SPECTRA OF EXTERIOR GROUP ALGEBRAS AND GIT QUOTIENTS

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Abstract. The skew group algebras $\bigwedge V \rtimes G$ are of natural interest throughout mathematics. Here G is a finite group, and V is a finite-dimensional representation over a field k of arbitrary characterstic. In this paper we first give a simple proof that the cohomology algebra $H^*(\bigwedge V \rtimes G, k)$ is finitely generated. This result subsumes Noether's theorem and the Evens-Venkov theorem on the finite generation of $S(\tilde{V})^G$ and $H^*(G,k)$ respectively. We then study the (graded-commutative) small Yoneda subalgebra and the (commutative) diagonal subalgebra which arise naturally in the theory of support varieties à la Balmer, and give two applications of our finiteness result. First, we infer the surjectivity of the two natural maps out of the Balmer spectrum of $D^b(\bigwedge V \rtimes G)$ to homogeneous Zariski spectra. Our second application is in number theory: If G is a compact p-adic Lie group and $I \lhd G$ is equi-p-valuable in the sense of Lazard, then $\mathrm{Ext}^*_{h(\mathcal{H}_I^\bullet)}(k,k)$ is Noetherian where $\mathcal{H}_I^\bullet$ denotes Schneider's differential graded Hecke algebra. When G is pro-p this says $h(\mathcal{H}_I^\bullet)$ is a complete intersection. In the Appendix we recast Koszul duality and the BGG correspondence in the tensor triangular framework.

Contents

1. Introduction

Hilbert's 14th problem asks whether $S(\tilde{V})^G$ is always a finitely generated k-algebra. Here G is a group and V is any finite-dimensional representation $G \to \mathrm{GL}(V)$ over a field k, and $\tilde{V}$ is its dual. We think of

$S(\tilde{V})^G$ as the algebra of G-invariant polynomial functions on V. The answer is no. In 1958 Nagata gave a famous counterexample, later simplified by Totaro (2008) who defined an action of $\mathbb{G}_a^3(k)$ on k^{18} with an infinitely generated algebra of invariants, see [Tot08]. On the bright side the answer is yes for finite groups G, as was shown by Noether. There is a nice survey [Hum78] by Humphreys detailing the history of the problem and subsequent developments. Apparently Noether first proved it for $\mathrm{char}(k) = 0$ in 1916, and found the general solution a decade later.

Customarily invariant theory is split in two cases: The non-modular case where $\mathrm{char}(k) \nmid |G|$ and the typically more difficult modular case where $\mathrm{char}(k) > 0$ divides $|G|$. In this article we will first prove a simple generalization of Noether's finiteness result in the modular case, and then give two applications.

To motivate our results we make some preliminary remarks. First some notation. For an augmented k-algebra A we let $H^*(A, k) = \mathrm{Ext}^*_A(k, k)$ denote its Yoneda algebra. What we refer to as exterior group algebras in the title are the skew group algebras $\bigwedge V \rtimes G$ (introduced in more detail in the main text) and one impulse to study their cohomology came from the following observation: When $\mathrm{char}(k) \nmid |G|$ one can canonically identify the invariant algebra $S(\tilde{V})^G$ with the cohomology algebra $H^*(\bigwedge V \rtimes G, k)$, and in the modular case at least we have a surjection $H^*(\bigwedge V \rtimes G, k) \twoheadrightarrow S(\tilde{V})^G$. See Remark 4.12 in the text below. This observation is surely well-known to the experts, but we have not been able to find it in the literature. In the modular case it is therefore natural to ask whether the sometimes bigger algebra $H^*(\bigwedge V \rtimes G, k)$ still enjoys the same finiteness properties as its quotient $S(\tilde{V})^G$. Our first result affirms this.

Theorem 1.1. *(Theorem 3.1.) Let G be a finite group and k an arbitrary field. Suppose $G \to GL(V)$ is a finite-dimensional representation over k. Then $H^*(\bigwedge V \rtimes G, k)$ is a Noetherian and (possibly non-commutative) finitely generated k-algebra.*

Since the cohomology algebra always surjects onto $S(\tilde{V})^G$ this immediately implies Noether's result. Moreover, in the extreme case $V = \{0\}$ Theorem 1.1 boils down to the Evens-Venkov theorem – to the effect that the group cohomology $H^*(G, k)$ is finitely generated. (This example also shows the surjection discussed above is not always an isomorphism in the modular case.) At the other extreme $G = \{e\}$ the cohomology algebra $H^*(\bigwedge V, k)$ is the Koszul dual $(\bigwedge V)^! = S(\tilde{V})$.

Although our result subsumes both Noether's theorem and the Evens-Venkov theorem, those are two main ingredients in our proof as we now explain.

Our basic tool is a variant of the Lyndon-Hochschild-Serre spectral sequence, whose multiplicative structure was explicated in [Neg15] (written in a much more general context). In our setup it takes the form

$$E_2^{s,t} = H^s(G, S^t(\tilde{V})) \Longrightarrow H^{s+t}(\bigwedge V \rtimes G, k).$$

Essentially by Koszulity of $\bigwedge V$ it already collapses on the E_2-page, and in particular the vertical edge map $H^*(\bigwedge V \rtimes G, k) \longrightarrow S(\tilde{V})^G$ is surjective. (Note that in the non-modular case it is an isomorphism since $E_2^{s,t} = 0$ for $s > 0$.) Our first step in Section 2 is to remove a freeness assumption in a finiteness criterion of Nguyen and Witherspoon [NW14, Thm. 3.1]. Instead of mimicking the proof of the Evens-Venkov theorem (as they do) we apply it directly and are able to strengthen their criterion. We then use Noether's theorem to check the hypotheses of the criterion hold for exterior group algebras.

We should point out right away that Theorem 1.1 itself is not new. The algebras $\bigwedge V \rtimes G$ carry a natural cocommutative Hopf structure, and 1.1 can be viewed as a special case of Drupieski's [Dru16, Cor. 5.5.2] which gives the finite generation of $H^*(A,k)$ for any finite-dimensional cocommutative Hopf superalgebra (when $\mathrm{char}(k) > 2$). Drupieski extends the arguments of [FS97] to supergroup schemes, and the proof is somewhat technical for the uninitiated (including the author of this paper). As a historical remark the finite generation of $H^*(A,k)$ for *connected* A (meaning $A^0 = k$) is a classical result of Wilkerson, see [Wil81, Thm. A]. We hope the simplicity of our approach in the special but important case of $\bigwedge V \rtimes G$ merits an exposition.

Since $\bigwedge V \rtimes G = k[G] \oplus \cdots$ is a graded algebra its cohomology $H^*(\bigwedge V \rtimes G, k)$ has an internal grading. Thus each $H^n(\bigwedge V \rtimes G, k)$ splits as a direct sum of components

$$H^{n,i}(\bigwedge V \rtimes G, k) = \mathrm{Ext}^n_{\bigwedge V \rtimes G-\mathrm{mod}}\big(k, k\langle -i\rangle\big)$$

for $i \geq 0$. Here $\bigwedge V \rtimes G - \mathrm{mod}$ is the category of finitely generated *graded* $\bigwedge V \rtimes G$-modules, and $k\langle -i\rangle$ denotes the trivial module k concentrated in degree i. This leads to two interesting subalgebras:

- $\mathrm{Ext}^*_{\bigwedge V \rtimes G-\mathrm{mod}}\big(k, k\big)$ – the small Yoneda algebra ($i = 0$);

- $H^{\mathrm{diag}}(\bigwedge V \rtimes G, k)$ – the diagonal subalgebra ($i = n$).

Although the full cohomology algebra $H^*(\bigwedge V \rtimes G, k)$ is neither commutative nor graded-commutative in general, as the two extreme cases $H^*(G,k)$ and $S(\tilde{V})$ show, the above subalgebras are. Note however that $H^*(\bigwedge V \rtimes G, k)$ is *braided* graded-commutative; see the last paragraph of Section 4.2.

One of the reasons we choose to focus on $\bigwedge V$ instead of a general finite-dimensional Koszul algebra is its comultiplication. We define a bialgebra structure on $\bigwedge V \rtimes G$ which equips $\bigwedge V \rtimes G - \mathrm{mod}$ with a monoidal structure. Thus the derived category $D^b(\bigwedge V \rtimes G - \mathrm{mod})$ becomes a tensor triangulated category in the sense of Balmer, cf. [Bal10b]. In particular a variation of the classical Eckmann-Hilton argument shows $\mathrm{Ext}^*_{\bigwedge V \rtimes G-\mathrm{mod}}\big(k, k\big)$ is always graded-commutative and $H^{\mathrm{diag}}(\bigwedge V \rtimes G, k)$ is commutative. In fact we verify that the edge map $H^*(\bigwedge V \rtimes G, k) \twoheadrightarrow S(\tilde{V})^G$ restricts to an isomorphism $H^{\mathrm{diag}}(\bigwedge V \rtimes G, k) \xrightarrow{\sim} S(\tilde{V})^G$. We believe this interpretation of $S(\tilde{V})^G$ is new in the modular case.

The small Yoneda algebra can also be viewed as a quotient of $H^*(\bigwedge V \rtimes G, k)$ in the obvious fashion. In particular $\mathrm{Ext}^*_{\bigwedge V \rtimes G-\mathrm{mod}}\big(k, k\big)$ is a finitely generated graded-commutative k-algebra ($\Rightarrow$ Noetherian), and this has applications to the theory of support varieties à la Balmer as we expound next.

Balmer attaches a topological space $\mathrm{Spc}(\mathcal{K})$ to any tensor triangular category $\mathcal{K}$. (It also has a structure sheaf which we will ignore here.) Its importance stems from the fact that there is an optimal way of defining supports $\mathcal{K} \ni x \rightsquigarrow \mathrm{supp}(x) \subset \mathrm{Spc}(\mathcal{K})$. Alas, the spectrum itself is notoriously hard to understand. For a choice of invertible object $u \in \mathcal{K}$ Balmer defines a graded ring $R^\bullet_{\mathcal{K},u}$ and a continuous map $\rho^\bullet_{\mathcal{K},u} : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathrm{Spec}^{\mathrm{h}} R^\bullet_{\mathcal{K},u}$ into the homogeneous spectrum. One of his main results is that $\rho^\bullet_{\mathcal{K},u}$ is surjective provided $R^\bullet_{\mathcal{K},u}$ is Noetherian (or even just coherent) – but only for $u = \Sigma(\mathbf{1})$. Examples show $\rho^\bullet_{\mathcal{K},u}$ is far from injective unless $\mathcal{K}$ is "algebraic enough" – see [Bal10b, Conj. 72].

We apply Balmer's theory to the category $\mathcal{K} = D^b(\bigwedge V \rtimes G - \mathrm{mod})$. In this case there are two natural choices of u, and the corresponding graded rings $R^\bullet_{\mathcal{K},u}$ turn out to be exactly $\mathrm{Ext}^*_{\bigwedge V \rtimes G-\mathrm{mod}}\big(k, k\big)$ and $H^{\mathrm{diag}}(\bigwedge V \rtimes G, k)$. As a byproduct of our finiteness results for these rings, Balmer's theory applies and gives the surjectivity of $\rho^\bullet_{\mathcal{K},u}$ for $u = \Sigma(\mathbf{1})$. This is our first application of Theorem 1.1.

Corollary 1.2. *The Balmer map $\rho^\bullet_{\mathcal{K},u}$ with $u = \Sigma(\mathbf{1})$ is surjective.*

In fact the other natural Balmer map into $\mathrm{Spec}^{\mathrm{h}}\big(S(\tilde{V})^G\big)$ is surjective as well, but this requires a direct non-trivial argument. The situation is summarized below:

$$\mathrm{Spec}^{\mathrm{h}}\big(\mathrm{Ext}^*_{\bigwedge V \rtimes G-\mathrm{mod}}(k,k)\big) \twoheadleftarrow \mathrm{Spc}\big(D^b(\bigwedge V \rtimes G - \mathrm{mod})\big) \twoheadrightarrow \mathrm{Spec}^{\mathrm{h}}\big(S(\tilde{V})^G\big).$$

Exterior group algebras crop up in distant areas of mathematics. Apart from their appearance in invariant theory of finite groups discussed above, examples arise in mirror symmetry and number theory. The former is very far from this authors expertise, but apparently the algebras $\bigwedge V \rtimes G$ with $k = \mathbb{C}$ and G finite abelian occur naturally in homological mirror symmetry for Calabi-Yau hypersurfaces, cf. [Sei15, Ch. 4]. In number theory examples emerge in the mod p Langlands program via derived Hecke algebras: In [Sch15] Schneider defines a Hecke differential graded algebra $\mathcal{H}_I^\bullet$ attached to a pair (G, I) comprising a p-adic Lie group G and a torsionfree pro-p subgroup I (plus a choice of injective resolution of $\mathrm{ind}_I^G(1)$). Using general results of Keller he gives an equivalence $D(G) \xrightarrow{\sim} D(\mathcal{H}_I^\bullet)$ where $D(G)$ is the derived category of smooth G-representations over a field k of characteristic p. In particular every complex $\pi^\bullet \in D(G)$ gives rise to a graded module $h(M_{\pi^\bullet})$ for the cohomology algebra $h(\mathcal{H}_I^\bullet)$. When G is a compact p-adic Lie group and $I \lhd G$ is a particularly nice pro-p subgroup ("equi-p-valuable") there is an isomorphism $h(\mathcal{H}_I^\bullet) \simeq \bigwedge \mathrm{Hom}(I, k) \rtimes G/I$. This gives our second application of Theorem 1.1.

Corollary 1.3. *$Ext^*_{h(\mathcal{H}_I^\bullet)}(k,k)$ is Noetherian when G is compact and $I \lhd G$ is equi-p-valuable.*

In view of [BH86] we like to interpret this as saying $h(\mathcal{H}_I^\bullet)$ is a (non-commutative) complete intersection, at least when G is pro-p in which case $h(\mathcal{H}_I^\bullet)$ is a local ring. This is in some sense the best one can hope for. Usually $h(\mathcal{H}_I^\bullet)$ has infinite global dimension, so it is not analogous to a regular local ring in this regard. To motivate the general reader we note that the complete intersection property for Hecke algebras is extremely powerful in the automorphic setting, cf. [TW95] which provided the last missing piece of the proof of Fermat's Last Theorem.

We have moved certain technicalities to Appendix A to not veer too far off track in the main text. We think the results therein are of interest in their own right. First we check that the Koszul duality functor

$$K : D^b(\bigwedge V - \mathrm{mod}) \xrightarrow{\sim} D^b(S(\tilde{V}) - \mathrm{mod})$$

preserves the natural monoidal structures on the two sides, see Proposition A.3. We suspect this is well-known (or at least not surprising) to the experts in the field, but we have not been able to find this fact in the literature. The painful verification is somewhat technical and involves juggling a plethora of signs. We hope the details are useful for future reference. We actually boost these results to a G-equivariant setting, where Koszul duality has been less explored.

Having established K is an equivalence of tensor triangulated categories allows us to use [DAS13] to infer that when $G = \{e\}$ the natural map

$$\mathrm{Spc}\big(D^b(\bigwedge V - \mathrm{mod})\big) \xrightarrow{\sim} \mathrm{Spec}^{\mathrm{h}}\big(S(\tilde{V})\big)$$

is a homeomorphism, see Proposition A.6. We think of this as a reformulation of Koszul duality from the tt-geometry perspective. In particular it shows that $D^b(\bigwedge V - \mathrm{mod})$ violates [Bal10b, Conj. 72] unless this category is not "algebraic enough" (which at the moment does not have a precise meaning, as far as we are aware). We hope this example will help sharpen the formulation of Balmer's conjecture, and we suggest a couple of possible directions for $\bigwedge V \rtimes G$ in general (questions 5.4 and 5.5).

2. A finiteness criterion for the cohomology of skew group algebras

In this section we revisit and strengthen a criterion of Nguyen and Witherspoon for the cohomology of certain skew group algebras to be Noetherian, cf. [NW14, Thm. 3.1]. We will show the following result:

Proposition 2.1. *Let G be a finite group, and let k be a field. Suppose G acts on a finite-dimensional augmented k-algebra A (preserving the augmentation) and let $A \rtimes G$ be the associated skew group algebra. Assume there exists a graded k-subalgebra*

$$R \subset im\big(Ext^*_{A\rtimes G}(k,k) \xrightarrow{\varrho} Ext^*_A(k,k)^G\big)$$

satisfying the following two properties:

(a) *R is commutative and Noetherian;*
(b) *$Ext^*_A(k,k)$ is a finite R-module.*

Then $Ext^*_{A\rtimes G}(k,k)$ is a Noetherian (possibly non-commutative) finitely generated k-algebra.

The restriction map ϱ will be defined in the main text below. We briefly point out how this strengthens the Nguyen-Witherspoon criterion: They assume R is a polynomial subalgebra, but more importantly they require $\mathrm{Ext}^*_A(k,k)$ to be a *free* R-module – and admitting a basis whose k-span is G-invariant. As Proposition 2.1 shows, this freeness assumption is unnecessary (Witherspoon has confirmed this in private communication with the author). The main difference between [NW14] and our approach is we *apply* the Evens-Venkov theorem whereas they mimic its proof. This simplifies the proof of the criterion somewhat. Other than that our proof is very similar to the one in [NW14], but we add some complementary details.

At the end of this section we focus on the case where A is Koszul. Under this assumption the Lyndon-Hochschild-Serre spectral sequence collapses on the E_2-page, and in particular ϱ is surjective. Thus we are looking for G-fixed commutative Noetherian subalgebras $R \subset A^!$ over which $A^!$ is finite. When $A^!$ is commutative $R = (A^!)^G$ works.

2.1. **Notation.** Let k be a field, and let A be an augmented k-algebra with augmentation map $\alpha : A \to k$. Suppose a finite group G acts on A by k-algebra automorphisms via a homomorphism $G \to \mathrm{Aut}_{k-\mathrm{alg}}(A)$. We assume G preserves the augmentation in the sense that $\alpha(g(a)) = \alpha(a)$ for all $g \in G$ and $a \in A$. The skew group algebra $A \rtimes G$ has underlying vector space $A \otimes k[G]$ and the algebra structure is given by

$$(a \otimes g) \cdot (a' \otimes g') = ag(a') \otimes gg'$$

for $g, g' \in G$ and $a, a' \in A$. Sending $a \mapsto a \otimes e$ identifies A with a subalgebra $A \hookrightarrow A \rtimes G$, and sending $a \otimes g \mapsto \alpha(a)g$ identifies $k[G]$ with a quotient $A \rtimes G \twoheadrightarrow k[G]$. In particular $A \rtimes G$ is endowed with a natural augmentation map $a \otimes g \mapsto \alpha(a)$ obtained by composition with $k[G] \twoheadrightarrow k$.

Lemma 2.2. *Suppose G is a p-group where $p = char(k) > 0$, and that A is a finite-dimensional local k-algebra (possibly non-commutative). Then $A \rtimes G$ is a local k-algebra with $\mathfrak{m}_{A\rtimes G} = \ker(A \rtimes G \to k)$.*

Proof. For $A = k$ this is a well-known easy exercise. For a general A we must show every $f \notin \mathfrak{m}_{A\rtimes G}$ is invertible. The projection of f in $k[G]$ is therefore not in $\mathfrak{m}_{k[G]}$ and thus invertible. That is, we may pick an $f' \in A \rtimes G$ such that $ff' \equiv f'f \equiv 1$ modulo $\mathfrak{m}_A \rtimes G$. It remains to observe $1 + \mathfrak{m}_A \rtimes G$ consists of units. Considering geometric sums this follows once we check $(\mathfrak{m}_A \rtimes G)^N = 0$ for large enough N. Since G preserves $\mathfrak{m}_A$ we have the inclusion $(\mathfrak{m}_A \rtimes G)^N \subset \mathfrak{m}_A^N \rtimes G$ and indeed $\mathfrak{m}_A^N = 0$ for $N \gg 0$ by Nakayama's lemma (for possibly non-commutative local Noetherian rings) as the powers $\mathfrak{m}_A^N$ stabilize. □

We will mostly be interested in the case where $p = \mathrm{char}(k)$ is positive and divides $|G|$, but we do not make that assumption unless it is stated explicitly.

Finally we note the following universal property of $A \rtimes G$. Suppose we are given a homomorphism of k-algebras $f : A \to B$ and a group homomorphism $u : G \to B^\times$. The pair (f, u) extends to a k-algebra map $A \rtimes G \to B$ precisely when $f(g(a)) = u(g)f(a)u(g)^{-1}$ holds for all $g \in G$ and $a \in A$.

2.2. **Cohomology of augmented algebras.** We view k as an A-module via α and consider the Yoneda algebra $H^*(A, k) = \mathrm{Ext}^*_A(k, k)$. It is functorial for flat ring maps. More precisely, a homomorphism of augmented k-algebras $A \to B$ such that B is flat over A on the right induces a map of graded k-algebras

$$\mathrm{Ext}^*_B(k, k) \longrightarrow \mathrm{Ext}^*_B(B \otimes_A k, k) \simeq \mathrm{Ext}^*_A(k, k).$$

The arrow comes from the augmentation map $B \otimes_A k \to k$, and the isomorphism uses the flatness.

As a right A-module $A \rtimes G \simeq \bigoplus_{g \in G} A^g$ where A^g denotes A with the right module structure twisted by g. This is easily checked to be flat over A. Consequently $A \hookrightarrow A \rtimes G$ yields a restriction map

$$\varrho : H^*(A \rtimes G, k) \longrightarrow H^*(A, k).$$

The G-action on A gives a G-action on $H^*(A, k)$ as follows. Choose a resolution $P^\bullet \to k$ by projective (left) A-modules which is G-equivariant, meaning each term of $P^\bullet$ carries a G-action compatible with the A-module structure – and the differentials are G-linear. (For instance take the bar resolution with G acting diagonally on $A \otimes \cdots \otimes A$.) Then $H^*(A, k)$ is the cohomology algebra of the differential graded algebra $\mathrm{End}^\bullet_A(P^\bullet)$ and G acts on the latter by conjugation.

Lemma 2.3. *$im(\varrho) \subset H^*(A, k)^G$.*

Proof. Start with a resolution $Q^\bullet \to k$ by projective $A \rtimes G$-modules. Since $A \rtimes G \simeq A \otimes k[G]$ as a left A-module, the terms of $Q^\bullet$ are still projective over A and we may take our $P^\bullet$ above to be $Q^\bullet$ (restricted to A). Then ϱ lifts to a map at the DG level $\mathrm{End}^\bullet_{A \rtimes G}(Q^\bullet) \to \mathrm{End}^\bullet_A(Q^\bullet)$ whose image is exactly the G-invariants. Passing to cohomology shows G acts trivially on $\mathrm{im}(\varrho)$. ☐

In the next subsection we will give a more precise spectral sequence interpretation of $\mathrm{im}(\varrho)$.

2.3. **Lyndon-Hochschild-Serre.** We keep the setup above, but now insist that $\dim_k(A)$ is finite. This guarantees $\dim_k H^t(A, k) < \infty$ for all t, and therefore $H^s(G, H^t(A, k))$ is also finite-dimensional. This ensures the convergence of the Lyndon-Hochschild-Serre spectral sequence (or rather a mild generalization thereof):

$$E_2^{s,t} = H^s(G, H^t(A, k)) \Longrightarrow H^{s+t}(A \rtimes G, k).$$

A good reference for this is [Neg15, Thm. 1.2] which emphasizes the multiplicative structure: Each sheet E_r carries a multiplication $E_r \otimes E_r \xrightarrow{m_r} E_r$ preserving the bigrading and satisfying the Leibniz rule with respect to the total degree. Moreover $H^*(E_r) \simeq E_{r+1}$ as bigraded algebras. By convergence the limiting sheet E_∞ gets a multiplication $E_\infty \otimes E_\infty \longrightarrow E_\infty$ of bidegree $(0, 0)$. Furthermore $H^*(A \rtimes G, k)$ is a filtered graded algebra; that is, it is endowed with a filtration by two-sided homogeneous ideals

$$H^*(A \rtimes G, k) = F^0 H^*(A \rtimes G, k) \supset \cdots \supset F^s H^*(A \rtimes G, k) \supset \cdots$$

such that $F^s \cdot F^{s'} \subset F^{s+s'}$ where F^s stands for $F^s H^*(A \rtimes G, k)$. The associated bigraded algebra is

$$\mathrm{gr}H^*(A \rtimes G, k) = \bigoplus_{s\geq 0} F^s/F^{s+1} = \bigoplus_{s,t\geq 0} \frac{F^s \cap H^{s+t}}{F^{s+1} \cap H^{s+t}} \xrightarrow{\sim} E_\infty.$$

where H^{s+t} is shorthand for $H^{s+t}(A \rtimes G, k)$ for ease of legibility.

Example 2.4. Suppose $p = \mathrm{char}(k)$ is positive and does *not* divide $|G|$. In this case the spectral sequence is concentrated on the vertical edge (both p and $|G|$ annihilate $E_2^{s,t}$ for $s > 0$ by restriction-corestriction). We deduce an isomorphism of graded algebras $H^*(A \rtimes G, k) \xrightarrow{\sim} H^*(A, k)^G$.

In general the restriction map ϱ from the previous subsection coincides with the vertical edge map of the spectral sequence:

$$H^*(A \rtimes G, k) \twoheadrightarrow E_\infty^{0,*} \hookrightarrow E_2^{0,*} = H^*(A, k)^G.$$

Thus $\ker(\varrho) = F^1$ and $\mathrm{im}(\varrho) = E_\infty^{0,*}$. From the horizontal edge we get a map $H^*(G, k) \longrightarrow H^*(A \rtimes G, k)$ whose image is $E_\infty^{*,0}$.

There is a bit more standard notation we wish to put in place here. As usual Z_r denotes the r-cocycles and B_r denotes the r-coboundaries, and one has the following inclusions among them:

$$0 = B_1 \subset B_2 \subset \cdots \subset B_r \subset \cdots \subset Z_r \subset \cdots \subset Z_2 \subset Z_1 = E_2.$$

Recall that $E_r \simeq Z_{r-1}/B_{r-1}$ and the terms above are defined recursively by writing $\ker(d_r) = Z_r/B_{r-1}$ and $\mathrm{im}(d_r) = B_r/B_{r-1}$ where $d_r : E_r \to E_r$ is the differential. Finally Z_∞ is the intersection of all the Z_r, and B_∞ is the union of the B_r. The limiting sheet is $E_\infty = Z_\infty/B_\infty$.

2.4. **Invoking Evens-Venkov.** The ensuing argument is based on the following classical result from group cohomology.

Theorem 2.5. *(Evens-Venkov) Let G be a finite group acting trivially on a commutative ring R. Let M be an $R[G]$-module which is Noetherian as an R-module. Then $H^*(G, M)$ is a Noetherian $H^*(G, R)$-module. In particular $H^*(G, R)$ is a finitely generated R-algebra provided R is a Noetherian ring.*

Proof. This is paraphrasing Theorem 7.4.1 and its Corollary 7.4.6 in [Eve91]. □

We apply this result to $M = H^*(A, k)$ and take R as in the following hypothesis:

Assumption 2.6. There is a graded k-subalgebra $R = \bigoplus_{t\geq 0} R^t \subset \mathrm{im}(\varrho)$ such that

(a) R is commutative and Noetherian;
(b) $H^*(A, k)$ is finite as an R-module.

(We stress that R is assumed to be commutative, not necessarily graded-commutative.)

Then Theorem 2.5 tells us $H^*(G, R)$ is a finitely generated R-algebra, and $E_2 = H^*(G, H^*(A, k))$ is a Noetherian $H^*(G, R)$-module. Note that $H^*(G, R) \simeq \bigoplus_{s,t\geq 0} H^s(G, k) \otimes R^t$ is a bigraded k-algebra.

Our next step is to turn E_r into an $H^*(G, R)$-module for all $r \geq 2$ (including $r = \infty$). First, from the horizontal edge we get projections $H^s(G, k) \twoheadrightarrow E_r^{s,0}$. Secondly, since $R \subset \mathrm{im}(\varrho) = E_\infty^{0,*}$ we have inclusion maps $R^t \hookrightarrow E_r^{0,t}$. Combining them gives maps $(\forall s, t)$

$$H^s(G, k) \otimes R^t \longrightarrow E_r^{s,0} \otimes E_r^{0,t} \xrightarrow{m_r} E_r^{s,t}$$

which we assemble into a single bigraded map $\psi_r : H^*(G, R) \longrightarrow E_r$.

Lemma 2.7. *The map $\psi_r : H^*(G,R) \longrightarrow E_r$ is a homomorphism of bigraded k-algebras. Thus the rth sheet E_r becomes an $H^*(G,R)$-module, and the differential $d_r : E_r \to E_r$ is graded $H^*(G,R)$-linear. That is, for all homogeneous $h \in H^*(G,R)$ and $x \in E_r$ we have the identity*

$$d_r(h \bullet x) = (-1)^{deg(h)} \cdot h \bullet d_r(x).$$

(Here $deg(h)$ denotes the total degree, and $h \bullet x = m_r(\psi_r(h) \otimes x)$ is the module structure.)

Proof. To see ψ_r preserves multiplication one draws the hopefully obvious diagram, and it remains to check the horizontal algebra $E_r^{*,0}$ commutes with the vertical algebra $E_r^{0,*}$ inside E_r. This is obviously true for $r = 2$ where the two algebras in question are $H^*(G,k)$ and $H^*(A,k)^G$ respectively. The result follows for all r using $H^*(E_r) \simeq E_{r+1}$. Thus E_r becomes an $H^*(G,R)$-module via ψ_r.

The graded-linearity of d_r follows from the Leibniz rule after observing $d_r(\psi_r(h)) = 0$. This is immediate since $d_r = 0$ on the horizontal edge, and $d_r = 0$ on $R \subset E_\infty^{0,*}$ viewed as a subring of $E_r^{0,*}$. □

One deduces Z_r and B_r are $H^*(G,R)$-submodules of E_2 for all r. We already observed that E_2 is a Noetherian module so $0 = B_1 \subset B_2 \subset B_3 \subset \cdots$ must become stationary. Say $B_r = B_{r+1} = \cdots = B_\infty$ for all $r \geq r_0$. Then $E_r = E_{r+1} = \cdots = E_\infty$ for all $r > r_0$. In other words the spectral sequence collapses at a finite stage. Since E_r is a subquotient of E_2 it is obviously also a Noetherian $H^*(G,R)$-module.

2.5. **Lifting the Noetherian property.** We have just seen that E_∞ is a Noetherian $H^*(G,R)$-module. So obviously E_∞ is a Noetherian ring (ideals are submodules). There is a standard result lifting this property to $H^*(A \rtimes G, k)$ which we quote here:

Lemma 2.8. *Let $H = \bigoplus_{n\geq 0} H^n$ be a graded algebra filtered by homogeneous two-sided ideals F^sH,*

$$H = F^0H \supset F^1H \supset \cdots \supset F^sH \supset \cdots \qquad F^sH = \bigoplus_{n\geq 0} H^n \cap F^sH,$$

such that $F^sH \cdot F^{s'}H \subset F^{s+s'}H$. Assume the following separation condition is fulfilled: $\forall n$,

$$H^n \cap F^sH = 0 \qquad \forall s \gg 0 \qquad (\textit{possibly depending on } n).$$

If $grH = \bigoplus_{s\geq 0} F^sH/F^{s+1}H$ is a Noetherian ring, then so is H.

Proof. One can reduce to the case of an ascending chain of homogeneous ideals in H. We recommend the reader tries the homogeneous case as an exercise. For hints see [Eve91, Lem. 7.4.5] for a more general result. □

We just have to check the filtration F^s of $H^*(A \rtimes G, k)$ satisfies the separation condition above, but this implied by the convergence of the spectral sequence. In fact $F^sH^n(A \rtimes G, k) = 0$ for $s > n$ since $E_\infty^{s,n-s} = 0$ in that range. Altogether this shows $H^*(A \rtimes G, k)$ is a Noetherian ring, cf. Proposition 2.1.

A standard argument now shows the finite generation of $H^*(A \rtimes G, k)$.

Lemma 2.9. *The cohomology $H^*(A \rtimes G, k)$ is a finitely generated k-algebra (that is, a quotient of the free algebra $k\langle S\rangle$ for some finite set S.)*

Proof. Consider the positive part $\mathcal{I} = H^+(A \rtimes G, k)$. This is a two-sided ideal of the Noetherian ring $H^*(A \rtimes G, k)$ and we pick homogeneous generators $y_1, \dots, y_N$ for $\mathcal{I}$ as a left ideal. We need to show every $h \in H^*(A \rtimes G, k)$ can be written as a finite sum $h = \sum_{\ell=0}^\infty \sum_{I:|I|=\ell} c_I y_{i_1} \cdots y_{i_\ell}$ where I ranges over all ordered tuples $I = (i_1, \dots, i_\ell)$ from $\{1, 2, \dots, N\}$ – and the coefficients c_I all lie in k. It suffices to do this for homogeneous h. Proceed by induction on $n = \deg(h)$. The case $n = 0$ is trivial. For $n > 0$ split h as a sum $h = \sum_{j=1}^N h_j y_j$ with $h_j \in H^*(A \rtimes G, k)$. Writing h_j as a sum of homogeneous elements we see that we may assume h_j is homogeneous of $\deg(h_j) = n - \deg(y_j) < n$. By induction each h_j can therefore be expressed as a linear combination of words from $\{y_1, \dots, y_N\}$. Inserting those expansions in $h = \sum_{j=1}^N h_j y_j$ shows h has the desired form. Let $S = \{X_1, \dots, X_N\}$ be any set with N elements. Sending $X_i \mapsto y_i$ extends to a surjection $k\langle X_1, \dots, X_N\rangle \twoheadrightarrow H^*(A \rtimes G, k)$ as required. □

2.6. **Koszul algebras.** In this subsection we restrict to the case where $A = k \oplus A^+ = \bigoplus_{i \geq 0} A^i$ is a Koszul algebra with a G-action (assumed to preserve the grading). Here the Yoneda algebra $H^*(A, k)$ coincides with the Koszul dual $A^!$.

Proposition 2.10. *If A is Koszul the Lyndon-Hochschild-Serre spectral sequence collapses on the first page; that is $E_2 = E_3 = \cdots = E_\infty$.*

Proof. In [Neg15] the spectral sequence is lifted to an isomorphism of differential graded algebras

$$R\mathrm{Hom}_{(A \rtimes G)^e}(A \rtimes G, k) \simeq R\mathrm{Hom}_G\big(k, R\mathrm{Hom}_{A^e}(A, k)\big), \tag{2.11}$$

see [Neg15, Thm. 4.3, Thm. 6.5] which are based on Negron's construction of a smash product resolution. The differentials of the complex $R\mathrm{Hom}_{A^e}(A, k) \simeq R\mathrm{Hom}_A(k, k)$ are all zero by minimality of the Koszul resolution $K^\bullet(A, k) \to k$. It therefore decomposes as a direct sum of complexes $\bigoplus_{t \geq 0} H^t(A, k)[-t]$, and using the compactness of k the right-hand side of (2.11) is identified with

$$R\mathrm{Hom}_G\big(k, \bigoplus_{t \geq 0} H^t(A, k)[-t]\big) \simeq \bigoplus_{t \geq 0} R\mathrm{Hom}_G\big(k, H^t(A, k)\big)[-t].$$

Passing to cohomology shows that $H^n(A \rtimes G, k) \simeq \bigoplus_{t \geq 0} H^{n-t}(G, H^t(A, k))$ for every n. By comparing dimensions we deduce that $\dim_k E_2^{s,t} = \dim_k E_\infty^{s,t}$ for all s, t. Therefore $E_2 = E_\infty$ as claimed. □

In particular the restriction map $\varrho : H^*(A \rtimes G, k) \to (A^!)^G$ is surjective, and Proposition 2.1 immediately reduces to the criterion below.

Corollary 2.12. *Let A be a finite-dimensional Koszul algebra over k with a G-action, whose Koszul dual $A^!$ is a* <u>*commutative*</u> *k-algebra (automatically finitely generated). Then $H^*(A \rtimes G, k)$ is a Noetherian and finitely generated k-algebra (a quotient of the free algebra on a finite set).*

Proof. By Noether's classical finiteness result [NS02, Thm. 2.1.4] we may take $R = (A^!)^G$ in Proposition 2.1: Noether's results imply $(A^!)^G$ is finitely generated *and* that $A^!$ is finite over its G-invariants. (To see the Koszul dual is automatically finitely generated present A and $A^!$ as quadratic algebras.) □

Experts have informed us the assumptions in Corollary 2.12 in fact force $A^!$ to be a symmetric algebra.

Remark 2.13. As noted in the proof of Proposition 2.10 there is an an isomorphism of graded algebras

$$H^*(A \rtimes G, k) \simeq \mathbb{H}^*(G, A^!)$$

where $A^! = R\mathrm{Hom}_A(k, k)$ is viewed as a DGA with zero differentials, and we are taking hypercohomology. This gives yet another way of obtaining the finite generation of $H^*(A \rtimes G, k)$ by invoking [TvdK10, Thm. 1.1] which says the rational cohomology $H^*(G, R)$ of any reductive group G is finitely generated, for every finitely generated commutative k-algebra R with G-action. As algebras there is no difference between $\mathbb{H}^*(G, A^!)$ and $H^*(G, A^!)$ although the gradings are of course different, so [TvdK10] applies. (We thank Negron for his help with this remark.)

3. Exterior group algebras and their Hopf structure

We apply Corollary 2.12 to the exterior algebra $\bigwedge V$ of a G-representation. This immediately implies the result below. As noted in the introduction this is also a special case of the main result of [Dru16].

Theorem 3.1. *Let G be a finite group and let k be an arbitrary field. Suppose V is a finite-dimensional G-representation over k, and consider the associated skew group algebra $\bigwedge V \rtimes G$ ("exterior group algebra"). Then the cohomology algebra $H^*(\bigwedge V \rtimes G, k)$ is a Noetherian and finitely generated k-algebra.*

A result of Bøgvad and Halperin says that a commutative local ring is a complete intersection if and only if its Yoneda algebra is Noetherian, see [BH86]. In light of this we interpret Theorem 3.1 as saying $\bigwedge V \rtimes G$ is a (non-commutative) complete intersection – at least when G is a p-group so that $\bigwedge V \rtimes G$ is local (and p is the characteristic of k). Note that $\bigwedge V \rtimes G$ is usually not of finite global dimension, so it is rarely a regular local ring in that sense.

When the field k is finite the Dickson invariants generate an explicit polynomial subalgebra $R \subset S(\tilde{V})^G$ satisfying the hypotheses of Corollary 2.12. We discuss this simple case in 3.3 below, although it is not part of the logical structure of our paper.

The main purpose of this section is to flesh out the Hopf structure on $\bigwedge V \rtimes G$.

3.1. **Preliminaries.** Let V be a d-dimensional vector space over the field k, with dual space $\tilde{V}$. We consider its exterior algebra $\bigwedge V = \bigwedge^0 V \oplus \cdots \oplus \bigwedge^d V$. This is a local k-algebra with maximal ideal $\bigwedge^+ V$ (sums of elements of positive degree) – to see this observe that the dth power of $\bigwedge^+ V$ is zero so the inverse of $1 - a$ is a finite geometric sum for $a \in \bigwedge^+ V$.

We now bring into play a representation $G \to \mathrm{GL}(V)$. This defines a degree-preserving action of G on $\bigwedge V$ and we can consider the skew group algebra $\bigwedge V \rtimes G$ as in the previous section. These are the objects we dub *exterior group algebras* in the title. In [AEG01, Rk. 2.3.3] they are referred to as supergroup algebras. They are a mix of exterior algebras and group algebras. In the extreme case $V = \{0\}$ we recover the group algebra $k[G]$; when $G = \{e\}$ we get $\bigwedge V$.

In general $\bigwedge V \rtimes G$ is a graded k-algebra with $k[G]$ sitting in degree 0. It has dimension $2^d|G|$ over k. As Lemma 2.2 shows, $\bigwedge V \rtimes G$ is a local algebra when G is a p-group and $\mathrm{char}(k) = p$.

To produce maps out of $\bigwedge V \rtimes G$ it is useful to note the following universal property: Suppose B is a k-algebra. Let $f : V \to B$ be a k-linear map such that $f(v)^2 = 0$ for all $v \in V$, and let $u : G \to B^\times$ be a group homomorphism. The pair (f, u) extends to a k-algebra map $\bigwedge V \rtimes G \to B$ precisely when the compatibility condition $f(g(v)) = u(g)f(v)u(g)^{-1}$ is fulfilled for all $g \in G$ and $v \in V$. Usually B will also be a non-negatively graded algebra. In that case the extension $\bigwedge V \rtimes G \to B$ is a homomorphism of graded k-algebras provided $\deg f(v) = 1$ and $\deg u(g) = 0$ for all $g \in G$ and $v \in V$.

3.2. **Comultiplication.** In this subsection we turn $\bigwedge V \rtimes G$ into a graded k-bialgebra by amalgamating the comultiplications on $\bigwedge V$ and $k[G]$, which we view as subalgebras of $\bigwedge V \rtimes G$ in the obvious fashion. To not overload the use of the symbol $\otimes$ we will denote the element $v \otimes g \in \bigwedge V \rtimes G$ simply by vg.

Our goal here is to construct a homomorphism of graded k-algebras

$$\Delta : \bigwedge V \rtimes G \longrightarrow (\bigwedge V \rtimes G) \otimes (\bigwedge V \rtimes G).$$

We emphasize that the right-hand side is a tensor product of graded algebras (meaning we adopt the Koszul sign convention in the definition of multiplication). Δ arises from the pair (f, u) where $f : V \to (\bigwedge V \rtimes G) \otimes (\bigwedge V \rtimes G)$ is the map $f(v) = v \otimes 1 + 1 \otimes v$ and $u : G \to ((\bigwedge V \rtimes G) \otimes (\bigwedge V \rtimes G))^{\times}$ is the map $u(g) = g \otimes g$. Note that $f(v)^2 = 0$ since $\deg(v) = 1$. We are left with checking the compatibility condition $f(g(v)) = u(g)f(v)u(g)^{-1}$ but this is an easy calculation left to the reader; use that $\deg(g) = 0$ and $gv = g(v)g$. By the universal property of $\bigwedge V \rtimes G$ this yields a graded algebra map Δ as desired. On elements vg it is given explicitly by the formula

$$\Delta(vg) = vg \otimes g + g \otimes vg.$$

There is also a canonical anti-involution

$$S : \bigwedge V \rtimes G \longrightarrow \bigwedge V \rtimes G$$

arising from the pair of maps (f, u) where $f : V \to (\bigwedge V \rtimes G)^{\mathrm{op}}$ is negation $v \mapsto -v$ and the unit morphism $u : G \to ((\bigwedge V \rtimes G)^{\mathrm{op}})^{\times}$ is inversion $g \mapsto g^{-1}$. Again, their compatibility is easily verified. On elements vg the resulting map is given by the recipe $S(vg) = -g^{-1}v$ where the multiplication on the right-hand side takes place in $\bigwedge V \rtimes G$. This gives an antipode. Indeed the usual hexagon diagram is easily checked to commute. For instance,

$$(m \circ (S \otimes \mathrm{Id}) \circ \Delta)(vg) = (m \circ (S \otimes \mathrm{Id}))(vg \otimes g + g \otimes vg) = m(-g^{-1}v \otimes g + g^{-1} \otimes vg) = 0$$

and vg augments to 0. Altogether this turns $\bigwedge V \rtimes G$ into a graded Hopf k-algebra. In particular Δ defines a monoidal structure on the category of graded $\bigwedge V \rtimes G$-modules.

Remark 3.2. When k is algebraically closed of characteristic zero [Dru16, Sect. 5.7] recalls the following classification result of Kostant: Any finite-dimensional cocommutative Hopf superalgebra over k is isomorphic to $\bigwedge V \rtimes G$ for some finite group G acting on a finite-dimensional superspace $V = V_{\mathrm{odd}}$. A more precise formulation of Kostant's theorem can be found in [AEG01, Cor. 2.3.5] (stated for $k = \mathbb{C}$).

Remark 3.3. When $\mathrm{char}(k) = p > 0$ Palmieri has a conjecture on the structure of $\mathrm{Ext}^*_B(k, k)$ mimicking the Quillen F-isomorphism theorem in group cohomology, see [Pal97, Conj. 1.5]. Here B is any finite-dimensional graded cocommutative Hopf algebra over k. The role of elementary abelian p-subgroups is played by quasi-elementary sub-Hopf algebras $E \subset B$ – and their subalgebras. The notion of being quasi-elementary is modeled on Serre's theorem on products of Bocksteins and is somewhat technical and difficult to verify in concrete examples, cf. [Pal97, p. 204]. Palmieri's conjecture states that restriction to all such E gives an F-isomorphism $\mathrm{Ext}^*_B(k, k) \longrightarrow \varprojlim_E \mathrm{Ext}^*_B(k, k)$. Palmieri proves his conjecture for connected B, see [Pal97, Thm. 1.4]. It would be interesting to have a complete description of all the quasi-elementary sub-Hopf algebras of $B = \bigwedge V \rtimes G$. As far as we know [Pal97, Conj. 1.5] is still open in this special case.

3.3. **Dickson invariants and a polynomial algebra R.** For the convenience of the reader we briefly recall Dickson's classical result from invariant theory. We are forced to assume our base field k is *finite* in this example. Let $q = |k|$ and $p = \mathrm{char}(k)$.

We keep the d-dimensional k-vector space V from before and let $S(V)$ be its symmetric algebra. The general linear group $\mathrm{GL}(V)$ acts on $S(V)$ by degree-preserving algebra-automorphisms, and Dickson wrote down explicit generators for the algebra of invariants $S(V)^{\mathrm{GL}(V)}$ as follows. The polynomial $\prod_{v\in V}(X-v)$ in $S(V)[X]$ can be shown to have the following form:

$$\prod_{v\in V}(X-v)=X^{q^d}+\sum_{i=0}^{d-1}c_{d,i}X^{q^i}$$

for suitable $c_{d,i}\in S(V)$ necessarily of degree q^d-q^i. Note that V is a finite set with $|V|=q^d$.

Theorem 3.4. *(Dickson) The coefficients $c_{d,i}$ are algebraically independent ($i=0,\dots,d-1$) and generate the k-subalgebra of $GL(V)$-invariants:*

$$S(V)^{GL(V)}=k[c_{d,0},\dots,c_{d,d-1}].$$

Proof. This is a standard result from invariant theory of finite groups, for which our go-to reference is [NS02]. (Steven Sam also has a very useful blog post detailing the proof.) □

By the very definition of the $c_{d,i}$ it is obvious that $S(V)$ is an integral extension of $k[c_{d,0},\dots,c_{d,d-1}]$. In particular $S(V)$ is a finite module over $S(V)^{\mathrm{GL}(V)}$. Applying this result to the dual representation $\tilde{V}$ we arrive at a suitable polynomial subalgebra R:

Corollary 3.5. $R=S(\tilde{V})^{GL(\tilde{V})}$ *is a polynomial subalgebra of* $S(\tilde{V})^G$ *over which* $S(\tilde{V})$ *is a finite module.*

This example will not be used in the remainder of this paper.

4. The small Yoneda algebra and the diagonal subalgebra

As we have seen, the restriction map ϱ exhibits $S(\tilde{V})^G$ as a quotient of $H^*(\bigwedge V\rtimes G,k)$. In this section we establish finiteness properties of a certain graded-commutative quotient which we call the small Yoneda algebra $\mathrm{ext}^*_{\bigwedge V\rtimes G}(k,k)$, and we verify that ϱ identifies a certain subalgebra $H^{\mathrm{diag}}(\bigwedge V\rtimes G,k)$ of the full cohomology with the algebra of G-invariants $S(\tilde{V})^G$.

4.1. **Internal grading.** Since $\bigwedge V\rtimes G$ is a graded algebra its cohomology inherits an internal grading defined as follows. We work in the abelian category $\bigwedge V\rtimes G-\mathrm{mod}$ of all graded (left) modules and degree 0 morphisms. If M is an object thereof its internal shifts $M\langle i\rangle$ are given by shifting the grading by the formula $M\langle i\rangle^j=M^{i+j}$. Thus $k\langle -i\rangle$ is the augmentation module k sitting in degree i. For finitely generated graded modules, taking Hom in the category of all $\bigwedge V\rtimes G$-modules coincides with the direct sum of all homogeneous homomorphisms. In particular

$$H^n(\bigwedge V\rtimes G,k)=\bigoplus_{i\in\mathbb{Z}}\mathrm{Ext}^n_{\bigwedge V\rtimes G-\mathrm{mod}}\big(k,k\langle -i\rangle\big).$$

First observe that this sum ranges over $i\geq 0$:

Lemma 4.1. $H^{n,i}(\bigwedge V\rtimes G,k)=\!=\mathit{Ext}^n_{\bigwedge V\rtimes G-mod}\big(k,k\langle -i\rangle\big)=0$ *unless* $i\geq 0$.

Proof. As $\bigwedge V\rtimes G$ is non-negatively graded, and obviously Noetherian, k admits a resolution in $\bigwedge V\rtimes G-\mathrm{mod}$ by finite free modules

$$\cdots\longrightarrow(\bigwedge V\rtimes G)\otimes W_2\longrightarrow(\bigwedge V\rtimes G)\otimes W_1\longrightarrow(\bigwedge V\rtimes G)\otimes W_0\longrightarrow k\longrightarrow 0$$

where each W_j is a graded vector space concentrated in non-negative degrees. (Note that $\bigwedge V \rtimes G$ is not connected unless G is trivial so we cannot assume W_j sits in degrees $\geq j$ unless we are in that case.) Then

$$\mathrm{Ext}^n_{\bigwedge V\rtimes G-\mathrm{mod}}\big(k, k\langle -i\rangle\big) = H^n\big(\mathrm{Hom}_{k-\mathrm{mod}}(W_\bullet, k\langle -i\rangle)\big)$$

vanishes when $i < 0$. Indeed a degree 0 map $W_j \to k\langle -i\rangle$ amounts to a k-linear map $W_j^i \to k$, but $W_j^i = 0$ for negative i. Hence $\mathrm{Hom}_{k-\mathrm{mod}}(W_\bullet, k\langle -i\rangle)$ is the zero-complex for such i. □

We define the small Yoneda algebra to be the $i = 0$ part of the full cohomology algebra:

Definition 4.2. $\mathrm{ext}^*_{\bigwedge V\rtimes G}(k, k) =\!= \mathrm{Ext}^*_{\bigwedge V\rtimes G-\mathrm{mod}}(k, k)$.

(Here we adopt the notation ext introduced in [BGS96, p. 479].)

Remark 4.3. Note that $\mathrm{ext}^*_{\bigwedge V\rtimes G}(k, k)$ can be much smaller than $H^n(\bigwedge V \rtimes G, k)$. For instance, when $G = \{e\}$ the full cohomology is $S(\tilde{V})$ but the small Yoneda algebra is trivial in positive degrees; $\mathrm{ext}^*_{\bigwedge V}(k, k) = k$. (This follows from the proof of Lemma 4.1. See also Example A.10 for a generalization thereof.) At the other extreme $V = \{0\}$ they both coincide with the group cohomology $H^*(G, k)$.

Below we will also consider the diagonal subalgebra of the full cohomology:

Definition 4.4. $H^{\mathrm{diag}}(\bigwedge V \rtimes G, k) =\!= \bigoplus_{i\geq 0} H^{i,i}(\bigwedge V \rtimes G, k) = \bigoplus_{i\geq 0} \mathrm{Ext}^i_{\bigwedge V\rtimes G-\mathrm{mod}}\big(k, k\langle -i\rangle\big)$.

Remark 4.5. When $G = \{e\}$ this is the whole cohomology $H^{\mathrm{diag}}(\bigwedge V, k) = S(\tilde{V})$ by Koszulity. On the contrary, when $V = \{0\}$ the diagonal subalgebra is trivial in positive degrees; $H^{\mathrm{diag}}(k[G], k) = k$ since $k[G]$ is concentrated in degree 0.

4.2. **Commutativity properties.** There is a natural automorphism of graded k-algebras

$$\tau : (\bigwedge V \rtimes G) \otimes (\bigwedge V \rtimes G) \xrightarrow{\sim} (\bigwedge V \rtimes G) \otimes (\bigwedge V \rtimes G)$$

interchanging the two factors (with a Koszul sign). The comultiplication Δ introduced in 3.2 is graded cocommutative in the sense that $\tau \circ \Delta = \Delta$. This gives the category $\bigwedge V \rtimes G-\mathrm{mod}$ a symmetric monoidal structure in the obvious way: If M, N are graded modules the switch $\sigma_{M,N} : M \otimes N \xrightarrow{\sim} N \otimes M$ sends $m \otimes n \mapsto (-1)^{\deg(m)\deg(n)} n \otimes m$ for homogeneous m, n. For homogeneous $a, b \in \bigwedge V \rtimes G$ the action of $a \otimes b$ is given by $(a \otimes b) \cdot (m \otimes n) = (-1)^{\deg(b)\deg(m)} am \otimes bn$. The $\bigwedge V \rtimes G$-module structure of $M \otimes N$ is via Δ. One checks easily that $\sigma_{M,N}$ is an isomorphism of graded $\bigwedge V \rtimes G$-modules.

Since $\otimes$ is exact in both factors it extends naturally to a bifunctor on the bounded derived category $\mathcal{K} = D^b(\bigwedge V \rtimes G - \mathrm{mod})$; even on the unbounded derived category but we take D^b to fix ideas. The unit object $\mathbf{1}$ is the complex $\cdots \to 0 \to k\langle 0\rangle \to 0 \to \cdots$ concentrated in degree 0. This turns the derived category into a tensor triangulated category $(\mathcal{K}, \otimes, \mathbf{1})$ in Balmer's terminology, cf. [Bal10a, Df. 1.2] and [Bal10b, Df. 3].

For every invertible object $u \in \mathcal{K}$ Balmer introduces the graded central ring $R^\bullet_{\mathcal{K},u} = \mathrm{End}^\bullet_{\mathcal{K}}(\mathbf{1})$ with nth graded piece $\mathrm{Hom}_{\mathcal{K}}(\mathbf{1}, u^{\otimes n})$. As shown in [Bal10a, Prop. 3.3(e)] the ring $R^\bullet_{\mathcal{K},u}$ is ϵ_u-commutative, where $\epsilon_u = \pm 1$ is the sign given by the switch $\sigma_{u,u} : u \otimes u \xrightarrow{\sim} u \otimes u$.

Example 4.6. For $\mathcal{K} = D^b(\bigwedge V \rtimes G - \mathrm{mod})$ there are two natural choices of invertible u:

- $u = \Sigma(\mathbf{1}) = \Sigma(k\langle 0\rangle)$. In this case $R^\bullet_{\mathcal{K},u} = \mathrm{ext}^*_{\bigwedge V\rtimes G}(k, k)$ is the small Yoneda algebra

- $v = \Sigma(k\langle -1\rangle)$. In this case $R^\bullet_{\mathcal{K},v} = H^{\text{diag}}(\bigwedge V \rtimes G, k)$ is the diagonal algebra.

Lemma 4.7. *$\epsilon_u = -1$ and $\epsilon_v = +1$.*

Proof. First observe that the switch $M^\bullet \otimes N^\bullet \xrightarrow{\sim} N^\bullet \otimes M^\bullet$ for complexes involves an additional Koszul sign. The degree n part $\sigma^n_{M^\bullet,N^\bullet}$ is the alternating direct sum $\bigoplus_{i+j=n}(-1)^{ij}\sigma_{M^i,N^j}$ of module switches.

Consider ϵ_u. Here $u \otimes u = \Sigma(\mathbf{1}) \otimes \Sigma(\mathbf{1}) = \Sigma^2(k\langle 0\rangle)$ sits in degree -2 and $i = j = -1$ in the previous paragraph. Since $\sigma_{k\langle 0\rangle,k\langle 0\rangle} = \text{Id}$ we find that $\sigma_{u,u} = -\text{Id}$.

Now consider ϵ_v. Here $v \otimes v = \Sigma^2(k\langle -2\rangle)$ is the graded module $k\langle -2\rangle$ sitting in degree -2. Again we have $i = j = -1$ above but now $\sigma_{k\langle -1\rangle,k\langle -1\rangle} = -\text{Id}$ so we conclude $\sigma_{v,v} = +\text{Id}$. □

In other words $\text{ext}^*_{\bigwedge V \rtimes G}(k,k)$ is graded-commutative, and $H^{\text{diag}}(\bigwedge V \rtimes G, k)$ is commutative. In the next subsection we will identify the latter with $S(\tilde{V})^G$ via the restriction map ϱ.

Remark 4.8. The proof of [Bal10a, Prop. 3.3(e)] is a bit short. Suarez-Alvarez has a more detailed account [SA04] where he adapts the Eckmann-Hilton argument to $\text{End}^\bullet_{\mathcal{K}}(\mathbf{1})$. He uses the term "suspended monoidal category" instead of tensor triangulated category, and for full disclosure he only deals with $\Sigma(\mathbf{1})$ not arbitrary invertible u. The graded-commutativity is [SA04, Thm. 1.7], and our observation that $\epsilon_u = -1$ is essentially his Hopf algebra example [SA04, Ex. 2.2].

In hindsight, the above commutativity properties (Lemma 4.7) fit under one umbrella. The full cohomology algebra $H^*(\bigwedge V \rtimes G, k)$ is bigraded, and we claim it is ε-commutative relative to the dot product $\varepsilon : \mathbb{Z}^2 \times \mathbb{Z}^2 \longrightarrow \mathbb{Z}$, cf. [DAS13, Df. 2.4]. I.e., two classes $x \in H^{n,i}(\bigwedge V \rtimes G, k)$ and $x' \in H^{n',i'}(\bigwedge V \rtimes G, k)$ commute up to a sign $(-1)^{nn'+ii'}$. This seems to be well-known. It is discussed in passing in [Dru16, Sect. 5.1] for arbitrary supergroup schemes, where the result is inferred from [MPSW10, Thm. 3.12].

4.3. **Finiteness properties.** We record the following consequence of Theorem 3.1.

Corollary 4.9. *$\text{ext}^*_{\bigwedge V \rtimes G}(k,k)$ is a finitely generated graded-commutative k-algebra (that is, a quotient of the free graded-commutative algebra $S(W^{even}) \otimes \bigwedge(W^{odd})$ for some finite-dimensional graded vector space W); in particular it is Noetherian.*

Proof. Introduce the ideal $\mathcal{J} = \bigoplus_{i>0} H^{*,i}(\bigwedge V \rtimes G, k)$ of the full cohomology $H^*(\bigwedge V \rtimes G, k)$. As the latter is Noetherian, so is the quotient

$$H^*(\bigwedge V \rtimes G, k)/\mathcal{J} \xrightarrow{\sim} \text{ext}^*_{\bigwedge V \rtimes G}(k,k).$$

Pick generators $x_1, \dots, x_N$ for the ideal $\text{ext}^+_{\bigwedge V \rtimes G}(k,k)$. An easy and standard inductive argument shows every element of $\text{ext}^*_{\bigwedge V \rtimes G}(k,k)$ is a k-polynomial expression in the x_is, cf. the proof of Lemma 2.9. Let $W = \bigoplus_{i=1}^N ke_i$ be a graded vector space with $\deg(e_i) = \deg(x_i)$ for all i. Sending $e_i \mapsto x_i$ then extends to a surjective map

$$S(W^{\text{even}}) \otimes \bigwedge(W^{\text{odd}}) \twoheadrightarrow \text{ext}^*_{\bigwedge V \rtimes G}(k,k)$$

as required. This proves finite generation of the small Yoneda algebra. □

Next we show $H^{\text{diag}}(\bigwedge V \rtimes G, k)$ is isomorphic to $S(\tilde{V})^G$ via the restriction map ϱ.

Corollary 4.10. *The restriction $\varrho : H^{diag}(\bigwedge V \rtimes G, k) \longrightarrow S(\tilde{V})^G$ is an isomorphism.*

Proof. The restriction map $\varrho^n : H^n(\bigwedge V \rtimes G, k) \longrightarrow H^n(\bigwedge V, k) \simeq S^n(\tilde{V})$ preserves the internal grading and decomposes as a direct sum of maps

$$\varrho^{n,i} : H^{n,i}(\bigwedge V \rtimes G, k) \longrightarrow H^{n,i}(\bigwedge V, k) \simeq \begin{cases} S^n(\tilde{V}) & \text{if } n = i \\ 0 & \text{if } n \neq i. \end{cases}$$

(Here we have used the Koszulity of $\bigwedge V$ – see the proof of Lemma 4.1: The resolution there has W_j concentrated in degree j.) We deduce that ϱ vanishes on $\bigoplus_{n\neq i} H^{n,i}(\bigwedge V \rtimes G, k)$, and in particular

$$H^*(\bigwedge V \rtimes G, k) = H^{\mathrm{diag}}(\bigwedge V \rtimes G, k) + \ker(\varrho).$$

Since ϱ is surjective on the full $H^*(\bigwedge V \rtimes G, k)$ it remains surjective upon restriction to $H^{\mathrm{diag}}(\bigwedge V \rtimes G, k)$.

To show $\varrho^{n,n}$ is bijective it therefore suffices to check that $\dim_k H^{n,n}(\bigwedge V \rtimes G, k) = \dim_k S^n(\tilde{V})^G$. This follows from another spectral sequence argument. Negron's isomorphism (2.11) of DGA's alluded to in the proof of Proposition 2.10 extends to the graded setting by a close inspection of the construction in [Neg15]. (Negron has confirmed this in private correspondence with the author.) Thus, for any graded $\bigwedge V \rtimes G$-bimodule M there is a quasi-isomorphism

$$R\mathrm{Hom}_{(\bigwedge V \rtimes G)^e-\mathrm{mod}}(\bigwedge V \rtimes G, M) \xrightarrow{\sim} R\mathrm{Hom}_G\big(k, R\mathrm{Hom}_{\bigwedge V^e-\mathrm{mod}}(\bigwedge V, M)\big),$$

where the $R\mathrm{Hom}$'s are now computed in the respective categories of graded bimodules. Fixing an $i \geq 0$ and taking M to be the trivial bimodule $k\langle -i\rangle$ gives a hypercohomology spectral sequence:

$$E_2^{s,t} = H^s\big(G, \mathrm{Ext}^t_{\bigwedge V-\mathrm{mod}}(k, k\langle -i\rangle)\big) \Longrightarrow \mathrm{Ext}^{s+t}_{\bigwedge V \rtimes G-\mathrm{mod}}(k, k\langle -i\rangle). \tag{4.11}$$

(Here we have used [CE56, p. 185] to go between Hochschild cohomology and cohomology of augmented algebras.) As noted in the first paragraph of the proof, Koszulity of $\bigwedge V$ implies (4.11) is concentrated on the horizontal line $t = i$. In particular there is an isomorphism of vector spaces

$$H^{n,i}(\bigwedge V \rtimes G, k) \simeq H^{n-i}(G, S^i(\tilde{V}))$$

for all $n \geq i$. The special case $n = i$ gives the result by comparing dimensions. □

The above proof works verbatim for any Koszul algebra A with G-action and gives an isomorphism $H^{\mathrm{diag}}(A \rtimes G, k) \xrightarrow{\sim} (A^!)^G$ where the diagonal subalgebra $H^{\mathrm{diag}}(A \rtimes G, k)$ is defined in the hopefully obvious way analogous to Definition 4.4.

Remark 4.12. When $p = \mathrm{char}(k)$ does <u>not</u> divide $|G|$ the restriction map is an isomorphism, cf. Example 2.4, so in this case we have

$$H^{\mathrm{diag}}(\bigwedge V \rtimes G, k) = H^*(\bigwedge V \rtimes G, k) \xrightarrow{\sim} S(\tilde{V})^G.$$

In other words $H^*(\bigwedge V \rtimes G, k)$ is concentrated on the diagonal, and in particular $\mathrm{ext}^*_{\bigwedge V \rtimes G}(k, k) = k$.

5. Applications

In this section we discuss two applications of our finiteness results. The first in the theory of support varieties, the other in number theory.

5.1. **Balmer spectra and support varieties.** We put ourselves in the setup above. Thus V is a fixed G-representation over a field k. We employ the notation introduced in 4.2. Recall the tensor triangulated category $\mathcal{K} = D^b(\bigwedge V \rtimes G - \text{mod})$ and the graded central rings $R^\bullet_{\mathcal{K},u}$ and $R^\bullet_{\mathcal{K},v}$ which we identify with the small Yoneda algebra and the diagonal algebra respectively. We now know they are both Noetherian (Corollary 4.9 and Corollary 4.10.)

Balmer associates a topological space $\text{Spc}(\mathcal{K})$. It is the set of all proper thick prime $\otimes$-ideals $\mathcal{P} \subset \mathcal{K}$ equipped with an analogue of the Zariski topology. One can even turn $\text{Spc}(\mathcal{K})$ into a locally ringed space, but this will not be relevant here. The key feature of $\text{Spc}(\mathcal{K})$ is it carries a support datum. That is, there is a sensible way of attaching a closed subset $\text{supp}(x) \subset \text{Spc}(\mathcal{K})$ to any object $x \in \mathcal{K}$. The definition is in fact quite simple; $\text{supp}(x) = \{\mathcal{P} : x \notin \mathcal{P}\}$. More importantly the pair $(\text{Spc}(\mathcal{K}), \text{supp})$ is in some sense best possible – it is the *final* support datum: Any other pair (X, σ) arises by pulling back supports via a continuous map $X \to \text{Spc}(\mathcal{K})$. His theory is very nicely surveyed in his ICM paper [Bal10b].

In complete generality, i.e. for any tensor triangulated category, Balmer defines continuous maps out of $\text{Spc}(\mathcal{K})$ into homogeneous spectra of graded central rings $R^\bullet_{\mathcal{K}}$ – see [Bal10a]:

$$\rho^\bullet_{\mathcal{K}} : \text{Spc}(\mathcal{K}) \longrightarrow \text{Spec}^{\text{h}}(R^\bullet_{\mathcal{K}})$$

The spectrum $\text{Spc}(\mathcal{K})$ is a spectral space (in the sense of Hochster) so it is natural expect a connection to the algebraic geometry of the various graded central rings $R^\bullet_{\mathcal{K}}$ one associates with $\mathcal{K}$. The homogeneous spectrum Spec^{h} is the set of all graded prime ideals (the difference with Proj being we allow those containing the irrelevant ideal $R^+_{\mathcal{K}}$). Take $R^\bullet_{\mathcal{K},u}$ for instance. The definition of $\rho^\bullet_{\mathcal{K},u}$ is not complicated: A prime $\otimes$-ideal $\mathcal{P} \subset \mathcal{K}$ is sent to the ideal $\rho^\bullet_{\mathcal{K},u}(\mathcal{P})$ generated by all homogeneous $f \in R^\bullet_{\mathcal{K},u}$ for which $\text{cone}(f) \notin \mathcal{P}$. (An f of degree n is a morphism $\mathbf{1} \to u^{\otimes n}$ in $\mathcal{K}$ and its cone fits in a distinguished triangle $\mathbf{1} \xrightarrow{f} u^{\otimes n} \to \text{cone}(f) \to \Sigma(\mathbf{1})$.)

One of Balmer's main results, see [Bal10a, Thm. 7.3], says that $\rho^\bullet_{\mathcal{K}}$ is surjective provided $R^\bullet_{\mathcal{K}}$ is Noetherian (or just coherent) – and $u = \Sigma(\mathbf{1})$. This last assumption is stated in [Bal10a, p. 1548] just prior to his Theorem 7.3, and is used in the proof of his Proposition 7.5. Combined with our results this gives surjectivity in our setup:

Corollary 5.1. *For $\mathcal{K} = D^b(\bigwedge V \rtimes G - mod)$ Balmer's map $\rho^\bullet_{\mathcal{K},u}$ is surjective where $u = \Sigma(\mathbf{1})$.*

We summarize our situation in the diagram:

$$\text{Spec}^{\text{h}}\big(\text{ext}^*_{\bigwedge V \rtimes G}(k,k)\big) \xleftarrow{\rho^\bullet_{\mathcal{K},u}} \text{Spc}\big(D^b(\bigwedge V \rtimes G - \text{mod})\big) \xrightarrow{\rho^\bullet_{\mathcal{K},v}} \text{Spec}^{\text{h}}\big(S(\tilde{V})^G\big).$$

The target of $\rho^\bullet_{\mathcal{K},v}$ has a nice geometric interpretation. Ignoring the extra irrelevant point it is the projective GIT quotient $\mathbb{P}(\tilde{V})/\!\!/G$. We have just observed $\rho^\bullet_{\mathcal{K},u}$ is surjective. Relying on the Appendix we will show below in Corollary 5.2 that also $\rho^\bullet_{\mathcal{K},v}$ is surjective in general.

- When $V = \{0\}$ our category is $\mathcal{K} = D^b(k[G] - \text{mod})$ and the Yoneda algebra is $H^*(G,k)$. By combining various results from modular representation theory (by Benson, Carlson, Rickard in [BCR97] – and others) Balmer shows

$$\rho^\bullet_{\mathcal{K},u} : \text{Spc}\big(D^b(k[G] - \text{mod})\big) \longrightarrow \text{Spec}^{\text{h}}\big(H^*(G,k)\big)$$

is a homeomorphism in [Bal10a, Prop. 8.5]. Here $\rho^\bullet_{\mathcal{K},v}$ collapses to the point $\text{Spec}^{\text{h}}(k)$.

- When $G = \{e\}$ our category is $\mathcal{K} = D^b(\bigwedge V - \mathrm{mod})$ and the small Yoneda algebra is just k, see Remark 4.3. In this case $\rho^\bullet_{\mathcal{K},u}$ collapses to the point $\mathrm{Spec}^{\mathrm{h}}(k)$, and $\rho^\bullet_{\mathcal{K},v}$ is a continuous map
$$\rho^\bullet_{\mathcal{K},v} : \mathrm{Spc}\big(D^b(\textstyle\bigwedge V - \mathrm{mod})\big) \longrightarrow \mathrm{Spec}^{\mathrm{h}}\big(S(\tilde{V})\big).$$
In the Appendix we verify that in fact $\rho^\bullet_{\mathcal{K},v}$ is a homeomorphism, see Proposition A.6. The proof is a bit involved so we opt to defer it to the Appendix to not go off on a tangent and disrupt the flow of the paper. We like to think of Proposition A.6 as a tensor triangular reformulation of Koszul duality; or rather the Bernstein-Gelfand-Gelfand correspondence – see [BGG78] and [BGS96]. See also Proposition A.9. This example is closely related to [BIK11, Thm. 6.4] which gives a complete classification of all localizing subcategories of $K(\mathrm{Inj} \bigwedge V)$.

Using that $\rho^\bullet_{\mathcal{K},v}$ is a homeomorphism when $G = \{e\}$ (as Proposition A.6 states) we infer that $\rho^\bullet_{\mathcal{K},v}$ is surjective in general:

Corollary 5.2. *For $\mathcal{K} = D^b(\bigwedge V \rtimes G - mod)$ Balmer's map $\rho^\bullet_{\mathcal{K},v}$ is surjective where $v = \Sigma(k\langle -1\rangle)$.*

Proof. For the purposes of this proof let $\mathcal{K}_e = D^b(\bigwedge V - \mathrm{mod})$. The forgetful functor $\mathcal{K} \to \mathcal{K}_e$ induces a continuous map of spectra $\mathrm{Spc}(\mathcal{K}_e) \to \mathrm{Spc}(\mathcal{K})$ by pulling back $\otimes$-ideals. It also induces a homomorphism $R^\bullet_{\mathcal{K},v} \to R^\bullet_{\mathcal{K}_e,v}$ which we identify with the inclusion $S(\tilde{V})^G \hookrightarrow S(\tilde{V})$. This gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Spc}(\mathcal{K}) & \xrightarrow{\rho^\bullet_{\mathcal{K},v}} & \mathrm{Spec}^{\mathrm{h}}\big(S(\tilde{V})^G\big) \\ \uparrow & & \uparrow \pi \\ \mathrm{Spc}(\mathcal{K}_e) & \xrightarrow{\rho^\bullet_{\mathcal{K}_e,v}} & \mathrm{Spec}^{\mathrm{h}}\big(S(\tilde{V})\big) \end{array}$$

where the bottom horizontal arrow $\rho^\bullet_{\mathcal{K}_e,v}$ is a homeomorphism by Proposition A.6. The map labelled π is surjective by the going-up theorem for graded algebras – which applies since $S(\tilde{V})^G \hookrightarrow S(\tilde{V})$ is finite by Noether's theorem. This implies $\rho^\bullet_{\mathcal{K},v}$ is surjective. $\square$

In the mixed case where both V and G are non-trivial we expect neither of the two maps $\rho^\bullet_{\mathcal{K},u}$ and $\rho^\bullet_{\mathcal{K},v}$ to be injective. For general categories $\rho^\bullet_{\mathcal{K},u}$ is not necessarily injective. However, for many categories of algebraic nature it is. In [Bal10b, Conj. 72] it is conjectured that $\rho^\bullet_{\mathcal{K},u}$ is (locally) injective when the category $\mathcal{K}$ is "algebraic enough" – whatever that means. The above example $D^b(\bigwedge V - \mathrm{mod})$ would violate the *global* conjecture unless this category is *not* algebraic enough. Experts have emphasized to us that the key word is really the parenthetical *locally*, but since $\mathrm{Spc}\big(D^b(\bigwedge V - \mathrm{mod})\big)$ is not discrete the meaning of locally is unclear as well – it cannot be in the literal topological sense. So there is some flexibility, and we hope this example will help make Balmer's conjecture more precise.

The following observation is already hinted at in the paragraph following [Bal10b, Conj. 72] ("It might also be necessary to add some hypothesis like $\mathcal{K}$ being locally generated by $\mathbf{1}$"). See also [Bal10c, Cor. 1.9] and the pertaining discussion of cellular subcategories.

Remark 5.3. We let $\mathcal{K}^\square = \langle\langle \mathbf{1} \rangle\rangle$ be the smallest strictly full thick triangulated subcategory of $\mathcal{K}$ containing the unit $\mathbf{1}$. This is preserved by $\otimes$. Taking the invertible object to be $u = \Sigma(\mathbf{1}) \in \mathcal{K}^\square$ in the definition of graded central rings shows $R^\bullet_{\mathcal{K}} = R^\bullet_{\mathcal{K}^\square}$. (Instead of $\mathcal{K}^\square$ one can take any full tt-subcategory containing $\mathbf{1}$ here.) Therefore, if $\rho^\bullet_{\mathcal{K}}$ is injective the pullback $\mathrm{Spc}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K}^\square)$ must be injective. In this sense $\mathcal{K}^\square$ should distinguish the prime $\otimes$-ideals $\mathcal{P} \subset \mathcal{K}$. When $\mathcal{K}^\square$ is *rigid* (that is, it admits internal homs adjoint to $\otimes$) the map $\mathrm{Spc}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K}^{\square})$ is known to be surjective; see [Bal10c, Cor. 1.8] for a more general statement. So for rigid categories it is less critical to distinguish $\mathcal{K}$ and $\mathcal{K}^{\square}$ when talking about spectra.

Our example $D^b(\bigwedge V - \mathrm{mod})$ is generated as a thick subcategory by $\mathbf{1} = k\langle 0\rangle[0]$ and all its $\langle\cdot\rangle$-shifts (this follows immediately from Koszul duality) so additional assumptions are needed to make [Bal10b, Conj. 72] exact in general. Perhaps one should impose compactness of $\mathbf{1}$ when $\mathcal{K}$ admits infinite direct sums. In the case of $D^b(\bigwedge V \rtimes G - \mathrm{mod})$ we offer a suggestion below. Our results indicate the following might be true in the mixed case.

Question 5.4. *Suppose $S(\tilde{V})^G$ has finite global dimension. Is the product map*

$$\rho^{\bullet}_{\mathcal{K},u} \times \rho^{\bullet}_{\mathcal{K},v} : Spc\big(D^b(\textstyle\bigwedge V \rtimes G - mod)^{\square}\big) \longrightarrow Spec^h\big(ext^*_{\bigwedge V \rtimes G}(k,k)\big) \times Spec^h\big(S(\tilde{V})^G\big)$$

always a homeomorphism? If not, is it at least injective? If this fails in general, can it be salvaged by assuming G acts trivially on V?

As noted above this holds in the extreme cases $V = \{0\}$ and $G = \{e\}$. Let us point out that the category $\mathcal{K} = D^b(k[G] - \mathrm{mod})$ is rigid (internal homs are given by the usual morphism complex). Therefore in this case $\mathrm{Spc}(\mathcal{K}) \xrightarrow{\sim} \mathrm{Spc}(\mathcal{K}^{\square})$ as explained in Remark 5.3. Generalizing the case $G = \{e\}$, in Proposition A.11 we show the first answer is yes in the non-modular case – provided $S(\tilde{V})^G$ has finite global dimension (note that $\mathrm{ext}^*_{\bigwedge V \rtimes G}(k,k) = k$ when $\mathrm{char}(k) \nmid |G|$ as pointed out in Remark 4.12).

This is somewhat meager evidence, and this prevents us from conjecturing an affirmative answer – which would unify Koszul duality and fundamental results in modular representation theory.

Perhaps a more promising approach would be to work with the full cohomology $H^*(\bigwedge V \rtimes G, k)$. As we noted in the last paragraph of Section 4.2 this is an ε-commutative bigraded algebra (with ε being the dot product on $\mathbb{Z}^2$) and it makes sense to talk about its homogeneous spectrum $\mathrm{Spec}^{\mathrm{h}} H^*(\bigwedge V \rtimes G, k)$. See [DAS13, Sect. 2] for instance. One can interpret the full cohomology as a combination of Balmer's rings $R^{\bullet}_{\mathcal{K},u}$ and $R^{\bullet}_{\mathcal{K},v}$ by introducing the *bigraded* central ring

$$R^{\bullet}_{\mathcal{K},u,v} == \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{K}}(\mathbf{1}, u^{\otimes i} \otimes v^{\otimes j}) = \bigoplus_{i,j} H^{i+j,j}(\textstyle\bigwedge V \rtimes G, k).$$

We are optimistic Balmer's definition of $\rho^{\bullet}_{\mathcal{K},u}$ and $\rho^{\bullet}_{\mathcal{K},v}$ works in the bigraded setting and gives a continuous map $\rho^{\bullet}_{\mathcal{K},u,v} : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathrm{Spec}^{\mathrm{h}}(R^{\bullet}_{\mathcal{K},u,v})$. Granted this makes sense it would be natural to ask the following.

Question 5.5. *Knowing $H^*(\bigwedge V \rtimes G, k)$ is Noetherian, does that imply $\rho^{\bullet}_{\mathcal{K},u,v}$ is surjective? As a follow-up question, is the map*

$$\rho^{\bullet}_{\mathcal{K}^{\square},u,v} : Spc\big(D^b(\textstyle\bigwedge V \rtimes G - mod)^{\square}\big) \longrightarrow Spec^h\big(H^*(\textstyle\bigwedge V \rtimes G, k)\big)$$

always a homeomorphism?

Results of Hovey and Palmieri trigger optimism. In [HP01, Cor. 3.5] they classify $\otimes$-ideals in (certain) stable homotopy categories $\mathcal{C}$ with the "tensor product property" in terms of specialization-closed subsets of $\mathrm{Spec}^{\mathrm{h}}\pi_{**}(\mathbf{1}_{\mathcal{C}})$. A finite-dimensional graded cocommutative Hopf algebra B gives a stable homotopy category $\mathcal{C} = K(\mathrm{Inj}B)$ with $\pi_{**}(\mathbf{1}_{\mathcal{C}}) = \mathrm{Ext}^*_B(k,k)$. When this category has the tensor product property [HP01, Df. 3.1] we get a bijection between $\otimes$-ideals of $K(\mathrm{Inj}B)$ and specialization-closed subsets of $\mathrm{Spec}^{\mathrm{h}}\mathrm{Ext}^*_B(k,k)$ – as recorded in [HP01, Cor. 3.7(c)]. This seems to suggest we at least have a homeomorphism $\mathrm{Spec}^{\mathrm{h}}\mathrm{Ext}^*_B(k,k) \longrightarrow \mathrm{Spc}K(\mathrm{Inj}B)$. As a first step in leveraging these results in the special case

$B = \bigwedge V \rtimes G$ one should check whether the tensor product property holds. This is known for elementary Hopf algebras B. We hope to return to this in future work.

5.2. **Schneider's DG Hecke algebras.** Let G be a p-adic Lie group and $I \subset G$ a torsionfree pro-p subgroup. In [Sch15] Schneider establishes a mod p analogue of a classical result of Borel and Bernstein by passing to derived categories. Thus $D(G)$ denotes the derived category of smooth G-representations on k-vector spaces for a fixed field k of characteristic p. The fundamental observation is that the compactly induced representation $\mathbf{X} = \mathrm{ind}_I^G(1)$ generates $D(G)$. The classical Hecke algebra $\mathcal{H}_I^0$ is replaced by a differential graded extension $\mathcal{H}_I^\bullet$, which depends on a choice of injective resolution of $\mathbf{X}$ (although only up to quasi-isomorphism). Based on results of Keller (and others) Schneider deduces a derived equivalence

$$D(G) \xrightarrow{\sim} D(\mathcal{H}_I^\bullet).$$

The precise definition of $\mathcal{H}_I^\bullet$ is immaterial in this article. We will only be concerned with its cohomology algebra $h(\mathcal{H}_I^\bullet) = \mathrm{Ext}_G^*(\mathbf{X}, \mathbf{X})$. With the usual conventions we should really take the opposite algebra on the right-hand side, but we will suppress that.

We now assume G is compact and $I \triangleleft G$ is normal. Then $h(\mathcal{H}_I^\bullet)$ can be identified with a skew group algebra (we learned this fact from Schneider):

Lemma 5.6. *There is an isomorphism of graded algebras $h(\mathcal{H}_I^\bullet) \simeq H^*(I, k) \rtimes G/I$.*

Proof. Since I is normal it acts trivially on $\mathbf{X}$. Therefore by Shapiro we have an isomorphism of graded vector spaces

$$h(\mathcal{H}_I^\bullet) \simeq H^*(I, \mathbf{X}) \simeq H^*(I, k) \otimes k[G/I].$$

As a very special case of [OS19, Prop. 5.3] the algebra structure on $h(\mathcal{H}_I^\bullet)$ corresponds to the skew algebra structure on the right-hand side. (This can also be checked directly by hand.) □

Most interesting I admit a p-valuation ω in the sense of Lazard, cf. the exposition [Sch11]. Best-case scenario is when (I, ω) is equi-p-valued, which means I admits an ordered basis $(g_1, \ldots, g_d)$ such that $\omega(g_i)$ is independent of i. For equi-p-valuable I Lazard identified the cohomology $H^*(I, k)$ with the exterior algebra $\bigwedge \mathrm{Hom}(I, k)$. Thus in this case $h(\mathcal{H}_I^\bullet)$ is an example of an exterior group algebra:

$$h(\mathcal{H}_I^\bullet) \simeq \bigwedge \mathrm{Hom}(I, k) \rtimes G/I.$$

Applying our main Theorem 3.1 to the G/I-representation $\mathrm{Hom}(I, k)$ yields the result below.

Corollary 5.7. *Let G be a compact p-adic Lie group and $I \triangleleft G$ an equi-p-valuable normal subgroup. Then the following holds.*

(1) *$Ext^*_{h(\mathcal{H}_I^\bullet)}(k, k)$ is a Noetherian and finitely generated (possibly non-commutative) k-algebra.*

(2) *$ext^*_{h(\mathcal{H}_I^\bullet)}(k, k)$ is a Noetherian and finitely generated (graded-commutative) k-algebra.*

At least when G is pro-p (possibly with torsion) we interpret (1) as saying $h(\mathcal{H}_I^\bullet)$ is a complete intersection, cf. the Bøgvad-Halperin criterion discussed in the paragraph after Theorem 3.1. Globally such ring-theoretic properties of Hecke algebras are of fundamental importance in number theory. Perhaps the most famous example is [TW95, Thm. 1] saying that a certain minimal Hecke algebra $\mathbb{T}_\varnothing$ is a complete intersection – which was a key part of the proof of Fermat's Last Theorem. This analogy should not be pushed too far though: $h(\mathcal{H}_I^\bullet)$ is usually non-commutative and of characteristic p, whereas $\mathbb{T}_\varnothing$ is commutative of characteristic 0 and defined via modular forms.

We also wish to point out the resemblance of $\mathrm{Ext}^*_{h(\mathcal{H}_I^\bullet)}(k,k)$ to the double Yoneda algebras considered in [BR86] for instance. The analogy becomes more transparent if we express it as $\mathrm{Ext}^*_{\mathrm{Ext}^*_G(\mathbf{X},\mathbf{X})}(k,k)$.

Example 5.8. When $G=I$ is equi-p-valuable the big Yoneda algebra in Corollary 5.7 is the symmetric algebra on the dual space of $\mathrm{Hom}(G,k)$. The small Yoneda algebra is k.

Finally let us emphasize that pairs (G,I) as in Corollary 5.7 arise naturally as follows. Let $G=\mathcal{G}(\mathbb{Z}_p)$ for a smooth affine group scheme $\mathcal{G}_{/\mathbb{Z}_p}$. Suppose $p>2$. As explained in [HKN11, Lem. 2.2.2] the kernel

$$I=\ker\big(\mathcal{G}(\mathbb{Z}_p)\longrightarrow\mathcal{G}(\mathbb{F}_p)\big)$$

is equi-p-valuable by identifying it with the Serre standard group associated with the formal group law for $\mathcal{G}$. One can generate a whole neighborhood basis of open normal equi-p-valuable subgroups of G by taking powers I^{p^r}. This example even works with $\mathbb{Z}_p$ replaced by $\mathcal{O}_F$ for any unramified extension $F/\mathbb{Q}_p$.

Appendix A. Koszul duality and tensor products

For ease of comparison with [BGS96] we let $A=\bigwedge V$ and $A^!=S(\tilde{V})$ in this appendix. Our primary goal is to show the Koszul duality functor

$$K:D^b(A-\mathrm{mod})\xrightarrow{\sim}D^b(A^!-\mathrm{mod})$$

commutes with tensor products. On the source $\otimes$ arises from the comultiplication Δ on A, as discussed in the main text, whereas on the target $\otimes$ is the derived tensor product $\otimes^L_{A^!}$. This is probably well-known, but we have not been able to find it in the literature. The closest we could get is the first remark after [BGG78, Thm. 1] which states the BGG functor $\Phi(\cdot)$ commutes with tensor products. They omit the proof, and their result is not quite sufficient for our purposes anyway. So we decided to include the somewhat technical details here. At the end we explain how these results generalize in the presence of a G-action on V.

We need to first recall the precise definition of K from [BGS96, p. 489]. The input M is a bounded complex of finitely generated graded A-modules

$$\cdots\longrightarrow M^{i-1}\xrightarrow{\partial}M^i\xrightarrow{\partial}M^{i+1}\longrightarrow\cdots.$$

The grading is indicated by subscripts as in $M^i=\bigoplus_{j\in\mathbb{Z}}M^i_j$. Analogously for A and $A^!$. The initial step is to turn the bigraded vector space $A^!\otimes M$ into a double complex

$$\begin{array}{ccc} A^!_r\otimes M^i_j & \xrightarrow{d''} & A^!_r\otimes M^{i+1}_j \\ \downarrow{\scriptstyle d'} & & \\ A^!_{r+1}\otimes M^i_{j+1} & & \end{array}$$

The two anti-commuting differentials are defined as follows. Once and for all we fix vectors $v_\alpha\in V$ and $\tilde{v}_\alpha\in\tilde{V}$ such that $\sum\tilde{v}_\alpha\otimes v_\alpha\mapsto\mathrm{Id}_V$ under the isomorphism $\tilde{V}\otimes V\xrightarrow{\sim}\mathrm{End}(V)$.

Definition A.1. The differentials d' and d'' are given by the formulas below, for $a\in A^!_r$ and $m\in M^i_j$.

$$d'(a\otimes m)=(-1)^{i+j}\sum a\tilde{v}_\alpha\otimes v_\alpha m,\qquad d''(a\otimes m)=a\otimes\partial(m).$$

(In the formula for d' we embed $V \hookrightarrow A$ and $\tilde{V} \hookrightarrow A^!$.)

The second step is to pass to the total complex $T(M)$ with differential $d = d' + d''$. This has nth term $T(M)^n = \bigoplus_{r+i=n} A^!_r \otimes M^i$ which carries a natural grading preserved by d. Observe that both d' and d'' fix the difference $r - j$. Thus, for $q \in \mathbb{Z}$ we let $T(M)^n_q = \bigoplus_{r+i=n, r-j=q} A^!_r \otimes M^i_j$ which gives $T(M)$ the structure of a graded complex. Note that the left action of an element of $A^!_\ell$ sends $T(M)^n_q \to T(M)^{n+\ell}_{q+\ell}$, and in particular $T(M)^n$ is not an $A^!$-module.

The third and final step in the construction of K is regrading in order to get a complex of $A^!$-modules. As in Step 1 of the proof of [BGS96, Thm. 2.12.1] we change the grading of the total complex and introduce $K(M)^s_q = T(M)^{s+q}_q$. In this notation $A^!_\ell$ sends $K(M)^s_q \to K(M)^s_{q+\ell}$. In other words $K(M)^s = \bigoplus_{q \in \mathbb{Z}} K(M)^s_q$ is a graded $A^!$-module. We summarize this definition below in a slightly different notation.

Definition A.2. For $s \in \mathbb{Z}$ introduce the graded vector space $M^{(s)} = \bigoplus_{j \in \mathbb{Z}} M^{s+j}_{-j}$, and the associated free graded $A^!$-module $K(M)^s = A^! \otimes M^{(s)}$ (finitely generated by our assumptions on M). The differentials of the complex

$$\cdots \longrightarrow K(M)^{s-1} \xrightarrow{d} K(M)^s \xrightarrow{d} K(M)^{s+1} \longrightarrow \cdots$$

are determined as follows. For elements $a \otimes m \in A^!_r \otimes M^i_j$ as above we have the formula

$$d(a \otimes m) = (-1)^{i+j} \sum a\tilde{v}_\alpha \otimes v_\alpha m + a \otimes \partial(m) \in (A^!_{r+1} \otimes M^i_{j+1}) \oplus (A^!_r \otimes M^{i+1}_j).$$

(Note that for $a \otimes m$ to lie in $K(M)^s_q$ the two requirements are that $i + j = s$ and $r - j = q$.)

Invoking the spectral sequence of the double complex shows $K(M)$ is acyclic granted M is, and hence K extends to a functor between the derived categories, cf. the proof of [BGS96, Thm. 2.12.1].

Now suppose we are given two complexes M, N of graded A-modules as before. On the one hand we can form the tensor product complex $M \otimes N$, whose terms become graded A-modules via $\Delta : A \to A \otimes A$, and consider $K(M \otimes N)$. On the other hand we can form $K(M) \otimes^L_{A^!} K(N)$ (we could just write $\otimes_{A^!}$ here since the two factors are perfect). We claim the resulting two complexes are naturally isomorphic.

Proposition A.3. *$K(M) \otimes^L_{A^!} K(N) \xrightarrow{\sim} K(M \otimes N)$.*

Proof. For simplicity let $C = K(M) \otimes^L_{A^!} K(N)$. It has nth term $C^n = \bigoplus_{s+t=n} K(M)^s \otimes_{A^!} K(N)^t$ and differential $d_C = d \otimes 1 + 1 \otimes d$. We fix a pair s, t with $s + t = n$ and consider the corresponding summand

$$K(M)^s \otimes_{A^!} K(N)^t = (A^! \otimes M^{(s)}) \otimes_{A^!} (A^! \otimes N^{(t)}) = A^! \otimes (M^{(s)} \otimes N^{(t)}).$$

Also let $D = K(M \otimes N)$. This is a complex with nth term $D^n = A^! \otimes (M \otimes N)^{(n)}$. By unwinding the definitions $(M \otimes N)^{(n)}$ is the direct sum of all $M^i_j \otimes N^{i'}_{j'}$ with $i + j + i' + j' = n$. Equivalently we have

$$(M \otimes N)^{(n)} = \bigoplus_{s+t=n} M^{(s)} \otimes N^{(t)}.$$

Tensoring with $A^!$ gives an isomorphism $C^n \xrightarrow{\sim} D^n$. However, these isomorphisms do not commute with the differentials. Since the definition of d' involves the sign $(-1)^{i+j}$ which depends on j, we are forced to make a sign change $\epsilon_{i,j,i',j'} \in \{\pm 1\}$ on the summand $M^i_j \otimes N^{i'}_{j'}$. We will determine the appropriate sign change below.

After modifying the canonical map by these signs we obtain an isomorphism $\phi : C^n \xrightarrow{\sim} D^n$. We will find signs $\epsilon_{i,j,i',j'}$ such that these isomorphisms commute with the differentials. To wit, such that the diagram below commutes.

$$\begin{array}{ccc} C^n \supset K(M)^s \otimes_{A^!} K(N)^t & \longrightarrow & (K(M)^{s+1} \otimes_{A^!} K(N)^t) \oplus (K(M)^s \otimes_{A^!} K(N)^{t+1}) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi+\phi} \\ D^n = K(M \otimes N)^n & \xrightarrow{\quad d \quad} & K(M \otimes N)^{n+1} \end{array}$$

Pick random elements $x = a \otimes m \in A^!_r \otimes M^i_j \subset K(M)^s$ and $y = 1 \otimes n \in A^!_0 \otimes N^{i'}_{j'} \subset K(N)^t$. Chase $x \otimes y$ through the diagram.

First let us see what happens to $x \otimes y \in C^n$ if we go right-then-down. Applying d_C yields

$$\begin{aligned} d_C(x \otimes y) &= d(x) \otimes y + (-1)^s x \otimes d(y) \\ &= d'(x) \otimes y + d''(x) \otimes y + (-1)^s x \otimes d'(y) + (-1)^s x \otimes d''(y). \end{aligned}$$

We now work out all four terms explicitly.

(1) $d'(x) \otimes y = (-1)^s \sum (a\tilde{v}_\alpha \otimes v_\alpha m) \otimes (1 \otimes n)$;

(2) $d''(x) \otimes y = (a \otimes \partial(m)) \otimes (1 \otimes n)$;

(3) $(-1)^s x \otimes d'(y) = (-1)^{s+t} \sum (a \otimes m) \otimes (1\tilde{v}_\alpha \otimes v_\alpha n)$;

(4) $(-1)^s x \otimes d''(y) = (-1)^s (a \otimes m) \otimes (1 \otimes \partial(n))$.

The downwards map $\phi + \phi$ sends these to:

(a) $\epsilon_{j+1}(-1)^s \sum a\tilde{v}_\alpha \otimes (v_\alpha m \otimes n)$;

(b) $\epsilon_{i+1}(a \otimes (\partial(m) \otimes n))$;

(c) $\epsilon_{j'+1}(-1)^{s+t} \sum a\tilde{v}_\alpha \otimes (m \otimes v_\alpha n)$;

(d) $\epsilon_{i'+1}(-1)^s (a \otimes (m \otimes \partial(n)))$.

Here we employ the notation $\epsilon_{i+1} = \epsilon_{i+1,j,i',j'}$ etc. (when all but one index remain the same we suppress them).

Next we map $x \otimes y \in C^n$ down-then-right. Its image in D^n via the altered canonical map ϕ is $\epsilon(a \otimes (m \otimes n))$ with the sign being $\epsilon = \epsilon_{i,j,i',j'}$. We work out the two components of $\epsilon d(a \otimes (m \otimes n))$.

(i) $\epsilon d'(a \otimes (m \otimes n)) = \epsilon(-1)^{s+t} \sum a\tilde{v}_\alpha \otimes (v_\alpha m \otimes n) + \epsilon(-1)^{s+t+j} \sum a\tilde{v}_\alpha \otimes (m \otimes v_\alpha n)$;

(ii) $\epsilon d''(a \otimes (m \otimes n)) = \epsilon(a \otimes (\partial(m) \otimes n)) + \epsilon(-1)^i (a \otimes (m \otimes \partial(n)))$.

In part (i) we have used the formula $\Delta(v_\alpha)(m \otimes n) = v_\alpha m \otimes n + (-1)^j m \otimes v_\alpha n$ giving the A-module structure on $M^i \otimes N^{i'}$. Note the sign $(-1)^j$ arising from the Koszul sign convention; we are interchanging v_α and m.

Comparing the four terms in (i) and (ii) to those in (a)–(d) we have agreement given the validity of the four recursive relations

- $\epsilon_{i+1} = \epsilon$
- $\epsilon_{j+1} = \epsilon(-1)^{i'+j'}$
- $\epsilon_{i'+1} = \epsilon(-1)^j$

- $\epsilon_{j'+1} = \epsilon(-1)^j$.

They are satisfied if we take our signs to be $\epsilon_{i,j,i',j'} = (-1)^{(i'+j')j} = (-1)^{tj}$. With this definition of ϕ the above diagram commutes. Done. □

This proof seems overly technical for such a simple statement. If someone has a more conceptual explanation why K should commute with $\otimes$ please inform us.

We finish by recording a few consequences of our last Proposition. First we evaluate the Koszul functor K on our two preferred invertible objects $u = \Sigma(\mathbf{1}) = \Sigma(k\langle 0\rangle)$ and $v = \Sigma(k\langle -1\rangle)$ of $D^b(A - \mathrm{mod})$. This is essentially the content of parts (ii) and (iii) of [BGS96, Thm. 2.12.5]:

Lemma A.4. *$K(u) = \Sigma(\mathbf{1})$ and $K(v) = \mathbf{1}\langle 1\rangle$. (Here the $\mathbf{1}$ on the right denotes the unit object $(A^!)[0]$ of the category $D^b(A^! - mod)$ and Σ is its suspension functor.)*

Proof. The first statement is immediate since K commutes with Σ and $K(\mathbf{1}) = \mathbf{1}$. We will briefly indicate why $K(k\langle -1\rangle) = \Sigma^{-1}(A^!)\langle 1\rangle$. Our complex M is the module $k\langle -1\rangle$ sitting in degree 0. Therefore $M^i_j = 0$ unless $i = 0$ and $j = 1$, and $M^0_1 = k$. It follows that $M^{(s)} = 0$ unless $s = 1$, and $M^{(1)} = k\langle +1\rangle$ (note the sign switch here). We infer that $K(M)^s = 0$ unless $s = 1$, and $K(M)^1 = A^! \otimes k\langle 1\rangle = (A^!)\langle 1\rangle$. □

We note the analogy between $K(v) = \mathbf{1}\langle 1\rangle$ and the twisting sheaf $\mathcal{O}(1)$ on $\mathbb{P}(\tilde{V})$.

To state our next result let $\mathcal{K} = D^b(A - \mathrm{mod})$ and $\tilde{\mathcal{K}} = D^b(A^! - \mathrm{mod})$. As we have just seen, the Koszul functor is an equivalence of tensor triangulated categories $K : \mathcal{K} \xrightarrow{\sim} \tilde{\mathcal{K}}$. It therefore induces isomorphisms between the central graded rings $\mathrm{End}^\bullet_{\mathcal{K}}(\mathbf{1})$ and $\mathrm{End}^\bullet_{\tilde{\mathcal{K}}}(\mathbf{1})$.

Proposition A.5. *$R^\bullet_{\tilde{\mathcal{K}},\Sigma(\mathbf{1})} = k$ and $R^\bullet_{\tilde{\mathcal{K}},\mathbf{1}\langle 1\rangle} = S(\tilde{V})$.*

Proof. By Proposition A.3 and Lemma A.4 the functor K induces isomorphisms of graded algebras

$$R^\bullet_{\mathcal{K},u} \xrightarrow{\sim} R^\bullet_{\tilde{\mathcal{K}},\Sigma(\mathbf{1})}, \qquad R^\bullet_{\mathcal{K},v} \xrightarrow{\sim} R^\bullet_{\tilde{\mathcal{K}},\mathbf{1}\langle 1\rangle}.$$

As already observed in Section 4 we have $R^\bullet_{\mathcal{K},u} = \mathrm{ext}_{\bigwedge V}(k,k) = k$ and $R^\bullet_{\mathcal{K},v} = H^{\mathrm{diag}}(\bigwedge V, k) = S(\tilde{V})$. □

Furthermore the two Balmer maps $\rho^\bullet_{\mathcal{K},v}$ and $\rho^\bullet_{\tilde{\mathcal{K}},\mathbf{1}\langle 1\rangle}$ are compatible; meaning they are the horizontal arrows in a commutative diagram

$$\begin{array}{ccc}
\mathrm{Spc}(D^b(\bigwedge V - \mathrm{mod})) & \longrightarrow & \mathrm{Spec}^{\mathrm{h}}(R^\bullet_{\mathcal{K},v}) \simeq \mathrm{Spec}^{\mathrm{h}}(S(\tilde{V})) \\
{\scriptstyle K}\downarrow\simeq & & {\scriptstyle K}\downarrow\simeq \\
\mathrm{Spc}(D^b(S(\tilde{V}) - \mathrm{mod})) & \longrightarrow & \mathrm{Spec}^{\mathrm{h}}(R^\bullet_{\tilde{\mathcal{K}},\mathbf{1}\langle 1\rangle}) \simeq \mathrm{Spec}^{\mathrm{h}}(S(\tilde{V}))
\end{array}$$

We proceed to showing $\rho^\bullet_{\mathcal{K},v}$ and $\rho^\bullet_{\tilde{\mathcal{K}},\mathbf{1}\langle 1\rangle}$ are homeomorphisms.

The fact that $\rho^\bullet_{\tilde{\mathcal{K}},\mathbf{1}\langle 1\rangle}$ is a homeomorphism essentially follows from [DAS13]. Dell'Ambrogio and Stevenson obtain a homeomorphism $\mu : \mathrm{Spec}^{\mathrm{h}}(S(\tilde{V})) \to \mathrm{Spc}(D^b(S(\tilde{V}) - \mathrm{mod}))$ going in the opposite direction. All we have to check is that $(\rho^\bullet_{\tilde{\mathcal{K}},\mathbf{1}\langle 1\rangle} \circ \mu)(\mathfrak{p}) = \mathfrak{p}$ for all graded prime ideals $\mathfrak{p} \subset S(\tilde{V})$. Then $\rho^\bullet_{\tilde{\mathcal{K}},\mathbf{1}\langle 1\rangle} = \mu^{-1}$. In fact [DAS13] applies to much more exotic ϵ-commutative graded rings S. The map μ arises from the universal property of $\mathrm{Spc}(D^b(S - \mathrm{mod}))$ with its final support datum supp, as we now briefly recall. Dell'Ambrogio and Stevenson study the small support $\sigma(x) = \{\mathfrak{p} : k(\mathfrak{p}) \otimes^L_S x \neq 0\}$ for $x \in D^b(S - \mathrm{mod})$.

They establish that $(\mathrm{Spec}^{\mathrm{h}}(S), \sigma)$ is a support datum, and therefore σ is the pullback of supp along a unique continuous map $\mu : \mathrm{Spec}^{\mathrm{h}}(S) \to \mathrm{Spc}(D^b(S-\mathrm{mod}))$. Their main result [DAS13, Thm. 5.1] implies that μ is a homeomorphism. Note that $D^b(S-\mathrm{mod}) = D^{\mathrm{perf}}(S)$ under the assumption that S is Noetherian and of finite global dimension. The map μ is given explicitly by the recipe $\mu(\mathfrak{p}) = \{x : \mathfrak{p} \notin \sigma(x)\}$ (this is a prime $\otimes$-ideal of $D^b(S-\mathrm{mod})$).

Proposition A.6. *The two horizontal Balmer maps $\rho^{\bullet}_{\mathcal{K},v}$ and $\rho^{\bullet}_{\tilde{\mathcal{K}},\mathbf{1}\langle 1\rangle}$ are homeomorphisms.*

Proof. We check $(\rho^{\bullet}_{\tilde{\mathcal{K}},\mathbf{1}\langle 1\rangle} \circ \mu)(\mathfrak{p}) = \mathfrak{p}$ by adapting the proof of [Bal10a, Prop. 8.1] to our graded setting. Unraveling the definitions shows that $(\rho^{\bullet}_{\tilde{\mathcal{K}},\mathbf{1}\langle 1\rangle} \circ \mu)(\mathfrak{p})$ is the ideal generated by homogeneous $f \in R^{\bullet}_{\tilde{\mathcal{K}},\mathbf{1}\langle 1\rangle}$ for which $\mathrm{cone}(f) \notin \mu(\mathfrak{p}) \Leftrightarrow \mathfrak{p} \in \sigma(\mathrm{cone}(f)) \Leftrightarrow k(\mathfrak{p}) \otimes^L_{S(\tilde{V})} \mathrm{cone}(f) \neq 0$. Here we are identifying $R^{\bullet}_{\tilde{\mathcal{K}},\mathbf{1}\langle 1\rangle} = S(\tilde{V})$ as in Proposition A.5. We make this identification more concrete. An $f \in R^{\bullet}_{\tilde{\mathcal{K}},\mathbf{1}\langle 1\rangle}$ of degree r is a morphism $f : \mathbf{1} \to \mathbf{1}\langle r\rangle$ in $D^b(S(\tilde{V})-\mathrm{mod})$. Since $\mathbf{1} = S(\tilde{V})[0]$ such an f corresponds to an $S(\tilde{V})$-linear map $S(\tilde{V}) \to S(\tilde{V})\langle r\rangle$ which is multiplication by an element of $S^r(\tilde{V})$. We abuse notation and call that element f as well. This gives an isomorphism of graded algebras $R^{\bullet}_{\tilde{\mathcal{K}},\mathbf{1}\langle 1\rangle} \xrightarrow{\sim} S(\tilde{V})$ (and an alternate proof of the second half of Proposition A.5). Now, if f is of degree r its mapping cone is simply the perfect complex

$$\cdots \longrightarrow 0 \longrightarrow S(\tilde{V}) \xrightarrow{\times f} S(\tilde{V})\langle r\rangle \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degrees -1 and 0. Applying the functor $k(\mathfrak{p}) \otimes^L_{S(\tilde{V})} (-)$ to this yields the complex

$$\cdots \longrightarrow 0 \longrightarrow S(\tilde{V})_{\mathfrak{p}}/\mathfrak{p} \xrightarrow{\times f} \big(S(\tilde{V})_{\mathfrak{p}}/\mathfrak{p}\big)\langle r\rangle \longrightarrow 0 \longrightarrow \cdots .$$

Since $S(\tilde{V})_{\mathfrak{p}}/\mathfrak{p}$ is a field the latter complex is non-acyclic iff $f = 0$ in $S(\tilde{V})_{\mathfrak{p}}/\mathfrak{p}$. That is iff $f \in \mathfrak{p}$. $\square$

Lastly we give a tt-geometric formulation of the Bernstein-Gelfand-Gelfand (BGG) correspondence, cf. [BGG78]. The latter is an equivalence between the stable module category $\bigwedge V - \mathrm{stab}$ and the bounded derived category of coherent sheaves $D^b(\mathrm{Coh}_{\mathbb{P}(\tilde{V})})$. Since $\bigwedge V$ is self-injective (indeed Frobenius) the graded generalization of [Ric89, Thm. 2.1] allows us to interpret $\bigwedge V - \mathrm{stab}$ as the Verdier quotient $D^b(\bigwedge V - \mathrm{mod})/D^{\mathrm{perf}}(\bigwedge V - \mathrm{mod})$ – also known as the stable derived category or the singularity category. The subcategory of perfect complexes $D^{\mathrm{perf}}(\bigwedge V - \mathrm{mod})$ is classically generated by $\bigwedge V$ (it is the smallest strictly full thick triangulated subcategory containing $\bigwedge V[0]$ and its $\langle\cdot\rangle$-shifts). We may thus view $\mathrm{Spc}(\bigwedge V - \mathrm{stab})$ as the subspace of $\mathrm{Spc}(D^b(\bigwedge V - \mathrm{mod}))$ of all prime $\otimes$-ideals $\mathcal{P}$ containing the perfect complexes – or equivalently just $\bigwedge V$. See [Bal10a, Rem. 1.4]. Our next goal is to show $\mathrm{Spc}(\bigwedge V - \mathrm{stab})$ only omits $\mathcal{P} = 0$ and $\rho^{\bullet}_{\mathcal{K},v}$ identifies it with $\mathbb{P}(\tilde{V})$. More graphically, we will build the diagram below.

$$\begin{array}{ccccc}
\mathrm{Spc}(\bigwedge V - \mathrm{stab}) & \longrightarrow & \mathrm{Spc}(D^b(\bigwedge V - \mathrm{mod})) & \longleftarrow & \{0\} \\
\simeq \downarrow & & \simeq \downarrow & & \downarrow \\
\mathbb{P}(\tilde{V}) & \longrightarrow & \mathrm{Spec}^{\mathrm{h}}(S(\tilde{V})) & \longleftarrow & \{S^{+}(\tilde{V})\}
\end{array}$$

This should be compared to the diagram in the proof of [Bal10a, Prop. 8.5] for $k[G] - \mathrm{stab}$. The next lemma shows that at least $\rho^{\bullet}_{\mathcal{K},v}$ takes $\mathrm{Spc}(\bigwedge V - \mathrm{stab})$ into $\mathbb{P}(\tilde{V})$.

Lemma A.7. *$\rho^{\bullet}_{\mathcal{K},v}(0) = S^{+}(\tilde{V})$.*

Proof. Note that 0 is in fact a prime $\otimes$-ideal (a tensor product over k vanishes only if one of the factors is zero). By definition $\rho^{\bullet}_{\mathcal{K},v}(0)$ is generated by homogeneous $f \in S(\tilde{V})$ for which $\mathrm{cone}(f) \neq 0$. We adopt the conventions from the proof of Proposition A.6. Thus we are asking that $\times f$ is not onto, that is f is not a unit. This is true as long as the degree of f is positive. So f lies in the irrelevant ideal $S^+(\tilde{V})$. □

To show $\rho^\bullet_{\mathcal{K},v}$ maps $\mathrm{Spc}(\bigwedge V - \mathrm{stab})$ onto $\mathbb{P}(\tilde{V})$ and hence identifies the two topologically, we need to check that every (thick) prime $\otimes$-ideal $\mathcal{P} \neq 0$ must contain $\bigwedge V$. In other words that all nonzero $\mathcal{P}$ lie in $\mathrm{Spc}(\bigwedge V - \mathrm{stab})$. Using Koszul duality we translate this into a question about prime $\otimes$-ideals in the category $D^b(S(\tilde{V}) - \mathrm{mod})$.

The following is essentially part (iii) of [BGS96, Thm. 2.12.5].

Lemma A.8. *$K(\bigwedge V[0]) = k\langle d\rangle[-d]$ where $d = \dim_k(V)$.*

Proof. Our input is $M = \bigwedge V[0]$. Thus $M^{(s)} = (\bigwedge^s V)\langle s\rangle$ and $K(M)^s = S(\tilde{V}) \otimes (\bigwedge^s V)\langle s\rangle$. Fix an isomorphism $\bigwedge^d V \xrightarrow{\sim} k$ where $d = \dim_k(V)$ and use it to identify $\bigwedge^s V$ with the dual space $(\bigwedge^{d-s} V)^*$. Then $K(M)$ is the complex

$$0 \longrightarrow S(\tilde{V}) \otimes (\bigwedge^{d} V)^*\langle 0\rangle \longrightarrow \cdots \longrightarrow S(\tilde{V}) \otimes (\bigwedge^{d-s} V)^*\langle s\rangle \longrightarrow \cdots \longrightarrow S(\tilde{V}) \otimes (\bigwedge^{0} V)^*\langle d\rangle \longrightarrow 0$$

concentrated in degrees $[0, d]$ whose differentials are given in Definition A.2 (with $\partial = 0$). A careful comparison of differentials shows $K(M)[d]$ is the $\langle d\rangle$-shift of the Koszul resolution of k, and therefore quasi-isomorphic to $k\langle d\rangle$. □

It follows that K restricts to an equivalence between $D^{\mathrm{perf}}(\bigwedge V - \mathrm{mod})$ and the category $\langle\langle k\rangle\rangle$ generated by the augmentation module (the smallest strictly full thick triangulated subcategory of $D^b(S(\tilde{V}) - \mathrm{mod})$ containing k and its $\langle\cdot\rangle$-shifts). Passing to quotients $\bigwedge V - \mathrm{stab}$ gets identified with $D^b(S(\tilde{V}) - \mathrm{mod})/\langle\langle k\rangle\rangle$. It remains to observe that every nonzero (thick) prime $\otimes$-ideal $\mathcal{P} \subset D^b(S(\tilde{V}) - \mathrm{mod})$ must contain k. This is essentially a reference to [DAS13] as we flesh out below.

Proposition A.9. *$\rho^\bullet_{\mathcal{K},v}$ restricts to a homeomorphism $\mathrm{Spc}(\bigwedge V - \mathrm{stab}) \xrightarrow{\sim} \mathbb{P}(\tilde{V})$.*

Proof. Let $\mathcal{P} \neq 0$ be a prime $\otimes$-ideal of $D^b(S(\tilde{V}) - \mathrm{mod})$. Write $\mathcal{P} = \mu(\mathfrak{p})$ for some $\mathfrak{p} \in \mathrm{Spec}^{\mathrm{h}}(S(\tilde{V}))$. By Lemma A.7 we know $\mathfrak{p} \neq S^+(\tilde{V})$ since $\mathcal{P}$ is nonzero. We want to show $k \in \mathcal{P}$. By definition of the map μ this amounts to verifying $k(\mathfrak{p}) \otimes^L_{S(\tilde{V})} k \simeq 0$ (see the paragraph just prior to Proposition A.6). Now [DAS13, Lem. 5.4] states that $k(\mathfrak{p})$ and $k(\mathfrak{q})$ have disjoint supports for any two homogeneous prime ideals $\mathfrak{p} \neq \mathfrak{q}$. In other words $\mathrm{supp}(k(\mathfrak{p}) \otimes^L_{S(\tilde{V})} k(\mathfrak{q})) = \mathrm{supp}(k(\mathfrak{p})) \cap \mathrm{supp}(k(\mathfrak{q})) = \varnothing$ which is equivalent to $k(\mathfrak{p}) \otimes^L_{S(\tilde{V})} k(\mathfrak{q}) \simeq 0$. Apply this observation to the irrelevant ideal $\mathfrak{q} = S^+(\tilde{V})$. □

There are G-equivariant refinements of these results. An $A \rtimes G$-module amounts to an A-module with a G-action such that $g(am) = g(a)g(m)$. Similarly for graded modules. A complex M in $D^b(A \rtimes G - \mathrm{mod})$ thus amounts to a complex in $D^b(A - \mathrm{mod})$ whose terms M^i have G-actions (preserving the grading) and with G-equivariant differentials ∂. Analogously for $A^!$. The sth term of the output complex $K(M)^s = A^! \otimes M^{(s)}$ carries the diagonal G-action and d is G-linear (the definition of d' is independent of the choice of v_α and $\tilde{v}_\alpha$ so one can take $g(v_\alpha)$ and $g(\tilde{v}_\alpha)$ instead). Similar remarks apply to the quasi-inverse of K defined in Step 2 on [BGS96, p. 489]. Thus K gives an equivalence of triangulated categories

$$K : D^b(A \rtimes G - \mathrm{mod}) \xrightarrow{\sim} D^b(A^! \rtimes G - \mathrm{mod})$$

Propostion A.3 tells us $\otimes^L_{A^!}$ on the right corresponds to the $\otimes$ defined via the comultiplication Δ for $A \rtimes G$ on the left. In particular the small Yoneda algebra $\mathrm{ext}^*_{A \rtimes G}(k, k)$ can be identified with $\mathrm{ext}^*_{A^! \rtimes G}(A^!, A^!)$.

Example A.10. Note that $\mathrm{ext}^*_{\bigwedge V \rtimes G}(k,k) \simeq H^*(G,k)$ for all *trivial* G-modules V. Indeed, if $P \to k$ is a resolution of k by free $k[G]$-modules then $S(\tilde{V}) \otimes P \to S(\tilde{V})$ is a free resolution in $S(\tilde{V})[G]-\mathrm{mod}$. Therefore

$$\mathrm{ext}^i_{S(\tilde{V})[G]}(S(\tilde{V}), S(\tilde{V})) = h^i\big(\mathrm{Hom}_{S(\tilde{V})[G]-\mathrm{mod}}(S(\tilde{V}) \otimes P, S(\tilde{V}))\big) = h^i\big(\mathrm{Hom}_{k[G]}(P, S^0(\tilde{V}))\big) = H^i(G,k).$$

When $G = \{e\}$ this of course recovers the observation (in remark 4.3) that $\mathrm{ext}^*_{\bigwedge V}(k,k) = k$ for all V.

We view $A^! = S(\tilde{V})$ as a $(A^! \rtimes G, (A^!)^G)$-bimodule (associativity holds because we take G-invariants). This gives two adjoint functors $\mathrm{Hom}_{A^! \rtimes G}(A^!, -)$ and $A^! \otimes_{(A^!)^G} (-)$ between the categories $A^! \rtimes G - \mathrm{mod}$ and $(A^!)^G - \mathrm{mod}$. Hom and $\otimes$ here refer to the graded versions. Note that $\mathrm{Hom}_{A^! \rtimes G}(A^!, -)$ is simply the functor $(-)^G$ taking G-invariants; in particular it is exact if $\mathrm{char}(k) \nmid |G|$.

In the remainder of this Appendix we assume we are in the non-modular case where $\mathrm{char}(k) \nmid |G|$. Moreover, we assume $(A^!)^G$ has finite global dimension. Under these assumptions $R\mathrm{Hom}_{A^! \rtimes G}(A^!, -)$ and $A^! \otimes^L_{(A^!)^G} (-)$ give an adjunction between the *bounded* derived categories

$$D^b(A^! \rtimes G - \mathrm{mod}) \leftrightarrows D^b((A^!)^G - \mathrm{mod}).$$

The two adjunction maps are

- $A^! \otimes^L_{(A^!)^G} R\mathrm{Hom}_{A^! \rtimes G}(A^!, X) \longrightarrow X$;
- $Y \longrightarrow R\mathrm{Hom}_{A^! \rtimes G}(A^!, A^! \otimes^L_{(A^!)^G} Y)$.

The bottom map is always an isomorphism since $(A^!)^G$ generates $D^b((A^!)^G - \mathrm{mod})$. The top map is an isomorphism at least for $X \in \langle\langle A^! \rangle\rangle$. The adjunction thus restricts to an equivalence of triangulated categories $\langle\langle A^! \rangle\rangle \xrightarrow{\sim} D^b((A^!)^G - \mathrm{mod})$. The two monoidal structures $\otimes^L_{A^!}$ and $\otimes^L_{(A^!)^G}$ clearly correspond. Altogether, composing with K gives an equivalence of tt-categories

$$D^b(A \rtimes G - \mathrm{mod})^{\square} \xrightarrow{\sim} D^b((A^!)^G - \mathrm{mod}).$$

Recall our $(-)^{\square}$ notation from Remark 5.3. We let $\mathcal{K}^{\square} = \langle\langle \mathbf{1} \rangle\rangle$ denote the smallest strictly full thick triangulated subcategory of $\mathcal{K}$ containing the unit object $\mathbf{1}$ <u>and</u> its $\langle \cdot \rangle$-shifts. Observe that $D^b(A \rtimes G - \mathrm{mod})^{\square}$ therefore contains both u and v.

This leads to the following generalization of Proposition A.6.

Proposition A.11. *Assume char$(k) \nmid |G|$ and $S(\tilde{V})^G$ has finite global dimension. Then the Balmer map*

$$\rho^{\bullet}_{\mathcal{K}^{\square}, v} : Spc\big(D^b(\textstyle\bigwedge V \rtimes G - mod)^{\square}\big) \longrightarrow Spec^h\big(S(\tilde{V})^G\big)$$

is a homeomorphism.

Proof. The proof is more or less identical to that of Proposition A.6 so we will only sketch it. We let $\mathcal{K}$ denote $D^b(A \rtimes G - \mathrm{mod})$ and $\tilde{\mathcal{K}} = D^b((A^!)^G - \mathrm{mod})$. We just established an equivalence $\mathcal{K}^{\square} \xrightarrow{\sim} \tilde{\mathcal{K}}$ by combining K with the adjunction. It takes v to $\mathbf{1}\langle 1 \rangle$ in $\tilde{\mathcal{K}}$ and identifies the rings $R^{\bullet}_{\mathcal{K},v}$ and $R^{\bullet}_{\tilde{\mathcal{K}},\mathbf{1}\langle 1 \rangle}$. They are both isomorphic to $(A^!)^G$. Thus the problem translates into showing $\rho^{\bullet}_{\tilde{\mathcal{K}},\mathbf{1}\langle 1 \rangle}$ is a homeomorphism. This is done by verifying the identity $\rho^{\bullet}_{\tilde{\mathcal{K}},\mathbf{1}\langle 1 \rangle} \circ \mu = \mathrm{Id}$ where $\mu : \mathrm{Spec}^{\mathrm{h}}\big((A^!)^G\big) \to \mathrm{Spc}\big(D^b((A^!)^G - \mathrm{mod})\big)$ is the homeomorphism from [DAS13] going in the opposite direction. This proceeds exactly as in the proof of Proposition A.6 by replacing $S(\tilde{V})$ with $(A^!)^G$ everywhere. □

Arguments very similar to the proofs of Lemmas A.7 and A.8, and Proposition A.9, show that $\rho^\bullet_{\mathcal{K}^\square,v}$ takes the 0-ideal to the irrelevant $S^+(\tilde{V})^G$ and furthermore that the nonzero prime $\otimes$-ideals $\mathcal{P} \subset D^b(\bigwedge V \rtimes G - \text{mod})^\square$ correspond to the primes of $D^b(S(\tilde{V})^G - \text{mod})$ containing k.

Remark A.12. Since $\bigwedge V \rtimes G$ is self-injective (Frobenius) it makes sense to speak of its stable module category $\bigwedge V \rtimes G - \text{stab}$. In [Pos19, p. 11] it is claimed in passing that it is equivalent to $D^b(G - \text{Coh}_{\mathbb{P}(\tilde{V})})$ – the bounded derived category of G-equivariant coherent sheaves on $\mathbb{P}(\tilde{V})$. Assuming $\text{Rep}_G(k)$ is semisimple. However, the argument in [Pos19] relies heavily on [Flø01, Thm. 9.1.2] and as far as we know Fløystad's 20 year old preprint was never published. When $\mathbb{P}(\tilde{V})/\!\!/G$ is smooth we suspect the claim in [Pos19] yields a homeomorphism

$$\text{(A.13)} \qquad \text{Spc}\big(\bigwedge V \rtimes G - \text{stab}\big) \xrightarrow{\sim} \text{Spc}\big(D^b(G - \text{Coh}_{\mathbb{P}(\tilde{V})})\big) = \text{Spc}\big(D^{\text{perf}}(\mathbb{P}(\tilde{V})/\!\!/G)\big) \xleftarrow{\sim} \mathbb{P}(\tilde{V})/\!\!/G$$

after passing to spectra, using [Bal10b, Thm. 54] (essentially due to Thomason). Note the direction of the last arrow; it arises from the universal property of Spc by verifying that homological support is a classifying support datum – this parts works for any Noetherian scheme X. Alas, the total map (A.13) does *not* seem to be the restriction of our $\rho^\bullet_{\mathcal{K}^\square,v}$. Indeed, let $\mathcal{K} = D^b(\bigwedge V \rtimes G - \text{mod})$. In our situation (A.11) we may compose $\rho^\bullet_{\mathcal{K}^\square,v}$ with the composed map

$$\text{Spc}\big(\bigwedge V \rtimes G - \text{stab}\big) \longrightarrow \text{Spc}(\mathcal{K}) \longrightarrow \text{Spc}(\mathcal{K}^\square).$$

For one thing, it is unclear to us why the resulting map would send $\text{Spc}\big(\bigwedge V \rtimes G - \text{stab}\big)$ into $\text{Proj}\big(S(\tilde{V})^G\big)$.

In light of [Bal10a, Rk. 8.2] we expect a negative answer: Suppose $X = \text{Proj}(S)$ satisfies the two requirements (i) $S^0 = \Gamma(X, \mathcal{O}_X)$ and (ii) $S^i = \Gamma(X, \mathcal{O}(i))$ for $i \gg 0$. Then the inverse of the homeomorphism $X \xrightarrow{\sim} \text{Spc}\big(D^{\text{perf}}(X)\big)$ is the map $\rho^\bullet_{D^{\text{perf}}(X),\mathcal{O}(1)}$. However, since $S(\tilde{V})^G$ is not in general generated by $S^1(\tilde{V})^G$ as a k-algebra (cf. Dickson invariants for example) the second requirement (ii) may fail for $X = \mathbb{P}(\tilde{V})/\!\!/G$ according the Hartshorne exercise (II.5.9) quoted in [Bal10a, Rk. 8.2] – indeed $\mathcal{O}(1)$ may fail to be invertible.

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