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On a refined Stark conjecture for function fields. (English summary)


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FEA TURED REVIEW.

The zeta-function \( \zeta_K(s) \) of a number field \( K \) has a zero of order \( r(K) = r_1 + r_2 - 1 \) at \( s = 0 \), \( r_1 \) and \( r_2 \) being respectively the number of real and complex places of \( K \). In fact, Dedekind’s evaluation of the residue of \( \zeta_K(s) \) at \( s = 1 \) together with the functional equation implies \( \zeta_K(s) = - (h_K R_K/w_K) s^{r(K)} + O(s^{r_1 + r_2}) \), where \( h_K \), \( R_K \), and \( w_K \) are the class number, the regulator, and the number of roots of unity of \( K \). The appearance of \( R_K \) in this equality is of particular interest because \( R_K \) is an \( r(K) \times r(K) \) determinant of logarithms of absolute values of units \( \varepsilon \in K \). The leading term in the Taylor expansion of \( \zeta_K(s) \) at \( s = 0 \) therefore encapsulates information about elements of \( K \).

Let \( K/k \) be an abelian extension of number fields with Galois group \( G \). By class field theory, \( K/k \) is determined by arithmetic invariants in the ring of integers of \( k \). However, the proofs of class field theory do not provide efficient algorithms for constructing \( K \) out of these invariants. Hilbert asked in Problem #12 of his famous list if special values of analytic functions, defined by arithmetic information in completions of \( k \), might generate \( K \). If this is possible in the completion \( k_v \) at a place \( v \) of \( k \), then we expect \( K \subset k_v \), and so \( v \) should split in \( K/k \).

The identity \( \zeta_K(s) = \prod_{\chi \in \hat{G}} L(s, \chi) \) factors the zeta-function of \( K \) as a product of \( L \)-functions that are defined over \( k \) by the same arithmetic invariants that determine \( K \) as a class field. In a series of papers [Advances in Math. 7 (1971), 301–343 (1971); MR0289429 (44 #6620); Advances in Math. 17 (1975), no. 1, 60–92; MR0382194 (52 #3082); Advances in Math. 22 (1976), no. 1, 64–84; MR0437501 (55 #10427); Adv. in Math. 35 (1980), no. 3, 197–235; MR0563924 (81f:10054)]. H. Stark proposed conjectures detailing how the leading terms of the Taylor series of these \( L \)-functions at \( s = 0 \) encapsulate information about \( S \)-units of \( K \), \( S \) being any finite set of \( k \)-places containing the Archimedean places and all places that ramify in \( K/k \). The conjectures
apply most naturally to the imprimitive functions $\zeta_{K,S}(s)$ and $L_S(s, \chi)$ obtained by removing Euler factors over $S$. If $r(\chi)$ is the order of vanishing of $L_S(s, \chi)$ at $s = 0$, then $L_S(s, \chi) = L(\chi) \cdot s^{r(\chi)} + O(s^{r(\chi)+1})$ defines a complex number $L(\chi) \neq 0$. Loosely speaking, Stark constructed an $r(\chi) \times r(\chi)$ determinant of $\mathbb{Q}$-linear forms in $\log |\varepsilon|_w$, the $\varepsilon$’s being $S$-units of $K$ and the $w$’s in the set $S_K$ of $K$-places dividing a place in $S$, and he conjectured that $L(\chi)/R(\chi) \in \mathbb{Q}(\chi)$, the subfield of $\mathbb{C}$ generated by the values of $\chi$.

When the minimal value of the $r(\chi)$ is unity and $S$ contains a distinguished split place $v$, Stark formulated a precise conjecture that identifies the linear form $R(\chi)$ in terms of an $S$-unit $\varepsilon_v \in K$ and its conjugates under $G$. Stark identified the denominator in $\mathbb{Q}(\chi)$ as $w_K$, and he conjectured that $K(\varepsilon^{1/w_K})/K$ is abelian. This “first order zero” conjecture implies an algorithm, currently implemented in PARI, for constructing $K$ when $k$ is totally real. J. T. Tate, [Les conjectures de Stark sur les fonctions L d’Artin en $s = 0$. Lecture notes edited by Dominique Bernardi and Norbert Schappacher, Progr. Math., 47, Birkhäuser Boston, Boston, MA, 1984; MR0782485 (86e:11112)(Chapter V)] applied the $l$-adic cohomological interpretation of $L$-functions to prove this conjecture in function fields.


In the paper under review, Popescu proves Rubin’s conjecture in function fields over $\mathbb{F}_q$ up to primes dividing $g$. He invokes the $l$-adic étale cohomology for every prime $l$, as well as crystalline $p$-adic cohomology at the characteristic $p$ of $\mathbb{F}_q$. Popescu proceeds by first proving a strong form of the conjecture over $\mathbb{Z}[1/g]$ for the case $r = 0$. Using functorial properties, he then finds a unique element in $\mathbb{Z}[1/g] \text{Fitt}_{\mathbb{Z}[G]}[A_{S,T}] \cdot \Lambda_{S,T}$ satisfying the conjecture for any $r \geq 0$. Here, $A_{S,T}$ is the $S$-class group of $K$ trivialized along $T$. His techniques involve tensoring the $l$-adic [resp. crystalline] cohomology with $\mathbb{C}_l$ [resp. $\mathbb{C}_p$] and then decomposing into $\chi$-components. When $K/k$ is a constant field extension, Popescu proves a stronger form of the full conjecture.

These results provide the first extensive non-classical evidence for Rubin’s conjecture for arbitrary $r$.

{See also the following review [ MR1711315 (2000m:11116)].}

**Reviewed by David R. Hayes**
[References]

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