\mathbb{Z}[G(L'/k)] \twoheadrightarrow \mathbb{Z}[G]$, for all L'.

Now, we are ready to define the relevant graded rings $\mathcal{R}(\Gamma)$ and $\mathcal{R}_\Gamma$.

Definition 5.2.2. *We define $\mathcal{R}(\Gamma)$ and $\mathcal{R}_\Gamma$ to be the graded rings*

$$\mathcal{R}(\Gamma) := \bigoplus_{n\geq 0} I(\Gamma)^n/I(\Gamma)^{n+1}\,, \qquad \mathcal{R}_\Gamma := \bigoplus_{n\geq 0} I_\Gamma^n/I_\Gamma^{n+1}\,,$$

with the obvious addition and multiplication.

Lemma 5.2.3. *The following hold.*

(1) *For all $n \in \mathbb{Z}_{\geq 0}$, we have a direct sum decomposition*

$$I_\Gamma^n = \bigoplus_{\sigma\in G} \widetilde{\sigma}\cdot I(\Gamma)^n\,,$$

where $\widetilde{\sigma}$ is a fixed element of $\mathcal{G}$ which maps to σ via the natural surjection $\mathcal{G} \twoheadrightarrow G$.

(2) *For all $n \in \mathbb{Z}_{\geq 0}$, we have an isomorphism of $\mathbb{Z}[G]$–modules*

$$I(\Gamma)^n/I(\Gamma)^{n+1} \otimes_{\mathbb{Z}} \mathbb{Z}[G] \simeq I_\Gamma^n/I_\Gamma^{n+1}, \qquad \widehat{\iota}\otimes\sigma \to \widehat{\widetilde{\sigma}\iota}\,,$$

where $\widehat{\iota}$ is the class of $\iota \in I(\Gamma_n)^n$ in the quotient $I(\Gamma)^n/I(\Gamma)^{n+1}$, $\widetilde{\sigma}$ is defined as in (1), for all $\sigma \in G$ and $\widehat{\widetilde{\sigma}\iota}$ is the class of $\widetilde{\sigma}\iota \in I_\Gamma^n$ in the quotient $I_\Gamma^n/I_\Gamma^{n+1}$.

(3) *We have isomorphisms of graded $\mathbb{Z}[G]$–algebras*

$$i : \mathcal{R}(\Gamma)[G] \simeq \mathcal{R}(\Gamma)\otimes_{\mathbb{Z}} \mathbb{Z}[G] \simeq \mathcal{R}_\Gamma$$

where the first isomorphism is $i^{-1}_{\mathcal{R}(\Gamma)}$ of §1.2 and the second is the direct sum over $n \in \mathbb{Z}_{\geq 0}$ of the isomorphisms in (2).

Proof. (1) Obviously, it suffices to prove the analogue of (1) for $I_{G(L'/K)}$ and $I(G(L'/K))$, for all L' as above and then take a projective limit. We leave this as an exercise for the reader. Part (2) is an immediate consequence of (1). Of course, one needs to note that the element $\widehat{\widetilde{\sigma}\iota}$ defined above does not depend on the lift $\widetilde{\sigma} \in \mathcal{G}$ of $\sigma \in G$ and that $I_\Gamma^n/I_\Gamma^{n+1}$ has a natural $\mathbb{Z}[G]$–module structure given by $\sigma \cdot \widehat{\iota} := \widehat{\widetilde{\sigma}\iota}$, for all $\sigma \in G$ and $\iota \in I_\Gamma^n$. Part (3) is a direct consequence of part (2). □

Remark 5.2.4. *In what follows, we will denote by $\mathcal{R}(\Gamma)^{(n)} := I(\Gamma)^n/I(\Gamma)^{n+1}$ and $\mathcal{R}_\Gamma^{(n)} := I_\Gamma^n/I_\Gamma^{n+1}$ the degree n homogeneous components of $\mathcal{R}(\Gamma)$ and $\mathcal{R}_\Gamma$, respectively. By abuse of notation, we let*

$$\mathcal{R}(\Gamma)^{(n)}[G] := i_{\mathcal{R}(\Gamma)}(\mathcal{R}(\Gamma)^{(n)} \otimes_{\mathbb{Z}} \mathbb{Z}[G]).$$

Obviously, the map i in part (3) of the Lemma above establishes an isomorphism of $\mathbb{Z}[G]$–modules $\mathcal{R}(\Gamma)^{(n)}[G] \simeq \mathcal{R}_\Gamma^{(n)}$, for all n.

5.3. The conjecture

We assume that the data $(L/K/k, S, T, r)$ satisfies the above hypotheses. As in the set-up for the Rubin-Stark conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$, we let $(v_1, \dots, v_r)$ be an ordered r–tuple of distinct primes in S which split completely in K/k. For each $i = 1, \dots, r$, we fix a prime w_i in K sitting above v_i and consider the ordered r–tuple $W := (w_1, \dots, w_r)$.

Let $v := v_i$ and $w := w_i$, for some $i \in \{1, \dots, r\}$. We denote by $G_w(L/K)$ the decomposition group associated to w in the abelian extension L/K. Note that, because v splits completely in K/k, we have $G_w(L/K) = G_v(L/k)$. We denote by K_w the completion of K in the metric $|\cdot|_w$. In what follows, we identify $G_w(L/K)$ with the Galois group $G(L\cdot K_w/K_w)$ in the usual way. Here, $L\cdot K_w$ denotes a field compositum of K_w and L. Furthermore, we let

$$\rho_w : K_w^\times \longrightarrow G_w(L/K) \subseteq \Gamma$$

denote the composition of the local Artin reciprocity map associated to the abelian extension $L\cdot K_w/K_w$ of the local field K_w with the inclusion $G_w(L/K) \subseteq \Gamma$.

Definition 5.3.1. *Following Gross, we define $\mathbb{Z}[G]$–linear morphisms ϕ_w and ϕ_w, for all w as above:*

$$\psi_w : U_{S,T} \xrightarrow{\phi_w} \mathcal{R}(\Gamma)[G] \xrightarrow[\sim]{i} \mathcal{R}_\Gamma\,, \qquad \phi_w(u) := \sum_{\sigma\in G} (\widehat{\rho_w(u^\sigma) - 1}) \cdot \sigma^{-1},$$

for all $u \in U_{S,T}$, where $(\widehat{\rho_w(u^\sigma) - 1})$ is the class of $(\rho_w(u^\sigma) - 1)$ in

$$I(\Gamma)/I(\Gamma)^2 = \mathcal{R}(\Gamma)^{(1)} \subseteq \mathcal{R}(\Gamma).$$

Now, let us assume that the Rubin-Stark conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$ holds, i.e. the Rubin-Stark element $\varepsilon := \varepsilon_{S,T}$ lies in $\Lambda_{S,T} \subseteq \mathcal{L}_{\mathbb{Z}[G]}(U_{S,T}, r)$. Since the maps $\phi_{w_1}, \dots, \phi_{w_r}$ defined above take values in $\mathcal{R}(\Gamma)^{(1)}[G]$, Remark 2 in §1.2 implies that

$$\mathrm{ev}_{\varepsilon,\mathcal{R}(\Gamma)}(\phi_1 \wedge \cdots \wedge \phi_r) \in \mathcal{R}(\Gamma)^{(r)}[G]\,.$$

This remark, combined with the grading-preserving ring isomorphism i of Lemma 5.2.3, implies that

$$\mathrm{ev}_{\varepsilon,\mathcal{R}_\Gamma}(\psi_{w_1} \wedge \cdots \wedge \psi_{w_r}) := i(\mathrm{ev}_{\varepsilon,\mathcal{R}(\Gamma)}(\phi_{w_1} \wedge \cdots \wedge \phi_{w_r})) \in \mathcal{R}_\Gamma^{(r)} = I_\Gamma^r/I_\Gamma^{r+1}\,.$$

Definition 5.3.2. *If L/k is finite, then $\Theta_{L/k,S,T}(0)$ denotes the value of the usual $G(L/k)$–equivariant L–function associated to the data $(L/k, S, T)$. If L/k is infinite, then $\Theta_{L/k,S,T}(0)$ is the unique element in $\mathbb{Z}[[\mathcal{G}]]$ which maps to $\Theta_{L'/k,S,T}(0)$ via the projections*

$$\pi_{L/L'} : \mathbb{Z}[[\mathcal{G}]] \twoheadrightarrow \mathbb{Z}[G(L'/k)]\,,$$

for all finite field extensions L'/k, with $k \subseteq L' \subseteq L$. (Note that the inflation property of Artin L–functions ensures that such an element exists and is unique.)

Now, we are finally fully prepared to state the Gross–type refinement of the Rubin-Stark conjecture announced in the introduction.

Conjecture 5.3.3. $\widetilde{A}_{\mathbb{Z}}(L/K/k, S, T, r)$. *Under the above hypotheses, the following hold true.*

(1) $\Theta_{L/k,S,T}(0) \in I_\Gamma^r$.
(2) *Assume that the Rubin-Stark Conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$ holds. Let $\varepsilon := \varepsilon_{S,T}$ denote the corresponding Rubin–Stark element. Then, we have an equality in $I_\Gamma^r/I_\Gamma^{r+1}$:*

$$\Theta_{L/k,S,T}(0) \bmod I_\Gamma^{r+1} = \mathrm{ev}_{\varepsilon,\mathcal{R}_\Gamma}(\psi_{w_1} \wedge \cdots \wedge \psi_{w_r}).$$

A more enlightening formulation, incorporating the Rubin-Stark conjecture, can be achieved as follows. Given $(L/K/k, S, T, r)$ and the ordered r–tuple W as above, we can define a new $\mathbb{Z}[G]$–linear regulator $R_{W,\Gamma}$ on the Rubin-Stark lattice $\Lambda_{S,T}$, by setting

$$R_{W,\Gamma} : \Lambda_{S,T} \longrightarrow I_\Gamma^r/I_\Gamma^{r+1}, \quad R_{W,\Gamma}(\epsilon) := \mathrm{ev}_{\epsilon,\mathcal{R}_\Gamma}(\psi_{w_1} \wedge \cdots \wedge \psi_{w_r}), \quad \text{for all } \epsilon \in \Lambda_{S,T}.$$

Also, note that if we view $\Theta_{K/k,S,T}(s)$ as an element of the power series ring $\mathbb{C}[G][[s]]$, we have

$$\mathrm{ord}_{s=0}\Theta_{K/k,S,T}(s) \geq r \;\Leftrightarrow\; \Theta_{K/k,S,T}(s) \in s^r\mathbb{C}[G][[s]]\,.$$

Further, since the above equivalent statements hold (under our current hypotheses), we have

$$\Theta_{K/k,S,T}(s) \mod s^{r+1}\mathbb{C}[G][[s]] = \frac{1}{r!}\frac{\mathrm{d^r}}{\mathrm{ds^r}}\Theta_{K/k,S,T}(0)\,,$$

under the isomorphism of $\mathbb{C}[G]$–modules $s^r\mathbb{C}[G][[s]]/s^{r+1}\mathbb{C}[G][[s]] \simeq \mathbb{C}[G]$ which sends $\widehat{s^r \cdot f}$ to $f(0)$, for all $f \in \mathbb{C}[G][[s]]$. Consequently, the following statement holds if and only if $A_{\mathbb{Z}}(K/k, S, T, r)$ and $\widetilde{A}_{\mathbb{Z}}(L/K/k, S, T, r)$ hold simultaneously.

Conjecture 5.3.4. $A_{\mathbb{Z}}(L/K/k, S, T, r)$. *Under the above hypotheses, there exists a unique $\varepsilon := \varepsilon_{S,T}$ in the Rubin-Stark lattice $\Lambda_{S,T}$, such that the following hold.*

(1) *We have $\Theta_{K/k,S,T}(s) \in s^r\mathbb{C}[G][[s]]$ and an equality*

$$R_W(\varepsilon) = \Theta_{K/k,S,T}(s) \mod s^{r+1}\mathbb{C}[G][[s]]$$

in $\mathbb{C}[G] \simeq s^r\mathbb{C}[G][[s]]/s^{r+1}\mathbb{C}[G][[s]]$.
(2) *We have $\Theta_{L/k,S,T}(0) \in I_\Gamma^r$ and an equality*

$$R_{W,\Gamma}(\varepsilon) = \Theta_{L/k,S,T}(0) \mod I_\Gamma^{r+1}$$

in $I_\Gamma^r/I_\Gamma^{r+1}$.

Remark 5.3.5. *Note that the Rubin-Stark conjecture $A_{\mathbb{Z}}(K/k, S, T, r)$ is the particular case $L = K$ of conjecture $A_{\mathbb{Z}}(L/K/k, S, T, r)$.*

The following proposition makes the link between Conjecture $\widetilde{A}_{\mathbb{Z}}(L/K/k,S,T,r)$ above and the classical conjectures of Gross (Conjectures 4.1 and 7.6 in [**20**] and Conjecture 3.13 in [**19**].)

Proposition 5.3.6. *Assume that the data $(L/K/k,S,T,r)$ satisfies the above hypotheses and* $\mathrm{char}(k)=0$. *Then, we have the following.*

1. *If $K=k$, then $\widetilde{A}_{\mathbb{Z}}(L/K/k,S,T,r)$ is equivalent to Gross's Global Conjecture 4.1 in* [**20**].
2. *If $r=1$, then $\widetilde{A}_{\mathbb{Z}}(L/K/k,S,T,r)$ is equivalent to Gross's Global Conjecture 7.1 in* [**20**].
3. *Assume that K is CM, k is totally real and $L:=K_{p^\infty}$ is the cyclotomic $\mathbb{Z}_p$–extension of K, for some prime number p. Then $\widetilde{A}_{\mathbb{Z}}(L/K/k,S,T,1)$ is equivalent to Gross's p–adic Conjecture 3.13 of* [**19**].

Proof. The reader is advised to look up the statements of Gross's conjectures in the references cited above. He will realize that (2) is immediate and (1) is a consequence of the explicit formula for the Rubin-Stark element given in Example 3.2.11. Note that, if $K=k$, then S contains $r+1$ primes which split completely in K/k, as required in Example 3.2.11.

The proof of (3) is more subtle and will be given in detail in the next section. □

5.4. Linking values of derivatives of p–adic and global L–functions

One of the main features of Conjecture $A_{\mathbb{Z}}(L/K/k,S,T,r)$ stated above is that, under the appropriate hypotheses, it establishes a deep connection between the values of the r–th derivatives of (equivariant) global and p–adic L–functions at $s=0$. We discuss this connection below. Once this connection is made, the proof of part (3) of Proposition 5.3.6 will be clear.

Let us assume that $\mathrm{char}(k)=0$, p is a prime number and $L:=K_{p^\infty}$ is the cyclotomic $\mathbb{Z}_p$–extension of K. Note that, under these assumptions, we have $S_p\subseteq S$, where S_p denotes the set of all primes in k sitting above p. Let us fix an isomorphism of (profinite) topological groups $\Gamma\simeq\mathbb{Z}_p$ and denote by γ the topological generator of Γ which is sent to $1\in\mathbb{Z}_p$ via this isomorphism. Since Γ is (topologically) cyclic, we have group isomorphisms

$$(\mathbb{Z}_p\simeq)\ \Gamma\simeq I(\Gamma)^n/I(\Gamma)^{n+1},\qquad \sigma\to\widehat{(\sigma-1)^n}\,, \tag{25}$$

for all $\sigma\in\Gamma$ and all $n\geq 1$. (Check this for finite quotients of Γ and pass to a projective limit.) the isomorphisms above, combined with Lemma 5.2.3 lead to $\mathbb{Z}_p[G]$–linear isomorphisms

$$I_\Gamma^n/I_\Gamma^{n+1}\simeq I(\Gamma)^n/I(\Gamma)^{n+1}\otimes_{\mathbb{Z}}\mathbb{Z}[G]\simeq\mathbb{Z}_p[G]\,,\qquad \text{for all } n\geq 1\,. \tag{26}$$

Since the algebras $\mathbb{Z}_p[[\Gamma]]$ and $\mathbb{Z}_p[[\mathcal{G}]]$ have well understood structures, it would be more convenient to work with $I(\Gamma)_p$ and $I_{\Gamma,p}$ (the analogues of $I(\Gamma)$ and I_Γ in $\mathbb{Z}_p[[\Gamma]]$ and $\mathbb{Z}_p[[\mathcal{G}]]$) instead of $I(\Gamma)$ and I_Γ, respectively. Fortunately, we have $I(\Gamma)_p^n\cap\mathbb{Z}[[\Gamma]]=I(\Gamma)^n$, $I_{\Gamma,p}^n\cap\mathbb{Z}[[\mathcal{G}]]=I_{\Gamma,p}^n$ and isomorphisms

$$I(\Gamma)^n/I(\Gamma)^{n+1}\simeq I(\Gamma)_p^n/I(\Gamma)_p^{n+1}\,,\qquad I_\Gamma^n/I_\Gamma^{n+1}\simeq I_{\Gamma,p}^n/I_{\Gamma,p}^{n+1}\,,$$

for all $n \geq 1$, as it can be easily checked by using (25) and (26) above. So, from now on, we will work with the algebras $\mathbb{Z}_p[[\Gamma]]$ and $\mathbb{Z}_p[[\mathcal{G}]]$ and their corresponding augmentation ideals and view the equivariant L–value $\Theta_{L/k,S,T}(0)$ inside $\mathbb{Z}_p[[\mathcal{G}]]$.

For simplicity, we will assume from this point on that k is totally real, $e^{2\pi i/p} \in K$ (and therefore K is totally complex) and k_{p^∞} and K are linearly disjoint over k. Consequently, Galois restriction leads to the following group and ring isomorphism, respectively.

$$\mathcal{G} \simeq \Gamma \times G\,, \qquad \mathbb{Z}_p[[\mathcal{G}]] \simeq \mathbb{Z}_p[G][[\Gamma]]\,.$$

Now, we have well–known (p–adically continuous) ring isomorphisms

$$\mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[t]]\,, \qquad \mathbb{Z}_p[G][[\Gamma]] \simeq \mathbb{Z}_p[G][[t]]\,,$$

which are $\mathbb{Z}_p$–linear and $\mathbb{Z}_p[G]$–linear, respectively, and map $(\gamma - 1)$ to t. Here, t is a variable. These lead to isomorphisms at the level of augmentation ideals

$$I(\Gamma)_p^n \simeq t^n\mathbb{Z}_p[[t]]\,, \qquad I_{\Gamma,p}^n \simeq t^n\mathbb{Z}_p[G][[t]]\,,$$

for all n. Via the above ring isomorphism, the equivariant L–value $\Theta_{L/k,S,T}(0)$ can be viewed as a power series $\Theta^{(p^\infty)}_{K/k,S,T}(t)$ in the variable t, with coefficients in the group ring $\mathbb{Z}_p[G]$. Consequently,

$$\Theta_{L/k,S,T}(0) \in I_{\Gamma,p}^r \;\Leftrightarrow\; \mathrm{ord}_{t=0}\Theta^{(p^\infty)}_{K/k,S,T}(t) \geq r\,.$$

If the above equivalent statements hold true, then the above established isomorphism $I_{\Gamma,p}^r/I_{\Gamma,p}^{r+1} \simeq \mathbb{Z}_p[G]$ maps

$$\Theta_{L/k,S,T}(0) \mod I_{\Gamma,p}^{r+1} \;\to\; \frac{1}{r!}\cdot\frac{\mathrm{d}^{\mathrm{r}}}{\mathrm{dt}^{\mathrm{r}}}(\Theta^{(p^\infty)}_{K/k,S,T}(t))\mid_{t=0}\,.$$

The regulator $R_{W,\Gamma}$ defined in the previous section gives rise to the p–adic regulator

$$R_{W,p}: \Lambda_{S,T} \xrightarrow{R_{W,\Gamma}} I_{\Gamma,p}^r/I_{\Gamma,p}^{r+1} \simeq t^r\mathbb{Z}_p[G][[t]]/t^{r+1}\mathbb{Z}_p[G][[t]] \simeq \mathbb{Z}_p[G]\,.$$

In light of these considerations, we can rewrite conjecture $A_{\mathbb{Z}}(K_{p^\infty}/K/k,S,T,r)$ as follows.

Conjecture 5.4.1. $A_{\mathbb{Z},p}(K/k,S,T,r)$. *Assume that p is a prime number and $(K_{p^\infty}/K/k,S,T,r)$ satisfy the above hypotheses. Then, there exists a unique $\varepsilon := \varepsilon_{S,T}$ in the Rubin-Stark lattice $\Lambda_{S,T}$, such that the following hold.*

(1) *We have $\Theta_{K/k,S,T}(s) \in s^r\mathbb{C}[G][[s]]$ and an equality in $\mathbb{C}[G]$:*

$$R_W(\varepsilon) = \frac{1}{r!}\cdot\frac{\mathrm{d}^{\mathrm{r}}}{\mathrm{ds}^{\mathrm{r}}}(\Theta_{K/k,S,T}(s))\mid_{s=0}$$

(2) *We have $\Theta^{(p^\infty)}_{K/k,S,T}(t) \in t^r\mathbb{Z}_p[G][[t]]$ and an equality in $\mathbb{Z}_p[G]$:*

$$R_{W,p}(\varepsilon) = \frac{1}{r!}\cdot\frac{\mathrm{d}^{\mathrm{r}}}{\mathrm{dt}^{\mathrm{r}}}(\Theta^{(p^\infty)}_{K/k,S,T}(t))\mid_{t=0}\,.$$

Now, the main point is that the power series $\Theta^{(p^\infty)}_{K/k,S,T}(t)$ can be viewed in fact as a G–equivariant p–adic L–function associated to the data $(K/k,S,T)$. We clarify this connection (essentially due to Iwasawa, Deligne-Ribet, and Pi. Cassou-Noguès and Barsky) below. For that purpose, we start by fixing an embedding

$\mathbb{C} \hookrightarrow \mathbb{C}_p$ (where $\mathbb{C}_p$ denotes the completion of an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$.) Via this embedding, we view the characters $\chi \in \widehat{G}(\mathbb{C})$ as taking values in $\mathbb{C}_p$. Let

$$\omega_p, c_p : \mathrm{Gal}(K_{p^\infty}/k) \longrightarrow \mathbb{Z}_p^\times$$

denote the p–adic Teichmüller and cyclotomic characters of $\mathrm{Gal}(K_{p^\infty}/k)$, respectively. Note that, under the assumption $e^{2\pi i/p} \in K$, the character ω_p factors through $G := \mathrm{Gal}(K/k)$. Let

$$u := c_p\omega_p^{-1}(\gamma) \in 1 + p\mathbb{Z}_p.$$

Now, let us split $\Theta^{(p^\infty)}_{K/k,S,T}(t)$ into its character components

$$\Theta^{(p^\infty)}_{K/k,S,T}(t) = \sum_{\chi \in \widehat{G}(\mathbb{C})} f_\chi(t) \cdot e_\chi\,,$$

where the $f_\chi(t) := \chi(\Theta^{(p^\infty)}_{K/k,S,T}(t))$ are power series in $\mathbb{Z}_p(\chi)[[t]]$ which are uniquely determined by and uniquely determine $\Theta^{(p^\infty)}_{K/k,S,T}(t)$. The following theorem, proved by Iwasawa in the case $k = \mathbb{Q}$, gives a remarkable p–adic interpolation property of the power series f_χ.

Theorem 5.4.2 (Deligne-Ribet [**12**], Cassou-Noguès [**8**], Barsky [**1**])**.** *Under the above assumptions, we have the following equalities*

$$f_\chi(u^{1-n} - 1) = L_{S,T}(\chi^{-1}\omega_p^{1-n}, 1-n)\,,$$

for all $\chi \in \widehat{G}(\mathbb{C})$ *and all* $n \in \mathbb{Z}_{\geq 1}$.

Definition 5.4.3. *For a character* $\chi \in \widehat{G}(\mathbb{C})$*, the* (S,T)*–modified* p*–adic* L*–function* $L_{S,T,p}(\chi, s)$ *of variable* $s \in \mathbb{Z}_p$ *associated to* χ *is defined by*

$$L_{S,T,p}(\chi, s) := f_{\chi^{-1}\omega_p}(u^s - 1)\,.$$

Equivalently (apply the Theorem above), $L_{S,T,p}(\chi, s)$ *is the unique* p*-adically analytic function defined on* $\mathbb{Z}_p$ *with values in* $\mathbb{C}_p$*, such that*

$$L_{S,T,p}(\chi, 1-n) = L_{S,T}(\chi\omega_p^{-n}, 1-n)\,, \qquad \text{for all } n \in \mathbb{Z}_{\geq 1}.$$

Next, we note that if $\log_p : 1 + p\mathbb{Z}_p \to \mathbb{Z}_p$ denotes the p–adic logarithm, then

$$\frac{1}{r!} \cdot \frac{\mathrm{d}^{\mathrm{r}}}{\mathrm{dt}^{\mathrm{r}}}(f_\chi(t))\,|_{t=0} = \frac{1}{(\log_p(u))^r}\frac{1}{r!} \cdot \frac{\mathrm{d}^{\mathrm{r}}}{\mathrm{ds}^{\mathrm{r}}}(f_\chi(u^s - 1))\,|_{s=0}\,,$$

for all $\chi \in \widehat{G}(\mathbb{C})$. Consequently, an equivalent, character-by-character formulation of Conjecture $A_{\mathbb{Z},p}(K/k, S, T, r)$ is the following.

Conjecture 5.4.4. $A_{\mathbb{Z},p}(K/k, S, T, r)$. *Assume that* p *is a prime number and* $(K_{p^\infty}/K/k, S, T, r)$ *satisfy the above hypotheses. Then, there exists a unique* $\varepsilon := \varepsilon_{S,T}$ *in the Rubin-Stark lattice* $\Lambda_{S,T}$*, such that the following hold, for all the characters* $\chi \in \widehat{G}(\mathbb{C})$.

(1) *We have* $\mathrm{ord}_{s=0}L_{S,T}(\chi, s) \geq r$ *and an equality in* $\mathbb{C}$*:*

$$\chi^{-1}(R_W(\varepsilon)) = \frac{1}{r!} \cdot \frac{\mathrm{d}^{\mathrm{r}}}{\mathrm{ds}^{\mathrm{r}}}(L_{S,T}(s))\,|_{s=0}$$

(2) *We have* $\mathrm{ord}_{s=0} L_{S,T,p}(\chi, s) \geq r$ *and an equality in* $\mathbb{C}_p$*:*

$$\chi^{-1}\omega_p(R_{W,p}(\varepsilon)) = \frac{1}{(\log_p(u))^r}\frac{1}{r!}\cdot\frac{\mathrm{d^r}}{\mathrm{ds^r}}(L_{S,T,p}(\chi, s))\mid_{s=0}.$$

Remark 5.4.5. *We conclude this section by noting that the statement of conjecture* $A_{\mathbb{Z},p}(K/k, S, T, r)$ *predicts the existence of a unique* **global***, arithmetically meaningful element* ε*, whose evaluation against the global regulator* R_W *essentially equals the* r*–th derivative of the* G*–equivariant global* L*–function at* $s = 0$ *and whose evaluation against the* p*–adic regulator* $R_{W,p}$ *essentially equals the* r*–th derivative of the* G*–equivariant* p*–adic* L*–function at* $s = 0$*. Another remarkable prediction is that if the* G*–equivariant global* L*–function has order of vanishing at least* r *at* $s = 0$ *due to the presence of* r *distinct primes in* S *which split completely in* K/k*, then the* G*–equivariant* p*–adic* L*-function has the same property.*

5.5. Sample evidence

In this final section, we discuss some evidence in support of $A_{\mathbb{Z}}(L/K/k, S, T, r)$ and, implicitly, its particular cases $A_{\mathbb{Z}}(K/k, S, T, r)$ (the Rubin-Stark conjecture, which is the case $L = K$) and $A_{\mathbb{Z},p}(K/k, S, T)$ (which is the case $L = K_{p^\infty}$, for some prime p.) We start with classical evidence, in the case $r = 1$, and then move on to very recent evidence, for arbitrary values of r.

The following theorem motivated Gross to formulate his p–adic conjecture (Conjecture 3.13 in [**19**].)

Theorem 5.5.1 (Gross-Koblitz [**21**])**.** *Assume that* $(K/\mathbb{Q}, S, T, 1)$ *satisfy the above hypotheses,* K *is imaginary and* p *is the distinguished splitting prime in* $K/\mathbb{Q}$*. Then, conjecture* $A_{\mathbb{Z},p}(K/\mathbb{Q}, S, T, 1)$ *holds true.*

First, it turns out that one can reduce the problem to the case $K = \mathbb{Q}(\zeta_m)$, where m is the conductor of $K/\mathbb{Q}$. In this case, the Rubin-Stark (Brumer-Stark) element $\varepsilon_{S,T}$ is more or less a Gauss sum $g(\mathfrak{P})_T$, where $\mathfrak{P}$ is the chosen prime in $\mathbb{Q}(\zeta_m)$ which sits above p (see Theorem 4.3.10 and its Corollary.) In our current notations, Gross and Koblitz [**21**] linked $g(\mathfrak{P})_T$ to a special value of Morita's p–adic Γ–function. In [**13**], Ferrero and Greenberg had already related the values of our regulators $(\chi^{-1}\omega_p) \circ R_{W,p}$ at this special p–adic Γ-value to the first derivatives at $s = 0$ of the p–adic L–functions $L_{S,T,p}(\chi, s)$, for all $\chi \in \widehat{G}(\mathbb{C})$. The proof of the Theorem above is the combination of these two results.

Building upon his explicit, Drinfeld-module theoretic proof of the Brumer-Stark conjecture in function fields (see Theorem 4.3.12 above), Hayes proved the following.

Theorem 5.5.2 (Hayes [**23**])**.** *If* k *is a function field, then* $A_{\mathbb{Z}}(L/K/k, S, T, 1)$ *holds true.*

More recent techniques, mostly stemming from efforts to prove the Equivariant Tamagawa Number Conjecture (ETNC) for Dirichlet motives and various Equivariant Main Conjectures in Iwasawa theory, have led to the following results.

Theorem 5.5.3 (Burns [**6**])**.** *Conjecture* $A_{\mathbb{Z}}(L/K/\mathbb{Q}, S, T, r)$ *holds true.*

In [**6**], Burns shows that the ETNC for abelian extensions of a global field k implies conjecture $A_{\mathbb{Z}}(L/K/k, S, T, r)$, as long as the data $(L/K/k, S, T, r)$ satisfies the above hypotheses. (See Burns's contribution to this volume as well.) In [**5**], Burns and Greither proved the ETNC for abelian extensions of $\mathbb{Q}$, away from its 2–primary part. Its 2–primary part was proved by Flach later (see [**14**] in [**7**].) As a consequence, the theorem above follows.

Theorem 5.5.4 (Greither-Popescu [**17**])**.** *If k and K are a totally real and a CM number field, respectively, p is an odd prime and the Iwasawa μ–invariant $\mu_{K,p}$ associated to K and p vanishes, then conjecture $A_{\mathbb{Z},p}(K/k, S, T, r)$ holds true.*

This result is a consequence of our proof of an Equivariant Main Conjecture in Iwasawa theory (mentioned in the paragraph below Theorem 4.3.9 as well.) Actually, we prove the p–primary part of the more general conjecture $A_{\mathbb{Z}}(L/K/k, S, T, r)$, for all odd primes p, under the additional hypotheses that $S_p \subseteq S$. This hypothesis is automatically satisfied if $L = K_{p^\infty}$, which leads to the Theorem above.

Theorem 5.5.5 (Burns [**6**], Greither-Popescu [**17**])**.** *Assume that k is a function field. Then, conjecture $A_{\mathbb{Z}}(L/K/k, S, T, r)$ holds true.*

Burns's proof is based on the fact that the ETNC for abelian extensions of k implies conjecture $A_{\mathbb{Z}}(L/K/k, S, T, r)$ and on his proof of the ETNC for abelian extensions of function fields (see [**6**] and the references therein, as well as Burns's contribution to this volume.)

The Greither-Popescu proof is a consequence of Theorem 4.3.14(1) above (the Equivariant Main Conjecture in function fields.) Of course, as mentioned before, the ETNC for function fields is a consequence of our Theorem 4.3.14(1) as well, but one can derive the Theorem above directly, without invoking the ETNC.

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