

# N-TORSION OF BRAUER GROUPS AS RELATIVE BRAUER GROUPS OF ABELIAN EXTENSIONS

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**Abstract.** It is now known ([KS2], [Po]) that if  $F$  is a global field, then the  $n$ -torsion subgroup  ${}_n\text{Br}(F)$  of its Brauer group  $\text{Br}(F)$  equals the relative Brauer group  $\text{Br}(L_n/F)$  of an abelian extension  $L_n/F$ , for all  $n \in \mathbb{Z}_{\geq 1}$ . We conjecture that this property characterizes the global fields within the class of infinite fields which are finitely generated over their prime fields. In the first part of this paper, we make a first step towards proving this conjecture. Namely, we show that if  $F$  is a non-global infinite field, which is finitely generated over its prime field and  $\ell \neq \text{char}(F)$  is a prime number such that  $\mu_{\ell^2} \subseteq F^\times$ , then there does not exist an abelian extension  $L/F$  such that  ${}_\ell\text{Br}(F) = \text{Br}(L/F)$ . The second and third parts of this paper are concerned with a close analysis of the link between the hypothesis  $\mu_{\ell^2} \subseteq F^\times$  and the existence of an abelian extension  $L/F$  such that  ${}_\ell\text{Br}(F) = \text{Br}(L/F)$ , in the case where  $F$  is a Henselian valued field.

## INTRODUCTION

This paper is concerned with the study of certain Galois theoretic properties of the  $n$ -torsion subgroup  ${}_n\text{Br}(F)$  of the Brauer group  $\text{Br}(F)$  of a field  $F$ . More precisely, in [AS], the authors raise the question whether  ${}_n\text{Br}(F)$  is equal to the relative Brauer group  $\text{Br}(L/F) := \ker(\text{Br}(F) \rightarrow \text{Br}(L))$  of a separable algebraic extension  $L/F$ . They showed that the answer to this question is negative in general, giving an example with  $F$  a power series field over a local field, and the present paper provides a systematic way of producing such examples (see Cor. 1.6 and Prop. 2.2 and Cor. 2.3 below). On the other hand, somewhat surprisingly, the answer turns out to be positive for global fields  $F$ :

**Theorem 0.1** ([KS2]). *If  $F$  is a global field (i.e. a finite extension of  $\mathbb{Q}$  or  $\mathbb{F}_p(T)$ , where  $T$  is a variable and  $p$  is prime), then for all integers  $n \geq 2$  there exists an abelian extension (necessarily of infinite degree)  $L_n/F$ , such that  ${}_n\text{Br}(F) = \text{Br}(L_n/F)$ .*

In [KS1], this result was proved for number fields  $F$  under certain restrictions on the pair  $(n, F)$  (in particular for all  $n$  when  $F = \mathbb{Q}$ ). In [Po], the result was proved for all global function fields  $F$  of characteristic  $p$  when  $n$  is a power of  $p$ . In [KS2] the result was proved in full for all number fields and all global function fields  $F$  of characteristic  $p$  with  $(n, p) = 1$ . This, together with the result in [Po] gives Theorem 0.1.

We conjecture that the conclusion in the theorem above characterizes global fields within the class of infinite fields which are finitely generated over their prime field.

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Although we are currently unable to fully confirm this conjecture, in §1 of the present paper we make encouraging steps towards doing so by proving the following (see Cor. 1.6 below).

**Theorem 0.2.** *Let  $F$  be an infinite field which is finitely generated over its prime field and is not a global field and let  $\ell$  be a prime number,  $\ell \neq \text{char}(F)$ , such that  $\mu_{\ell^2} \subseteq F^\times$ . Then, there does not exist an abelian extension  $L/F$  such that  ${}_\ell\text{Br}(F) = \text{Br}(L/F)$ .*

As usual,  $\mu_n$  denotes the group of roots of unity of order dividing  $n$ , for all  $n \in \mathbb{Z}_{\geq 1}$ . Although at present we are unable to remove the condition  $\mu_{\ell^2} \subseteq F^\times$  in the theorem above, in §§2–3 below we study more closely the link between this condition and the existence of  $L/F$  algebraic such that  ${}_\ell\text{Br}(F) = \text{Br}(L/F)$ , in the case where  $F$  is a Henselian valued field (and therefore not finitely generated over its prime field) of residue characteristic different from  $\ell$ . Examples of Henselian valued fields include (but are not restricted to) all fields which are complete with respect to a discrete valuation, in particular all local fields and all fields of iterated power series in finitely many variables over any field. It turns out that in this case, if the  $\ell$ -rank of the value group  $\Gamma_F$  of  $F$  is at least 2 (i.e.  $\dim_{\mathbb{Z}/\ell\mathbb{Z}}(\Gamma_F/\ell\Gamma_F) \geq 2$ ) and the residue field of  $F$  is finite, then the existence of an extension  $L/F$  as above is equivalent to  $\mu_{\ell^2} \not\subseteq F^\times$  (see Prop. 2.2 below). Obviously, if the  $\ell$ -rank of  $\Gamma_F$  is equal to 1, then this is not an equivalence, as the example of local fields shows (see the introduction to §2 below). The case where the residue field of  $F$  is infinite is more complicated. We give a partial answer to the question of the existence of  $L/F$  in this case (see Prop. 2.1), which allows us to construct explicit examples of extensions  $L/F$  in the particular case where  $F = \mathbb{Q}((t))$ , for all primes  $\ell$  (see §3).

In what follows we use standard notation. In particular, if  $F$  is a field endowed with a valuation  $v$ , then we denote by  $V_v$ ,  $M_v$  and  $\overline{F}_v$  the corresponding valuation ring, maximal ideal and residue field, respectively. If  $x \in V_v$ , we denote by  $\overline{x}$  its image in the residue field  $\overline{F}_v$ . If the valuation  $v$  is discrete of rank 1, then  $\widehat{F}_v$  denotes the completion of  $F$  in the  $v$ -adic topology. For the basic properties of central simple algebras and Brauer groups used in this paper, the reader may consult [Pi] and [Se]. We will use repeatedly the fact that if  $K/F$  is a cyclic field extension,  $a \in F^\times$  and  $\sigma$  generates the Galois group  $\mathcal{G}(K/F)$ , then the cyclic  $F$ -algebra  $(K/F, \sigma, a)$  is split if and only if  $a$  belongs to the image of the norm map  $N_{K/F} : K \rightarrow F$  (see [Pi, §15.1, Lemma]).

## 1. FINITELY GENERATED FIELDS

Our main goal in this section is to prove Theorem 0.2 announced in the introduction (see Corollary 1.6 below).

**Lemma 1.1.** *Let  $\ell$  be a prime number. Let  $F \subseteq L$  be fields such that  $L$  is Galois over  $F$  with Galois group  $\mathcal{G}(L/F)$  abelian of exponent  $\ell$ . Let  $v$  be a discrete valuation on  $F$  with  $\text{char}(\overline{F}_v) \neq \ell$ , and let  $w$  be any extension of  $v$  to  $L$ . Then,*

- (i) *The ramification index  $e_{w/v} = 1$  or  $\ell$ . If  $e_{w/v} = \ell$ , then there is a field  $K$  with  $F \subseteq K \subseteq L$ ,  $[K : F] = \ell$ , and  $K$  is totally ramified over  $F$  with respect to  $w$ . If, further,  $\mu_\ell \subseteq F^\times$ , then there is  $\pi \in F$  with  $v(\pi) = 1$  and  $K = F(\sqrt[\ell]{\pi})$ .*
- (ii)  *$\overline{L}_w$  is a Galois extension of  $\overline{F}_v$  with  $\mathcal{G}(\overline{L}_w/\overline{F}_v)$  abelian of exponent dividing  $\ell$ . Furthermore, there is a field  $M$  with  $F \subseteq M \subseteq L$ ,  $\overline{M}_w = \overline{L}_w$ , with  $v$*

- inert and unramified in  $M$  (so  $\mathcal{G}(M/F) \cong \mathcal{G}(\overline{L}_w/\overline{F}_v)$ , canonically).*
- (iii) Suppose  $\mu_\ell \subseteq F^\times$ . If  $\tilde{a} \in \overline{F}_v$  with  $\sqrt[\ell]{\tilde{a}} \in \overline{L}_w$ , then there is  $a \in F^\times$  with  $v(a) = 0$  such that  $\bar{a} = \tilde{a}$  and  $\sqrt[\ell]{a} \in L$ .

*Proof.* Since  $L$  is a direct limit of finite degree extensions which satisfy the same hypotheses as  $L$ , it suffices to prove the lemma when  $[L:F] < \infty$ . Assume this. Let  $G = \mathcal{G}(L/F)$ , and let  $D$  and  $I$  be the decomposition group and the inertia subgroups of  $G$  relative to  $w$ .

(i) Since  $[L:F]$  is a power of  $\ell$ , it is prime to  $\text{char}(\overline{F}_v)$ . Hence,  $I \cong \mathbb{Z}/e_{w/v}\mathbb{Z}$ , so  $e_{w/v} = |I| = \exp(I) \mid \ell$ . Suppose  $e_{w/v} = \ell$ , and let  $I'$  be a complement of  $I$  in the elementary abelian  $\ell$ -group  $G$ . Let  $K$  be the fixed field of  $I'$ , and let  $w_0$  be the restriction of  $w$  to  $K$ . Then,  $[K:F] = |G/I'| = |I| = \ell$ . The inertia group of  $w_0/v$  is  $II'/I' = G/I'$ , so  $e_{w_0/v} = |II'/I'| = [K:F]$ . Thus,  $w_0$  is totally ramified over  $v$ , and is the unique extension of  $v$  to  $K$ . Now, suppose  $\mu_\ell \subseteq F^\times$ . By Kummer theory,  $K = F(\sqrt[\ell]{c})$  for some  $c \in F^\times$ . Take any  $\pi_v \in F^\times$  with  $v(\pi_v) = 1$ . Suppose  $\ell \mid v(c)$ , say  $v(c) = k\ell$ ; then  $K = F(\sqrt[\ell]{d})$ , where  $d = c(\pi_v^{-k})^\ell$ , with  $v(d) = 0$ . If  $\bar{d} \notin \overline{F}_v^{\times\ell}$ , then  $\sqrt[\ell]{d} \in \overline{K}_{w_0} \setminus \overline{F}_v$ , a contradiction to  $K$  being totally ramified over  $F$ . But, if  $\bar{d} \in \overline{F}_v^{\times\ell}$ , then for the ring  $R = V_v[\sqrt[\ell]{d}]$  we have  $R/M_v R \cong \overline{F}_v[x]/(x^\ell - \bar{d})$ , which is a direct sum of  $\ell$  fields by the Chinese Remainder Theorem. Therefore, the integral closure  $T$  of  $V_v$  in  $K$ , which is integral over  $R$ , must contain a distinct maximal ideal lying over each of the  $\ell$  maximal ideals of  $R$ . Localizing  $T$  with respect to each maximal ideal gives a distinct discrete valuation ring of  $K$  extending  $V_v$ . This contradicts the uniqueness of  $w_0$  extending  $v$  to  $K$ . These contradictions force  $\ell \nmid v(c)$ . Hence, writing  $1 = i\ell + jv(c)$ , we can set  $\pi = c^j(\pi_v^i)^\ell$ . Then,  $v(\pi) = 1$  and  $F(\sqrt[\ell]{\pi}) = F(\sqrt[\ell]{c^j}) = F(\sqrt[\ell]{c}) = K$ , as  $\ell \nmid j$ .

(ii) Let  $J/I$  be a complement of  $D/I$  in the elementary abelian  $\ell$ -group  $G/I$ , and let  $M$  be the fixed field of  $J$ , and  $w_1$  the restriction of  $w$  to  $M$ . The decomposition group of  $w_1$  over  $v$  is  $DJ/J = G/J$ . Hence,  $v$  is inert in  $M$ . The inertia group of  $w_1$  over  $v$  is  $IJ/J = (1)$ ; so,  $w_1$  is unramified over  $v$ . Let  $N$  be the fixed field of  $I$ , which is the inertia field of  $w$  over  $v$ . The decomposition group of  $w|_N$  over  $w_1$  is  $(D/I) \cap (J/I) = (1)$ . Hence,  $w_1$  is totally decomposed in  $N$ ; so  $\overline{M}_{w_1} = \overline{N}_{w|_N} = \overline{L}_w$ . We have  $\mathcal{G}(M/F) \cong \mathcal{G}(\overline{M}_{w_1}/\overline{F}_v) = \mathcal{G}(\overline{L}_w/\overline{F}_v)$ .

(iii) Suppose  $\mu_\ell \subseteq F^\times$  and  $\tilde{a} \in \overline{F}_v$  with  $\sqrt[\ell]{\tilde{a}} \in \overline{L}_w$ . We may assume that  $\sqrt[\ell]{\tilde{a}} \notin \overline{F}_v$ . From the isomorphism of Galois groups in (ii), there is a field  $M_0$  with  $F \subseteq M_0 \subseteq M$  and  $\overline{M}_0 = \overline{F}_v(\sqrt[\ell]{\tilde{a}})$  in  $\overline{M}_{w_1}$ . We have  $[M_0:F] = [\overline{F}_v(\sqrt[\ell]{\tilde{a}}) : \overline{F}_v] = \ell$ . By Kummer theory,  $M_0 = F(\sqrt[\ell]{b})$  for some  $b \in F^\times$ . We have  $\ell \mid v(b)$  since  $M_0$  is unramified over  $F$ ; hence, we may assume  $v(b) = 0$ . If  $\bar{b} \in \overline{F}_v^\ell$ , then  $v$  would be totally decomposed in  $M_0$ , as we saw in the proof of (i). This cannot happen since  $v$  is inert in  $M$ , and so in  $M_0$ . Hence,  $\bar{b} \notin \overline{F}_v^\ell$ . As  $\overline{F}_v(\sqrt[\ell]{b}) \subseteq \overline{M}_{0w}$  and  $[\overline{F}_v(\sqrt[\ell]{b}) : \overline{F}_v] = \ell = [\overline{M}_{0w} : \overline{F}_v]$ , we have  $\overline{F}_v(\sqrt[\ell]{b}) = \overline{M}_{0w} = \overline{F}_v(\sqrt[\ell]{\tilde{a}})$ . By Kummer theory,  $\tilde{a} = \bar{b}c^\ell$  for some  $c \in F^\times$  with  $v(c) = 0$ . Set  $a = \bar{b}c^\ell$ . Then,  $v(a) = 0$  and  $\bar{a} = \bar{b}c^\ell = \tilde{a}$  and  $F(\sqrt[\ell]{a}) = F(\sqrt[\ell]{b}) = M_0 \subseteq L$ .  $\square$

**Proposition 1.2.** *Let  $\ell$  be a prime number, and let  $F$  be a field with  $\mu_{\ell^2} \subseteq F^\times$ . Suppose that  $L$  is an abelian Galois extension field of  $F$  with  $\exp(\mathcal{G}(L/F)) = \ell$  and  ${}_\ell\text{Br}(F) = \text{Br}(L/F)$ . Let  $v$  be a discrete valuation of  $F$  with  $\text{char}(\overline{F}_v) \neq \ell$ , and let*

$w$  be any extension of  $v$  to  $L$ . Then,

- (i) If  $w$  is ramified over  $v$ , then  $\bar{L}_w = \bar{F}_v$ .
- (ii) If  $w$  is unramified over  $v$ , then  $\bar{F}_v \subseteq (\bar{L}_w)^\ell$  and  ${}_\ell\text{Br}(\bar{F}_v) = {}_\ell\text{Br}(\bar{L}_w)$ .

*Proof.* (i) Since  $w$  is ramified over  $v$ , there is  $\pi \in F^\times$  with  $v(\pi) = 1$  and  $\sqrt[\ell]{\pi} \in L$  (see Lemma 1.1(i)). Since  $\bar{L}_w$  is an  $\ell$ -Kummer extension of  $\bar{F}_v$ , if  $\bar{F}_v \subsetneq \bar{L}_w$ , there is  $\tilde{a} \in \bar{F}_v \setminus (\bar{F}_v)^\ell$  with  $\sqrt[\ell]{\tilde{a}} \in \bar{L}_w$ . By Lemma 1.1(iii), there is  $a \in F^\times$  with  $v(a) = 0$ ,  $\bar{a} = \tilde{a}$ , and  $\sqrt[\ell]{a} \in L$ . Let  $A$  be the symbol algebra  $(a, \pi/F)_\ell$ . Then  $\exp(A) = \ell^2$  since  $A^{\otimes \ell} \sim (a, \pi/F)_\ell$  and  $(a, \pi/F)_\ell$  is nonsplit, as  $\pi \notin \text{im} N_{F(\sqrt[\ell]{a})/F}$  (see the last paragraph of the introduction). (Since  $v$  is inert and unramified in  $F(\sqrt[\ell]{a})/F$ , we have  $v(\text{im}(N_{F(\sqrt[\ell]{a})/F})) \subseteq \ell\mathbb{Z}$ .) But  $A$  is split by its maximal subfield  $F(\sqrt[\ell]{a}, \sqrt[\ell]{\pi}) \subseteq L$ , so  $L$  splits  $A$ . This contradicts  $\text{Br}(L/F) = {}_\ell\text{Br}(F)$ .

(ii) Suppose  $w$  is unramified over  $v$ . Let  $\pi \in F^\times$  with  $v(\pi) = 1$ . Take any  $b \in F^\times$  with  $v(b) = 0$ , and let  $B = (b, \pi/F)_\ell$ . By hypothesis,  $(b, \pi/L)_\ell = B \otimes_F L$  is split. Since  $w(\pi) = v(\pi) = 1$ , this implies that  $b \in \text{im}(N_{L(\sqrt[\ell]{\pi})/L})$ , so  $\bar{b} \in (\bar{L}_w)^\ell$ . Thus,  $\bar{F}_v \subseteq (\bar{L}_w)^\ell$ , as asserted. Since  $\mu_{\ell^2} \subseteq \bar{F}_v^\times$ , by the Merkurjev-Suslin Theorem (see [Sr, §8]),  ${}_\ell\text{Br}(\bar{F}_v)$  is generated by symbol algebras of degree  $\ell^2$ . Let  $\tilde{C} = (\tilde{c}, \tilde{d}/\bar{F}_v)_{\ell^2}$ , and suppose  $\exp(\tilde{C}) = \ell^2$  in  $\text{Br}(\bar{F}_v)$ . Since we just proved that  $\sqrt[\ell]{\tilde{c}}, \sqrt[\ell]{\tilde{d}} \in \bar{L}_w$ , by Lemma 1.1(iii) there are  $c, d \in F^\times$  with  $v(c) = v(d) = 0$ ,  $\bar{c} = \tilde{c}$ ,  $\bar{d} = \tilde{d}$ , and  $\sqrt[\ell]{c}, \sqrt[\ell]{d} \in L$ . Let  $C = (c, d/F)_{\ell^2}$ . Let  $p := \text{char}(\bar{F}_v)$ . In what follows, if  $G$  is an abelian group,  $G'$  denotes the ‘‘prime to  $p$ -part’’ of  $G$ , i.e.  $G' := G \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ . Witt’s Theorem (see [Se, Ch. XII, Th. 2 and Ex. 3]) gives an explicit group isomorphism

$$\text{Br}(\hat{F}_v)' \cong \text{Br}(\bar{F}_v)' \oplus \text{Hom}_c(G_{\bar{F}_v}, \mathbb{Q}/\mathbb{Z})',$$

where  $\text{Hom}_c(G_{\bar{F}_v}, \mathbb{Q}/\mathbb{Z})$  denotes continuous homomorphisms (i.e. homomorphisms with open kernel in the Krull topology on the absolute Galois group  $G_{\bar{F}_v}$  of  $\bar{F}_v$ .) In the composite homomorphism

$$\text{Br}(F)' \rightarrow \text{Br}(\hat{F}_v)' \rightarrow \text{Br}(\bar{F}_v)' \oplus \text{Hom}(G_{\bar{F}_v}, \mathbb{Q}/\mathbb{Z})',$$

$[C]$  maps to  $[\tilde{C}] \in \text{Br}(\bar{F}_v)$ . Hence,  $\ell^2 = \exp(\tilde{C}) \mid \exp(C) \mid \ell^2$ , so equality holds throughout. However,  $L$  splits  $C$ , since it contains the maximal subfield  $F(\sqrt[\ell]{c}, \sqrt[\ell]{d})$  of  $C$ . This contradicts  $\text{Br}(L/F) = {}_\ell\text{Br}(F)$ . Hence, every symbol algebra of  $\bar{F}_v$  of degree  $\ell^2$  has exponent at most  $\ell$ . So,  ${}_\ell\text{Br}(\bar{F}_v) = {}_\ell\text{Br}(\bar{L}_w)$ .  $\square$

We remind the reader of the following definition (see [FJ] for more details).

**Definition 1.3.** A field  $F$  is called *Hilbertian* if for every irreducible polynomial  $f(X, Y) \in F[X, Y]$  there exist infinitely many  $x_0 \in F$ , such that the specialization  $f(x_0, Y)$  is irreducible in  $F[Y]$ .

It is well known that global fields are Hilbertian and that if  $F$  is a finite extension of a rational function field  $K(X)$  over an arbitrary field  $K$ , then  $F$  is Hilbertian (see [FJ, p. 155]). In particular, any infinite field which is finitely generated over its prime field is Hilbertian. For the following, see [FSS, proof of Thm. 2.6] as well.

**Lemma 1.4.** *Let  $k$  be a Hilbertian field, let  $F$  be a finite separable extension of  $k(t)$ , where  $t$  is transcendental over  $k$ , and let  $K$  be a cyclic Galois extension of  $F$ . Then, there is a discrete valuation  $v$  of  $F$  with an extension  $w$  to  $K$  such that  $\overline{K}_w$  is cyclic Galois over  $\overline{F}_v$  and  $[\overline{K}_w : \overline{F}_v] = [K : F]$ . Also,  $\overline{F}_v$  is a finite degree extension of  $k$ .*

*Proof.* Since  $K$  is separable over  $k(t)$ , there is  $\alpha \in K$  with  $K = k(t)(\alpha)$ . We can adjust  $\alpha$  if necessary to assure that  $\alpha$  is integral over  $k[t]$ . Let  $f = f(t, x) \in k[t][x]$  be the minimal polynomial of  $\alpha$  over  $k(t)$ . Because  $k$  is Hilbertian, there is  $a \in k$  such that  $f_a = f(a, x)$  is irreducible in  $k[x]$ . Let  $v$  be the  $(t - a)$ -adic valuation on  $k(t)$ , which has residue field  $\overline{k(t)}_v = k$ . If  $w$  is any extension of  $v$  to  $K$ , then the integrality of  $\alpha$  over the valuation ring of  $v$  implies that  $w(\alpha) \geq 0$ . The image  $\overline{\alpha}$  of  $\alpha$  in  $\overline{K}_w$  satisfies  $f_a(\overline{\alpha}) = \overline{f(\alpha)} = 0$ . Hence, by the irreducibility of  $f_a$ , we have  $[\overline{K}_w : \overline{k(t)}_v] \geq \deg(f_a) = \deg(f) = [K : k(t)]$ . Therefore,  $v$  is inert and unramified in  $K$ . Consequently, for the (unique) extension of  $v$  to  $F$ , again denoted  $v$ , we have  $v$  is inert and unramified in  $K$ . Since  $K$  is Galois over  $F$ ,  $K_w$  is Galois over  $F_v$  and  $\mathcal{G}(\overline{K}_w/\overline{F}_v) \cong \mathcal{G}(K/F)$ , as desired.  $\square$

**Theorem 1.5.** *Let  $F$  be a field which is not a finite field nor a global field, but which is finitely generated over its prime field. Let  $\ell$  be a prime number with  $\mu_\ell \subseteq F^\times$  (so  $\text{char}(F) \neq \ell$ ). Let  $\mathcal{N} = \text{im}(N_{F(\mu_{\ell^2})/F})/F^{\times\ell}$ . Suppose that there is an abelian Galois algebraic extension  $L$  of  $F$  with  ${}_\ell\text{Br}(F) = \text{Br}(L/F)$ . Let  $\mathcal{K} = (L^{\times\ell} \cap F^\times)/F^{\times\ell}$ . Then,  $\mathcal{N} \cap \mathcal{K} = (1)$ .*

*Proof.* Let  $L_1$  be the maximal  $\ell$ -primary subextension of  $F$  in  $L$ . Then,  $\text{Br}(L_1/F)$  is the  $\ell$ -primary torsion subgroup of  $\text{Br}(L/F)$ . Thus, by replacing  $L$  by  $L_1$ , we may assume that  $\mathcal{G}(L/F)$  is an  $\ell$ -primary abelian group. (This replacement does not change  $\mathcal{K}$ .) Because  $F$  is finitely generated over its prime field, it is separably generated (though not algebraic) over the prime field (see [Mat, §27.E, Cor. to Lemma 2, p. 194]). Therefore there is a subfield  $k \subseteq F$  and an element  $t \in F$  such that  $t$  is transcendental over  $k$  and  $F$  is a finite separable extension of  $k(t)$ . Since  $k$  is finitely generated over a global field,  $k$  is Hilbertian. We claim that  $\mathcal{G}(L/F)$  is actually an  $\ell$ -torsion group. For, if not, there is a field  $K$ , with  $F \subseteq K \subseteq L$  and  $K$  cyclic Galois over  $F$  with  $[K : F] = \ell^2$ . Choose valuations  $v$  for  $F$  and  $w$  for  $K$  as in Lemma 1.4. Let  $A$  be the cyclic algebra  $(K/F, \sigma, \pi_v)$ , where  $\pi_v \in F$  is a uniformizer for  $v$  and  $\sigma$  is a generator of  $\mathcal{G}(K/F)$ . Then,  $\exp(A) = \ell^2$  because  $v(N_{K/F}(K^\times)) \subseteq \ell^2\mathbb{Z}$ , as  $v$  is inert and unramified in  $K/F$ , so  $\pi_v^{\ell^2}$  is the smallest power of  $\pi_v$  lying in  $\text{im}(N_{K/F})$ . But  $L$  splits  $A$ , as  $K \subseteq L$ , contradicting the choice of  $L$ . This proves the claim.

Suppose there is a nontrivial element  $aF^{\times\ell} \in \mathcal{N} \cap \mathcal{K}$ . Then  $[F(\sqrt[\ell]{a}) : F] = \ell$  and  $F(\sqrt[\ell]{a}) \subseteq L$ . Because  $aF^{\times\ell} \in \mathcal{N}$ , the symbol algebra  $(\omega_\ell, a/F)_\ell$  is split, where  $\omega_\ell$  is a primitive  $\ell$ -th root of unity. Hence,  $\omega_\ell \in F^\times$  is a norm from  $F(\sqrt[\ell]{a})$ . Albert's Theorem (see [A, Th. 11, Ch. IX, §6]) then says that there is a field  $K \supseteq F(\sqrt[\ell]{a})$  with  $K$  cyclic Galois over  $F$  and  $[K : F] = \ell^2$ . Let  $v$  be a valuation on  $F$  and  $w$  a valuation on  $K$  as in Lemma 1.4.

Suppose first that  $v$  is ramified in  $L$ . Then, by Lemma 1.1(i), there is  $\pi \in F^\times$  with  $v(\pi) = 1$  and  $\sqrt[\ell]{\pi} \in L$ . Let  $B = (K/F, \sigma, \pi)$ . Then  $\exp(B) = \ell^2$ , just as for  $A$  above, and  $L$  contains the maximal subfield  $F(\sqrt[\ell]{a}, \sqrt[\ell]{\pi})$  of  $B$ . So,  $L$  splits  $B$ , contradicting the choice of  $L$ .

Thus,  $v$  must be unramified in  $L$ . Now, the field  $\overline{F}_v$  is finite over  $k$  so is finitely generated over its prime field, but is not a finite field. If  $\overline{F}_v$  is a global field, there is a cyclic division algebra  $C$  over  $\overline{F}_v$  of degree and exponent  $\ell^2$  which is split by  $\overline{K}_w$ , say  $C = (\overline{K}_w/\overline{F}_v, \tau, \tilde{d})$  for some  $\tilde{d} \in \overline{F}_v^\times$  and a generator  $\tau$  of  $(\overline{K}_w/\overline{F}_v)$ . If  $\overline{F}_v$  is not a global field, it is still finitely generated over its prime field. Therefore, Lemma 1.4 shows that there is a discrete valuation  $u$  on  $\overline{F}_v$  with a unique extension to  $\overline{K}_w$  such that its residue field  $\overline{K}_w$  is cyclic Galois of degree  $\ell^2$  over the  $u$ -residue field  $\overline{F}_v$  of  $\overline{F}_v$ . Choose any  $\tilde{d} \in \overline{F}_v$  with  $u(\tilde{d}) = 1$ , and again let  $C = (\overline{K}_w/\overline{F}_v, \tau, \tilde{d})$ . The same argument as for  $A$  and  $B$  above shows that  $\exp(C) = \ell^2$ . By Prop. 1.2(ii),  $\sqrt[\ell]{\tilde{d}} \in \overline{L}_{w'}$  for any extension  $w'$  of  $v$  to  $L$ . So, Lemma 1.1(iii) shows that there is  $d \in F^\times$  with  $\sqrt[\ell]{d} \in L^\times$ ,  $v(d) = 0$ , and  $\bar{d} = \tilde{d}$  in  $\overline{F}_v$ . Let  $D = (K/F, \rho, d)$ , an algebra of degree  $\ell^2$  over  $F$ . Because  $D$  specializes to  $C$  with respect to the  $v$ -adic valuation on  $F$ , we have  $\ell^2 = \exp(C) \mid \exp(D)$ . But  $L$  contains the maximal subfield  $F(\sqrt[\ell]{a}, \sqrt[\ell]{d})$  of  $D$ . Hence,  $L$  splits  $D$ , contradicting the choice of  $L$ .  $\square$

**Corollary 1.6.** *Let  $F$  be a field which is not a global or a finite field, but which is finitely generated over its prime field. Let  $\ell$  be a prime number with  $\mu_{\ell^2} \subseteq F^\times$ . Then, there is no abelian Galois extension  $L$  of  $F$  with  ${}_\ell\text{Br}(F) = \text{Br}(L/F)$ .*

*Proof.* First, we will show that  ${}_\ell\text{Br}(F) \neq (1)$ . For this, let  $F_0$  be a global subfield of  $F$ . We assert that the canonical map  $\text{res}_{F/F_0} : {}_\ell\text{Br}(F_0) \rightarrow {}_\ell\text{Br}(F)$  is non-trivial. Since  ${}_\ell\text{Br}(F_0) \neq (1)$  (see [Pi, §18.5, Thm. and Ex. 5]), this will imply that the image of this map is non-trivial, which implies that  ${}_\ell\text{Br}(F) \neq (1)$ . We prove our assertion by induction on the transcendence degree  $\text{trdeg}(F/F_0)$  of  $F$  over  $F_0$ . If  $\text{trdeg}(F/F_0) = 0$ , then  $F/F_0$  is a finite extension. We may assume that  $F/F_0$  is Galois. The structure theorem for  $\text{Br}(F)$  shows that if  $\text{res}_{F/F_0}$  is trivial, then  $F/F_0$  has local degree divisible by  $\ell$  at all but possibly one finite prime of  $F_0$  (see [Pi, loc.cit.]). However, Chebotarev's density theorem shows that the set of finite primes in  $F_0$  which have local degree 1 in  $F/F_0$  has density  $1/[F : F_0] > 0$  and it is therefore infinite. This is a contradiction. Assume that we have proved our assertion for  $\text{trdeg}(F/F_0) < n$ , for some  $n \in \mathbb{Z}_{\geq 1}$ . If  $\text{trdeg}(F/F_0) = n$ , then let  $v$  be a discrete valuation on  $F$  trivial on  $F_0$  and whose residue field  $\overline{F}_v$  is a finitely generated extension of  $F_0$  satisfying  $\text{trdeg}(\overline{F}_v/F_0) < n$ . (It is an easy exercise to show that such  $v$  exists.) Then,  $\text{res}_{\overline{F}_v/F_0}$  can be written as the composition

$$\text{res}_{\overline{F}_v/F_0} : {}_\ell\text{Br}(F_0) \xrightarrow{\text{res}_{F/F_0}} {}_\ell\text{Br}(F) \rightarrow {}_\ell\text{Br}(\overline{F}_v),$$

where the last map in the composition above is the specialization map. (We remind the reader that in this context the specialization map is the composition of the restriction  $\text{res}_{\widehat{F}_v/F} : {}_\ell\text{Br}(F) \rightarrow {}_\ell\text{Br}(\widehat{F}_v)$  with projection of  ${}_\ell\text{Br}(\widehat{F}_v)$  onto the first component in the Witt decomposition of  $\text{Br}(\widehat{F}_v)'$  described in the proof of Prop. 1.2(ii).) Since by the induction hypothesis  $\text{res}_{\overline{F}_v/F_0}$  is non-trivial, so is  $\text{res}_{F/F_0}$ . This proves our assertion.

Suppose that there were a field  $L$  as in the statement of the Corollary. Because  $\mu_{\ell^2} \subseteq F^\times$ , the  $\mathcal{N}$  of the theorem is all of  $F^\times/F^{\times\ell}$ . The condition  $\mathcal{N} \cap \mathcal{K} = (1)$  then forces  $\mathcal{K} = (1)$ , hence  $F$  has no  $\ell$ -Kummer extension in  $L$ . Therefore,  $[L:F]$  is prime to  $\ell$ ; so  ${}_\ell\text{Br}(F)$  injects into  $\text{Br}(L)$ . Since  ${}_\ell\text{Br}(F)$  is nontrivial, we have  ${}_\ell\text{Br}(F) \neq \text{Br}(L/F)$ , a contradiction.  $\square$

2. HENSELIAN VALUED FIELDS

In this section, we study more closely the relationship between the hypothesis  $\mu_{\ell^2} \subseteq F^\times$  and the existence of a field extension  $L/F$ , such that  ${}_{\ell}\text{Br}(F) = \text{Br}(L/F)$ , in the case where  $F$  is a Henselian valued field (i.e. a field endowed with a Henselian valuation  $v$ ) and  $\ell$  is a prime different from the residual characteristic  $\text{char}(\overline{F}_v)$ . Although Henselian valued fields are not finitely generated over their prime fields, they occur naturally, for example, as completions of finitely generated fields with respect to any of their discrete valuations. In particular, a local field and a field of iterated power series in finitely many variables with coefficients in any field is a Henselian valued field. Note that if  $F$  is a local field and  $n \in \mathbb{Z}_{\geq 1}$ , then  ${}_n\text{Br}(F) = \text{Br}(L_n/F)$ , where  $L_n$  is any extension of degree  $n$  of  $F$ , in particular the (Galois, cyclic) unramified extension of  $F$  of degree  $n$  ([Se, Ch. XIII, §3, Cor. 1]).

**Proposition 2.1.** *Let  $\ell$  be a prime number, and let  $k$  be a field with  $\text{char}(k) \neq \ell$ . Let  $F$  be a field with Henselian valuation  $v$  with residue field  $\overline{F}_v = k$  and value group  $\Gamma_F$ , such that  $\mu_{\ell^2} \not\subseteq F^\times$ . Let  $\{\gamma_i\}_{i \in I} \subseteq \Gamma_F$  map to a  $\mathbb{Z}/\ell\mathbb{Z}$  vector space base of  $\Gamma_F/\ell\Gamma_F$ , and choose  $\{\pi_i\}_{i \in I}$  such that  $v(\pi_i) = \gamma_i$ . Suppose there is a field  $M$  algebraic over  $k$  such that  ${}_{\ell}\text{Br}(k) = \text{Br}(M/k)$  and no subfield of  $M$  which is a cyclic Galois extension of degree  $\ell$  over  $k$  lies in a cyclic Galois extension of  $k$  of degree  $\ell^2$ . Let  $M'$  be the unramified extension of  $F$  with  $\overline{M}'_v \cong M$ , and let  $L = M'(\{\sqrt[\ell]{\pi_i}\}_{i \in I})$ . Then,  ${}_{\ell}\text{Br}(F) = \text{Br}(L/F)$ .*

*Proof.* The Henselian valuation  $v$  on  $F$  yields a direct sum decomposition for the  $\ell$ -primary component  $\text{Br}(F)(\ell)$  of  $\text{Br}(F)$

$$\text{Br}(F)(\ell) \cong \text{Br}(k)(\ell) \oplus \text{Hom}_c(G_k, \Delta/\Gamma_F)(\ell) \oplus T, \quad (*)$$

where  $G_k$  is the absolute Galois group of  $k$ ;  $\Delta = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_F$ ;  $\text{Hom}_c$  denotes the group of continuous homomorphisms (where  $G_k$  has the profinite group topology and  $\Delta/\Gamma_F$  the discrete topology). Since  $\mu_{\ell^2} \not\subseteq F^\times$ ,  $T$  has the following description: If  $\mu_{\ell} \subseteq F^\times$ , then, after the index set  $I$  is given some arbitrary total ordering,  $T$  is the  $\mathbb{Z}/\ell\mathbb{Z}$ -vector space with base consisting of the (Brauer classes of the)  $\ell$ -symbol algebras  $(\pi_i, \pi_j/F)_{\ell}$  for all  $i < j$  in  $I$ . If  $\mu_{\ell} \not\subseteq F^\times$ , then  $T = (0)$ . See [ASW, §3, Thm. 3.2 and Prop. 3.5] for the decomposition above as well as the description of  $T$ . This decomposition of  $\text{Br}(F)(\ell)$  is compatible with the scalar extension to  $L$ , in that there is a commutative diagram (see [ASW, Prop. 3.3 (a)–(b)]):

$$\begin{array}{ccccccc} \text{Br}(F)(\ell) & \longrightarrow & \text{Br}(k)(\ell) & \oplus & \text{Hom}_c(G_k, \Delta/\Gamma_F)(\ell) & \oplus & T \\ \text{res} \downarrow & & \text{res} \downarrow & & \text{can} \downarrow & & 0 \downarrow \\ \text{Br}(L)(\ell) & \longrightarrow & \text{Br}(M)(\ell) & \oplus & \text{Hom}_c(G_M, \Delta/\Gamma_L)(\ell) & \oplus & T' \end{array}$$

Here, the first and second vertical maps are restriction (i.e., extension of scalars), the third vertical map is the one induced by inclusion  $G_M \rightarrow G_k$  and the canonical epimorphism  $\Delta/\Gamma_F \rightarrow \Delta/\Gamma_L$ , and the last vertical map is zero, since  $L$  splits each generator of  $T$ . Note that  $L$  is a totally ramified extension of  $M'$ , with  $\Gamma_L = \Gamma_F + \sum_{i \in I} \frac{1}{\ell} \gamma_i$ . Hence,  $\Gamma_L/\Gamma_F = \frac{1}{\ell} \Gamma_F/\Gamma_F$ , which is the  $\ell$ -torsion subgroup of  $\Delta/\Gamma_F$ .

For an element  $[A] \in \text{Br}(F)(\ell)$  write its components in the direct sum decomposition above as  $([B], \chi, [S])$ . Suppose that  $\exp(A) = \ell$ . Then  $\exp(B) \mid \ell$ ; so,  $M$

splits  $B$ , by hypothesis. Since  $\exp(\chi) \mid \ell$ , we have  $\text{im}(\chi) \subseteq \Gamma_L/\Gamma_F$ ; so,  $\text{can}(\chi) = 0$ . The commutative diagram shows that  $L$  splits  $A$ . Thus,  ${}_\ell\text{Br}(F) \subseteq \text{Br}(L/F)$ .

For the reverse inclusion, suppose now instead that  $L$  splits  $A$ . We may assume that  $\exp(A) = \ell^2$ . The commutative diagram shows that  $[B] \in \text{Br}(M/F) = {}_\ell\text{Br}(F)$ . Also,  $\exp(S) \mid \ell$ . Therefore,  $\exp(\chi) = \ell^2$ . Consider the fixed field  $K$  of  $\ker(\chi)$ . Then,  $K$  is an abelian Galois field extension of  $k$  of exponent  $\ell^2$ . Let  $\chi'$  be the image of  $\chi$  in  $\text{Hom}(G_k, \Delta/\Gamma_L)$ , and let  $N$  be the fixed field of  $\ker(\chi')$ . Then,  $k \subseteq N \subseteq K$  and  $N$  is the smallest subfield of  $K$  containing  $k$  such that  $\exp(\mathcal{G}(K/N)) = \ell$ . Because  $\text{can}(\chi) = 0$ , the restriction of  $\chi'$  to  $G_M$  is trivial; that is,  $M \cdot N = M$ , so  $N \subseteq M$ . Because  $\mathcal{G}(K/k)$  has exponent  $\ell^2$  there is a field  $K_0$  with  $k \subseteq K_0 \subseteq K$  and  $K_0$  cyclic of degree  $\ell^2$  over  $k$ . Let  $N_0 = K_0 \cap N$ , which is the subfield of  $K_0$  of degree  $\ell$  over  $k$ . We have  $N_0 \subseteq N \subseteq M$  and  $N_0$  lies in the cyclic extension  $K_0$  of degree  $\ell^2$  over  $k$ . This contradicts the hypothesis on  $M$ . Thus,  $\text{Br}(L/F) \subseteq {}_\ell\text{Br}(F)$ , completing the proof.  $\square$

**Proposition 2.2.** *Let  $\ell$  be a prime number, and let  $F$  be a field with Henselian valuation  $v$ , with residue field  $\overline{F}_v$  and value group  $\Gamma_F$ . Suppose  $\overline{F}_v$  is a finite field with  $\text{char}(\overline{F}_v) \neq \ell$  and  $\dim_{\mathbb{Z}/\ell\mathbb{Z}}(\Gamma_F/\ell\Gamma_F) \geq 2$ .*

- (i) *If  $\mu_{\ell^2} \subseteq F^\times$ , then  ${}_\ell\text{Br}(F) \neq \text{Br}(L/F)$  for any field  $L$  algebraic over  $F$ .*
- (ii) *If  $\mu_{\ell^2} \not\subseteq F^\times$ , then  ${}_\ell\text{Br}(F) = \text{Br}(L/F)$  for some abelian exponent  $\ell$  Galois extension  $L$  of  $F$ .*

*Proof.* (i) Since  $v$  is Henselian, we have the direct sum decomposition of  $\text{Br}(F)(\ell)$  as in (\*), where  $\overline{F}_v := k$  and  $T$  is generated by certain totally ramified symbol algebras of exponent  $\ell^r$ , where  $r$  is maximal such that  $\mu_{\ell^r} \subseteq F^\times$ . Of course,  $\text{Br}(k) = (0)$ , as  $k$  is finite. Suppose there were a field  $L$  algebraic over  $F$  with  ${}_\ell\text{Br}(F) = \text{Br}(L/F)$ . Since  $v$  is Henselian,  $v$  has a unique extension to  $L$ , which we again call  $v$ , and  $v$  on  $L$  is also Henselian. So, there is a decomposition of  $\text{Br}(L)(\ell)$  like (\*) for  $\text{Br}(F)(\ell)$ . Now, take any  $\pi, \rho \in F^\times$  such that  $v(\pi)$  and  $v(\rho)$  are  $\mathbb{Z}/\ell\mathbb{Z}$ -linearly independent in  $\Gamma_F/\ell\Gamma_F$ , and let  $A := (\pi, \rho/F)_\ell$ . Since  $v$  is indecomposed in  $F(\sqrt[\ell]{\pi})/F$ , we have  $v(\text{im}(N_{F(\sqrt[\ell]{\pi})/F})) \subseteq \ell\Gamma_{F(\sqrt[\ell]{\pi})} = \langle v(\pi) \rangle + \ell\Gamma_F$ . This contains  $v(\rho^j)$  if and only if  $\ell \mid j$ . Hence  $\exp(A) = \ell$ . Since  $L$  splits  $A$ ,  $v(\pi)$  and  $v(\rho)$  must be  $\mathbb{Z}/\ell\mathbb{Z}$ -linearly dependent in  $\Gamma_L/\ell\Gamma_L$ . For, otherwise the same argument as over  $F$  would show that  $A \otimes_F L$  has exponent  $\ell$ . Thus, there is  $s \in F$  with  $v(s) \in \ell\Gamma_L$  but  $v(s) \notin \ell\Gamma_F$ . Write  $s = uy^\ell$  with  $u, y \in L^\times$  and  $v(u) = 0$ .

Now, let us choose any  $\tilde{a} \in k^\times \setminus k^{\times\ell}$ , and any  $a \in F^\times$  with  $v(a) = 0$  and  $\bar{a} = \tilde{a}$ . Let  $B := (a, s/F)_{\ell^2}$ . Because  $v$  is inert (i.e. indecomposed and unramified) in  $F(\sqrt[\ell^2]{\tilde{a}})/F$ , we have  $v(\text{im}(N_{F(\sqrt[\ell^2]{\tilde{a}})/F})) \subseteq \ell^2\Gamma_F$ . But, since  $v(s) \notin \ell\Gamma_F$ , we have  $\ell v(s) \notin \ell^2\Gamma_F$ , hence,  $\exp(B) = \ell^2$ .

Suppose that  $\sqrt[\ell]{\tilde{a}} \in \overline{L}_v$ . Then,  $\sqrt[\ell]{\tilde{a}} \in L$ , by Hensel's Lemma. So, in  $\text{Br}(L)(\ell)$  we have  $B \otimes_F L \sim (\sqrt[\ell]{\tilde{a}}, uy^\ell/L)_\ell \sim (\sqrt[\ell]{\tilde{a}}, u/L)_\ell$ . In the isomorphism like (\*) for  $\text{Br}(L)(\ell)$ ,  $(\sqrt[\ell]{\tilde{a}}, u/L)_\ell$  has image  $(\sqrt[\ell]{\tilde{a}}, \bar{u}/\overline{L}_v)_\ell$  in  $\text{Br}(\overline{L}_v)$ . However, since  $\overline{L}_v$  is algebraic over a finite field,  $\text{Br}(\overline{L}_v) = (0)$ . Hence  $L$  splits  $B$ , which contradicts the hypothesis on  $L$ . Therefore,  $\sqrt[\ell]{\tilde{a}} \notin \overline{L}_v$ ; hence,  $\overline{L}_v(\sqrt[\ell]{\tilde{a}})$  is the unique extension of  $\overline{L}_v$  of degree  $\ell$ .

We claim: For any  $x \in L^\times$ , if  $v(x) \in \ell\Gamma_L$ , then  $x = a^i c^\ell$  for some  $i \in \mathbb{Z}$  and  $c \in L^\times$ . Indeed,  $x = bd^\ell$  for some  $b \in L^\times$  with  $v(b) = 0$ . From Kummer theory, we have  $\bar{b} = \bar{a}^i \bar{y}^\ell$  for some  $y \in L^\times$  with  $v(y) = 0$ . Then  $b/a^i y^\ell = 1 + m$  for some  $m \in L$

with  $v(m) > 0$  (or  $m = 0$ ). By Hensel's Lemma,  $1 + m = z^\ell$  for some  $z \in L^\times$ . Thus,  $x = a^i(yzd)^\ell$ , as claimed.

Now, let  $C := (a, \pi/F)_\ell$ . Since  $v$  is inert in  $F(\sqrt[\ell]{a})/F$ , and  $v(\pi) \notin \ell\Gamma_F$ , we have  $\exp(C) = \ell$ . Hence  $L$  splits  $C$ . So, as  $v$  is inert in  $L(\sqrt[\ell]{a})/L$ , we must have  $v(\pi) \in v(\text{im}(N_{L(\sqrt[\ell]{a})/L})) \subseteq \ell\Gamma_L$ . The claim above shows that  $\pi = a^i c^\ell$  for some  $c \in L^\times$ . The same argument as for  $\pi$  shows that  $\rho = a^j e^\ell$  for some  $e \in L^\times$ . Let  $D = (\pi a^{-i}, \rho a^{-j}/F)_{\ell^2}$ . Since  $v(\pi a^{-i}) = v(\pi)$  and  $v(\rho a^{-j}) = v(\rho)$ , the same argument as for  $A$  above shows that  $\exp(D) = \ell^2$ . However,  $D \otimes_F L \sim (c^\ell, e^\ell/L)_{\ell^2}$ , which is clearly trivial in  $\text{Br}(L)$ . Therefore  $D$  is split by  $L$ , which contradicts the choice of  $L$ .

(ii) This is a direct consequence of Proposition 2.1. Indeed, since  $k := \overline{F}_v$  is finite, we have  ${}_\ell\text{Br}(k) = \text{Br}(k) = 0$ , so in the hypotheses of Proposition 2.1 we may take  $M/k$  to be the trivial extension.  $\square$

**Corollary 2.3.** *Let  $F = k((t))$ , where  $k$  is a local field. Let  $\ell$  be a prime number with  $\text{char}(\overline{k}) \neq \ell$ .*

- (i) *If  $\mu_{\ell^2} \subseteq F^\times$ , then  ${}_\ell\text{Br}(F) \neq \text{Br}(L/F)$  for any field  $L$  algebraic over  $F$ .*
- (ii) *If  $\mu_{\ell^2} \not\subseteq F^\times$ , then  ${}_\ell\text{Br}(F) = \text{Br}(L/F)$  for some abelian Galois extension  $L$  of  $F$  with  $\ell$ -torsion Galois group.*

*Proof.* Let  $w$  be the usual complete discrete, hence Henselian, valuation on  $k$ . We use here the valuation  $v$  on  $F$  given by  $v(\sum_{i \geq N} c_i t^i) = (w(c_j), j)$ , where  $j$  is minimal

such that  $c_j \neq 0$ . Then,  $\overline{F}_v = \overline{k}_w$ , which is a finite field, and  $\Gamma_F = \mathbb{Z} \times \mathbb{Z}$ , which has  $\ell$ -rank 2. Note that  $v$  is the composite valuation built from the complete discrete (so Henselian)  $t$ -adic valuation  $u$  on  $F$  (given by  $u(\sum_{i \geq N} c_i t^i) = \min\{j \mid c_j \neq 0\}$ ) and

the valuation  $w$  on the residue field  $\overline{F}_u = k$ , cf. [B, Ch. VI, §4.1]. Because  $u$  and  $w$  are each Henselian,  $v$  is also Henselian, by [EP, p. 90, Cor. 4.1.4]. With this  $v$ , Cor. 2.3 follows immediately from Prop. 2.2.  $\square$

**Remark.** Cor. 2.3(i) in the case  $\ell = 2$  is essentially the example discussed in [AS].

### 3. A CONCRETE EXAMPLE

In this section we show that the Henselian valued field  $F := \mathbb{Q}((t))$  (rank one discrete valuation of uniformizer  $t$  and residue field  $\mathbb{Q}$ ) satisfies the hypotheses of Prop. 2.1, for all prime numbers  $\ell$ . This will allow us to construct explicit Galois extensions  $L_\ell/\mathbb{Q}((t))$  for which  ${}_\ell\text{Br}(\mathbb{Q}((t))) = \text{Br}(L_\ell/\mathbb{Q}((t)))$ , for all primes  $\ell$ .

The following is a refinement of the main result in [KS1] for base-field  $\mathbb{Q}$ .

**Proposition 3.1.** *Let  $\ell$  be a prime number. There exists an abelian Galois extension  $L$  of  $\mathbb{Q}$  of exponent  $\ell$  such that*

- (i)  ${}_\ell\text{Br}(\mathbb{Q}) = \text{Br}(L/F)$ ; and
- (ii) *no cyclic subextension of  $L/\mathbb{Q}$  of degree  $\ell$  lies in a cyclic Galois extension of  $\mathbb{Q}$  of degree  $\ell^2$ .*

*Proof.* Recall the following fact, which we will use frequently in the proof: Let  $\mathbb{Q} \subseteq M \subseteq K$  be fields with  $K$  Galois over  $\mathbb{Q}$  and  $[K:\mathbb{Q}] < \infty$ . Let  $p$  be any prime number. Then,  $p$  splits completely in  $M$  if and only if  $p$  splits completely in the

normal closure of  $M$  over  $\mathbb{Q}$ . In order to see this, let  $P$  be a prime of  $K$  lying over  $p$ , and let  $D_P$  be the decomposition field of  $P$  over  $p$ . Now, take into account that for any  $\sigma \in \mathcal{G}(K/\mathbb{Q})$ , the decomposition field  $D_{P^\sigma}$  of  $P^\sigma$  over  $p$  satisfies  $D_{P^\sigma} = \sigma(D_P)$  and that  $p$  splits completely in  $M$  if and only if  $M \subseteq D_{P^\sigma}$ , for all  $\sigma \in \mathcal{G}(K/\mathbb{Q})$  (see [Mar, Ch. 4, p. 108 and Thm. 29(i), p. 104]).

*Case I.* Assume  $\ell$  is odd.

For any prime number  $p$  with  $p \equiv 1 \pmod{\ell}$  let  $L^{(p)}$  denote the unique subfield of  $\mathbb{Q}(\mu_p)$  with  $[L^{(p)}:\mathbb{Q}] = \ell$ . We will need the following ‘‘Reciprocity Lemma.’’

**Lemma 3.2.** *Let  $p$  and  $q$  be distinct prime numbers with  $p \equiv 1 \pmod{\ell}$ . Then,  $q$  splits completely in  $L^{(p)}$  iff  $p$  splits completely in  $\mathbb{Q}(\sqrt[\ell]{q})$ .*

*Proof.*  $q$  splits completely in  $L^{(p)} \Leftrightarrow [\mathbb{F}_q(\mu_p):\mathbb{F}_q] \mid (p-1)/\ell \Leftrightarrow$  the order of  $q$  in  $(\mathbb{Z}/p\mathbb{Z})^\times$  divides  $(p-1)/\ell \Leftrightarrow q$  is an  $\ell$ -th power in  $(\mathbb{Z}/p\mathbb{Z})^\times \Leftrightarrow x^\ell - q$  has a root in  $\mathbb{Z}/p\mathbb{Z} \Leftrightarrow x^\ell - q$  factors into linear factors in  $\mathbb{Z}/p\mathbb{Z}[x] \Leftrightarrow p$  splits completely in  $\mathbb{Q}(\sqrt[\ell]{q})$ .  $\square$

We return to the proof of Prop. 3.1.

Step 1. We claim: There is a prime number  $p_1$  satisfying

- (a)  $p_1 \equiv 1 \pmod{\ell}$  (i.e.,  $p_1$  splits completely in  $\mathbb{Q}(\mu_\ell)$ );
- (b)  $p_1 \not\equiv 1 \pmod{\ell^2}$  (i.e.,  $p_1$  does not split completely in  $\mathbb{Q}(\mu_{\ell^2})$ );
- (c)  $\ell$  is inert in  $L^{(p_1)}$ .

Note that by the Reciprocity Lemma condition (c) is equivalent to the condition that  $p_1$  does not split completely in  $\mathbb{Q}(\sqrt[\ell]{\ell})$ . By the fact noted at the beginning of the proof, this is equivalent to

- (c')  $p_1$  does not split completely in  $\mathbb{Q}(\mu_\ell, \sqrt[\ell]{\ell})$ .

We show that these ‘‘Chebotarev conditions’’ are compatible. Consider the field  $K = \mathbb{Q}(\mu_{\ell^2}, \sqrt[\ell]{\ell})$ . We have  $\mathbb{Q}(\mu_{\ell^2}) \neq \mathbb{Q}(\mu_\ell, \sqrt[\ell]{\ell})$ , since the second field is not abelian Galois over  $\mathbb{Q}$ . Therefore,  $\mathcal{G}(K/\mathbb{Q}(\mu_\ell)) \cong \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$ . Choose an element  $\sigma$  of order  $\ell$  in this group which lies neither in  $\mathcal{G}(K/\mathbb{Q}(\mu_{\ell^2}))$  nor in  $\mathcal{G}(K/\mathbb{Q}(\mu_\ell, \sqrt[\ell]{\ell}))$ . Let  $p_1$  be any prime number whose Frobenius class is that of  $\sigma$ . Then, there is a prime  $P$  of  $K$  lying over  $p_1$  such that the fixed field  $D$  of  $\sigma$  is the decomposition field of  $P$  over  $p_1$ . By the fact noted at the beginning of the proof,  $p_1$  satisfies conditions (a), (b), and (c'). Set  $L_1 = L^{(p_1)}$ . Suppose there were a cyclic Galois extension  $M$  of  $\mathbb{Q}$  of degree  $\ell^2$  with  $L_1 \subseteq M$ . Because  $L_1 \cdot \mathbb{Q}_{p_1}$  is totally and tamely ramified over  $\mathbb{Q}_{p_1}$  of degree  $\ell$ ,  $M \cdot \mathbb{Q}_{p_1}$  must be cyclic of degree  $\ell^2$  over  $\mathbb{Q}_{p_1}$ , and is necessarily totally ramified over  $\mathbb{Q}_{p_1}$ . But, this cannot occur, as  $\mu_{\ell^2} \not\subseteq \mathbb{Q}_{p_1}$ . So, there is no such  $M$ .

Step 2. Let  $q_2$  be a prime number different from  $p_1$  and from  $\ell$ . We claim: There exists a prime number  $p_2$  satisfying

- (a<sub>2</sub>)  $p_2 \equiv 1 \pmod{\ell}$  (i.e.,  $p_2$  splits completely in  $\mathbb{Q}(\mu_\ell)$ );
- (b<sub>2</sub>)  $p_2 \not\equiv 1 \pmod{\ell^2}$  (i.e.,  $p_2$  does not split completely in  $\mathbb{Q}(\mu_{\ell^2})$ );
- (c<sub>2</sub>)  $p_2$  splits completely in  $L_1$ ;
- (d<sub>2</sub>)  $p_1$  splits completely in  $L^{(p_2)}$  (which is equivalent, by the Reciprocity Lemma to:  $p_2$  splits completely in  $\mathbb{Q}(\sqrt[\ell]{p_1})$ );
- (e<sub>2</sub>)  $q_2$  is inert in  $L^{(p_2)}$  (which is equivalent to:  $p_2$  does not split completely in  $\mathbb{Q}(\sqrt[\ell]{q_2})$ ).

Let  $K_2 = L_1(\mu_\ell, \sqrt[\ell]{p_1})$ . Since  $\ell$  does not ramify in  $L_1(\sqrt[\ell]{p_1})$ , we have  $\mu_{\ell^2} \not\subseteq K_2^\times$ ; likewise, since  $q_2$  does not ramify in  $K_2(\mu_{\ell^2})$ , we have  $\sqrt[\ell]{q_2} \notin K_2(\mu_{\ell^2})$ . Consequently, the field  $N_2 = K_2(\mu_{\ell^2}, \sqrt[\ell]{q_2})$  is abelian noncyclic Galois over  $K_2$  of degree

$\ell^2$ . Choose any element  $\sigma_2$  of order  $\ell$  in  $\mathcal{G}(N_2/K_2)$  not lying in  $\mathcal{G}(N_2/K_2(\mu_\ell^2))$  nor in  $\mathcal{G}(N_2/K_2(\sqrt[\ell]{q_2}))$ . Let  $p_2$  be any prime whose Frobenius class is that of  $\sigma_2$ . Then,  $p_2$  satisfies conditions (a<sub>2</sub>)–(e<sub>2</sub>). ( $p_2$  splits completely in  $K_2$  but not in  $K_2(\mu_{\ell^2})$ , so it cannot split completely in  $\mathbb{Q}(\mu_{\ell^2})$ ; likewise,  $p_2$  cannot split completely in  $\mathbb{Q}(\sqrt[\ell]{q_2})$ .) Set  $L_2 = L^{(p_2)}$ .

Continue in this fashion:

Step  $j$ . Let  $q_j$  be a prime number different from  $\ell, p_1, \dots, p_{j-1}, q_2, \dots, q_{j-1}$ . We next find a prime number  $p_j$  different from the primes we already have, satisfying

- (a <sub>$j$</sub> )  $p_j \equiv 1 \pmod{\ell}$  (i.e.,  $p_j$  splits completely in  $\mathbb{Q}(\mu_\ell)$ );
- (b <sub>$j$</sub> )  $p_j \not\equiv 1 \pmod{\ell^2}$  (i.e.,  $p_j$  does not split completely in  $\mathbb{Q}(\mu_{\ell^2})$ );
- (c <sub>$j$</sub> )  $p_j$  splits completely in  $L_1 \cdot L_2 \cdot \dots \cdot L_{j-1}$ ;
- (d <sub>$j$</sub> )  $p_1, \dots, p_{j-1}$  each split completely in  $L^{(p_j)}$  (which is equivalent to:  $p_j$  splits completely in  $\mathbb{Q}(\sqrt[\ell]{p_1}, \dots, \sqrt[\ell]{p_{j-1}})$ );
- (e <sub>$j$</sub> )  $q_j$  is inert in  $L^{(p_j)}$  (which is equivalent to:  $p_j$  is not split completely in  $\mathbb{Q}(\sqrt[\ell]{q_j})$ ).

Let  $K_j = L_1 \dots L_{j-1}(\mu_\ell, \sqrt[\ell]{p_1}, \dots, \sqrt[\ell]{p_{j-1}})$ . Since  $\ell$  does not ramify in the field  $L_1 \dots L_{j-1}(\sqrt[\ell]{p_1}, \dots, \sqrt[\ell]{p_{j-1}})$ , we have  $\mu_{\ell^2} \notin K_j^\times$ ; likewise, since  $q_j$  is unramified in  $K_j(\mu_{\ell^2})$ , we have  $\sqrt[\ell]{q_j} \notin K_j(\mu_{\ell^2})$ . Consequently, the field  $N_j = K_j(\mu_{\ell^2}, \sqrt[\ell]{q_j})$  is noncyclic abelian of degree  $\ell^2$  over  $K_j$ . Choose any element  $\sigma_j$  of order  $\ell$  in  $\mathcal{G}(N_j/K_j)$  not lying in  $\mathcal{G}(N_j/K_j(\mu_\ell^2))$  nor in  $\mathcal{G}(N_j/K_j(\sqrt[\ell]{q_j}))$ . Let  $p_j$  be any prime whose Frobenius class is that of  $\sigma_j$ . Then,  $p_j$  satisfies conditions (a <sub>$j$</sub> )–(e <sub>$j$</sub> ). Set  $L_j = L^{(p_j)}$ . Let  $S_i = L_1 L_2 \dots L_{i-1} L_{i+1} \dots L_j$  for  $1 \leq i \leq j$ . Take any cyclic Galois extension  $K$  of  $\mathbb{Q}$  with  $K \subseteq L_1 \dots L_j$ . Then,  $K \not\subseteq S_i$  for some  $i$ , since  $S_1 \cap \dots \cap S_j = \mathbb{Q}$ ; so  $K \cdot S_i = L_1 \dots L_j$ . Because  $p_i$  splits completely in  $S_i$  but is totally ramified in  $L_i$ , and hence in  $L_1 \dots L_j$ , this  $p_i$  must be totally ramified in  $K$ . Therefore, the same argument as in Step 1 shows that  $K$  does not embed in any cyclic Galois extension of  $\mathbb{Q}$  of degree  $\ell^2$ .

By continuing this process, we obtain two sequences of distinct prime numbers  $\{p_1, p_2, \dots\}$ ,  $\{q_2, q_3, \dots\}$  satisfying:  $p_i \equiv 1 \pmod{\ell}$  for all  $i$ ;  $p_i \not\equiv 1 \pmod{\ell^2}$  for all  $i$ ;  $p_i$  splits completely in  $L_j = L^{(p_j)}$  for all  $j \neq i$ ;  $q_j$  is inert in  $L_j$  for all  $j \geq 2$ . Clearly the  $q_j$  can be chosen so that  $\{\ell, p_1, p_2, \dots, q_2, q_3, \dots\}$  is the set of all prime numbers. We claim that  $L = L_1 L_2 \dots$  has local degree  $\ell$  at all the finite primes of  $\mathbb{Q}$ . At  $p_i$  this is true because  $p_i$  ramifies in  $L_i$  and splits completely in  $L_j$  for all  $j \neq i$ . At  $\ell$  and at each  $q_i$ ,  $L/\mathbb{Q}$  is unramified of exponent  $\ell$ , hence locally of degree  $\ell$ , since  $\ell$  is inert in  $L_1$  and  $q_i$  is inert in  $L_i$ . It now follows from the fundamental theorem for the Brauer group of a global field that  $L$  splits exactly the  $\ell$ -torsion of  $\text{Br}(\mathbb{Q})$ . Note that no cyclic Galois extension  $K$  of  $\mathbb{Q}$  lying in  $L$  embeds in a cyclic Galois extension of  $\mathbb{Q}$  of degree  $\ell^2$ , since we saw in step  $j$  that this is true for subfields of each  $L_1 \dots L_j$ . Thus, our  $L$  has the required properties, completing Case I.

Case II. Assume now that  $\ell = 2$ .

Step 1. Choose prime numbers  $p_1$  and  $p_2$  with  $p_1 \equiv 1 \pmod{8}$  and  $p_2 \equiv 3 \pmod{8}$ , and set  $L_1 = \mathbb{Q}(\sqrt{-p_1 p_2})$ . Note that 2 is inert in  $L_1$ , as  $-p_1 p_2 \equiv -3 \pmod{8}$ .

Step 2. Let  $q_2$  be a prime number different from 2,  $p_1$ , and  $p_2$ . Let  $L_2 = \mathbb{Q}(\sqrt{p_3 p_4})$ , where prime numbers  $p_3$  and  $p_4$  are chosen different from 2,  $p_1$  and  $p_2$  and are required to satisfy:

- (a)  $p_3 \equiv p_4 \equiv 3 \pmod{8}$  (so 2 splits in  $L_2$ );
- (b)  $p_3$  and  $p_4$  split in  $L_1$ ;
- (c)  $p_1$  and  $p_2$  split in  $L_2$ ;
- (d)  $q_2$  is inert in  $L_2$ .

To assure that these conditions can hold, we choose  $p_3$  satisfying  $p_3 \equiv 3 \pmod{8}$ , and  $p_3$  is inert in  $\mathbb{Q}(\sqrt{p_1})$ , split in  $\mathbb{Q}(\sqrt{p_2})$ , and inert in  $\mathbb{Q}(\sqrt{q_2})$ . Choose  $p_4$  satisfying  $p_4 \equiv 3 \pmod{8}$  and  $p_4$  is inert in  $\mathbb{Q}(\sqrt{p_1})$ , split in  $\mathbb{Q}(\sqrt{p_2})$ , and split in  $\mathbb{Q}(\sqrt{q_2})$ . Note that the condition  $p_3 \equiv 3 \pmod{8}$  is equivalent to: The Frobenius automorphism for  $p_3$  in  $\mathcal{G}(\mathbb{Q}(\mu_8)/\mathbb{Q})$  has fixed field  $\mathbb{Q}(\sqrt{-2})$ . To find a suitable  $p_3$ , let  $K_3 = \mathbb{Q}(\sqrt{-2}, \sqrt{p_2})$  and  $N_3 = K_3(\mu_8, \sqrt{p_1}, \sqrt{q_2})$ . So,  $N_3$  is abelian Galois over  $K_3$  of degree 8 and exponent 2. Choose the  $\sigma \in \mathcal{G}(N_3/K_3)$  with  $\sigma(\sqrt{-1}) = -\sqrt{-1}$ ,  $\sigma(\sqrt{p_1}) = -\sqrt{p_1}$ , and  $\sigma(\sqrt{q_2}) = -\sqrt{q_2}$ ; choose  $p_3$  with Frobenius  $\sigma$  for  $N_3/\mathbb{Q}$ . The argument for  $p_4$  is analogous, with  $K_4 = \mathbb{Q}(\sqrt{-2}, \sqrt{p_2}, \sqrt{q_2})$  and  $N_4 = N_3$ . These conditions yield the following in terms of Legendre symbols:  $\left(\frac{-1}{p_3}\right) = \left(\frac{p_1}{p_3}\right) = -1$ ,  $\left(\frac{p_2}{p_3}\right) = 1$ , and  $\left(\frac{q_2}{p_3}\right) = -1$ ;  $\left(\frac{-1}{p_4}\right) = \left(\frac{p_1}{p_4}\right) = -1$ ,  $\left(\frac{p_2}{p_4}\right) = \left(\frac{q_2}{p_4}\right) = 1$ . Hence,  $\left(\frac{-p_1 p_2}{p_3}\right) = \left(\frac{-p_1 p_2}{p_4}\right) = 1$ , which verifies condition (b). Also,  $\left(\frac{p_3 p_4}{p_1}\right) = \left(\frac{p_3}{p_1}\right)\left(\frac{p_4}{p_1}\right) = \left(\frac{p_1}{p_3}\right)\left(\frac{p_1}{p_4}\right) = 1$  and  $\left(\frac{p_3 p_4}{p_2}\right) = \left(\frac{p_3}{p_2}\right)\left(\frac{p_4}{p_2}\right) = (-1)^2 \left(\frac{p_2}{p_3}\right)\left(\frac{p_2}{p_4}\right) = 1$ , showing that condition (c) holds. Further,  $\left(\frac{p_3 p_4}{q_2}\right) = \left(\frac{p_3}{q_2}\right)\left(\frac{p_4}{q_2}\right) = (-1)^{q_2-1} \left(\frac{q_2}{p_3}\right)\left(\frac{q_2}{p_4}\right) = -1$ , verifying condition (d). Thus, we do have  $p_3$  and  $p_4$  with the specified properties.

Continue in this fashion.

Step  $j$ . At this point, we have chosen primes  $p_1, \dots, p_{2j-2}$  and  $q_2, \dots, q_{j-1}$ . Let  $q_j$  be an odd prime number different from any of these primes. Let  $L_j = \mathbb{Q}(\sqrt{p_{2j-1} p_{2j}})$ , where  $p_{2j-1}$  and  $p_{2j}$  are distinct odd primes different from any of the  $p_i$  and  $q_i$  we already have, and satisfying the conditions:

- (a<sub>j</sub>)  $p_{2j-1} \equiv p_{2j} \equiv 3 \pmod{8}$  (so 2 splits in  $L_j$ );
- (b<sub>j</sub>)  $p_{2j-1}$  and  $p_{2j}$  split completely in  $L_1, L_2, \dots, L_{j-1}$ ;
- (c<sub>j</sub>)  $p_1, p_2, \dots, p_{2j-2}$  each split in  $L_j$ ;
- (d<sub>j</sub>)  $q_j$  is inert in  $L_j$ .

To assure that these conditions can be satisfied, we choose  $p_{2j-1}$  so that  $p_{2j-1} \equiv 3 \pmod{8}$ , and  $p_{2j-1}$  is inert in  $\mathbb{Q}(\sqrt{p_1})$  but split in each of  $\mathbb{Q}(\sqrt{p_2}), \dots, \mathbb{Q}(\sqrt{p_{2j-2}})$ , and inert in  $\mathbb{Q}(\sqrt{q_j})$ ; we choose  $p_{2j}$  so that  $p_{2j} \equiv 3 \pmod{8}$ , and  $p_{2j}$  is inert in  $\mathbb{Q}(\sqrt{p_1})$  but split in each of  $\mathbb{Q}(\sqrt{p_2}), \dots, \mathbb{Q}(\sqrt{p_{2j-2}})$ , and split in  $\mathbb{Q}(\sqrt{q_j})$ . The Chebotarev density theorem assures the existence of such  $p_{2j-1}$  and  $p_{2j}$ . (For  $p_{2j-1}$ , let  $K_{2j-1} = \mathbb{Q}(\sqrt{-2}, \sqrt{p_2}, \dots, \sqrt{p_{2j-2}})$  and  $N_{2j-1} = K_{2j-1}(\mu_8, \sqrt{p_1}, \sqrt{q_j})$ , and argue as in Step 2. The case of  $p_{2j}$  is handled analogously.) Routine calculations with Legendre symbols as in Step 2 show that these primes satisfy conditions (a<sub>j</sub>)–(d<sub>j</sub>).

Now let  $L = L_1 L_2 \dots$ . Arguments like those in Case I show that we can choose the  $q_j$  so that the  $p_i$  and  $q_j$  are all the prime numbers. Again as in Case I, we find that  $L$  has local degree 2 at each prime number; it also has local degree 2 at  $\infty$ , since this is the case for  $L_1$ . Furthermore, since  $p_{2j}$  is totally ramified of degree 2 in  $L_j$  but split in the  $L_i$  for  $i \neq j$ , and  $\mu_4 \not\subseteq L$  the argument of Case I applies to show that no quadratic extension of  $\mathbb{Q}$  within  $L$  lies in a cyclic Galois extension of  $\mathbb{Q}$  of degree 4.  $\square$

**Corollary 3.3.**  ${}_2\text{Br}(\mathbb{Q}((t))) = \text{Br}(L_0((\sqrt{t}))/\mathbb{Q}((t)))$  for a suitable 2-Kummer extension  $L_0$  of  $\mathbb{Q}$ .

**Corollary 3.4.** *For any odd prime number  $\ell$ ,  ${}_{\ell}\mathrm{Br}(\mathbb{Q}((t))) = \mathrm{Br}(L_0((\sqrt[\ell]{t}))/\mathbb{Q}((t)))$  for a suitable exponent  $\ell$  abelian Galois extension extension  $L_0$  of  $\mathbb{Q}$ .*

Corollaries 3.3 and 3.4 follow immediately from Propositions 2.1 and 3.1.

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