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Locally algebraic vectors in the Breuil–Herzig ordinary part

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Abstract. For a fairly general reductive group $G_{/\mathbb{Q}_p}$, we explicitly compute the space of locally algebraic vectors in the Breuil–Herzig construction $\Pi(\rho)^{ord}$, for a potentially semi-stable Borel-valued representation ρ of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. The point being we deal with the whole representation, not just its socle—and we go beyond $\mathrm{GL}_n(\mathbb{Q}_p)$. In the case of $\mathrm{GL}_2(\mathbb{Q}_p)$, this relation is one of the key properties of the p-adic local Langlands correspondence. We give an application to p-adic local-global compatibility for $\Pi(\rho)^{ord}$ for modular representations, but with no indecomposability assumptions.

Contents

1. Introduction

The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ takes a continuous representation $\rho : \Gamma_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$, defined over a finite extension $E/\mathbb{Q}_p$, and associates a Banach E-space $\Pi(\rho)$ with a unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-action. See [14] for the latest developments. One of its main features is its compatibility with the *classical* local

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Langlands correspondence, in the following sense. The locally algebraic vectors $\Pi(\rho)^{alg}$ are those which lie in an algebraic sub-representation of a compact open subgroup, and they may very well all be trivial. In fact, $\Pi(\rho)^{alg} \neq 0$ precisely when ρ is potentially semistable with distinct Hodge-Tate weights[1], in which case $\Pi(\rho)^{alg}$ is given by the formula (if the Hodge-Tate weights are $HT(\rho) = \{0, k-1\}$)

$$\Pi(\rho)^{alg} = \mathrm{Sym}^{k-2}(E^2) \otimes_E \pi_{sm}(\rho), \tag{1.1}$$

where $\pi_{sm}(\rho)$ corresponds to the Weil-Deligne representation $WD(\rho)$, defined by Fontaine, via the local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$—or rather a "generic" extension thereof (a subtlety we can safely ignore for now). This compatibility is one of the main reasons why $\Pi(-)$ plays such a fundamental role in Emerton and Kisin's recent progress on the Fontaine-Mazur conjecture for two-dimensional representations of $\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, cf. [17,22] for example.

For $\mathrm{GL}_n(\mathbb{Q}_p)$ one still doesn't quite know how to define the p-adic local Langlands correspondence, although a globally defined candidate $\Pi(\rho)$ has recently been proposed (with very promising properties), see [13], which employs an intricate version of the Taylor-Wiles-Kisin patching argument at infinite level. When the representation $\rho : \Gamma_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$ is triangular, i.e. maps into the Borel $\hat{B}(E)$, Breuil and Herzig give a purely local construction of a Banach representation $\Pi(\rho)^{ord}$ of $\mathrm{GL}_n(\mathbb{Q}_p)$, which is expected to be the part of $\Pi(\rho)$ which can built out of principal series. In fact, the construction in [4] is more general, and deals with fairly general reductive p-adic groups $G(\mathbb{Q}_p)$. The purpose of this note is to prove the analogue of 1.1 for the Breuil–Herzig construction $\Pi(\rho)^{ord}$, for any G satisfying the usual weak assumption that Z_G and $Z_{\hat{G}}$ are connected. Throughout the text $\hat{G}$ denotes the Langlands dual group of G, defined by taking the dual root datum (cf. Sect. 2.1).

We now present the main result of this paper in more detail. We begin by briefly introducing the notation and terminology in use throughout.

Throughout, $G_{/\mathbb{Z}_p}$ is a split connected reductive group, with a choice of Borel pair $B \supset T$. As in [4], we always assume that Z_G is connected and G^{der} is simply connected (assumption 2.1 below). Once and for all, we pick a "twisting element" $\theta \in X(T)$. That is, a character such that $\langle \theta, \alpha^\vee \rangle = 1$ for all simple roots α (for example, take θ to be the sum of the fundamental weights). The terminology is taken from [3] (their Definition 5.2.1, p. 27). We consider continuous $\hat{B}(E)$-valued representations,

$$\rho : \Gamma_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \hat{B}(E) \subset \hat{G}(E),$$

where $E/\mathbb{Q}_p$ is a finite extension. To avoid degeneracies, we always assume ρ is *good*—which means the Zariski closure of $\rho(\Gamma_{\mathbb{Q}_p})$ is as small as possible among all $\hat{B}(E)$-conjugates (see Definition 3.2.4 in [4], or Section 2.3 for details). Following [4], we consider the semi-simplification $\hat{\chi}_\rho : \Gamma_{\mathbb{Q}_p} \to \hat{T}(E)$, and the character

[1] For irreducible ρ this is Proposition VI.5.1 on page 165 in [15]. For reducible ρ with scalar endomorphisms, see (the proof of) Proposition 4.14 in [25]. The scalar semi-simplification case is part of Proposition 4.3 in [27].

$\chi_\rho : T(\mathbb{Q}_p) \to E^\times$ corresponding to $\hat{\chi}_\rho$ via class field theory (see 2.2 below). We say ρ is *generic* if $\alpha^\vee \circ \hat{\chi}_\rho \notin \{1, \epsilon^{\pm 1}\}$ for all roots α (where ϵ is the p-adic cyclotomic character). To a good generic ρ, the Breuil–Herzig construction attaches a unitary continuous representation $\Pi(\rho)^{ord}$ of $G(\mathbb{Q}_p)$ on an E-Banach space—which is expected to account for the "ordinary" part of the (yet to be defined) p-adic local Langlands correspondence $\Pi(\rho)$. That is, $\Pi(\rho)^{ord}$ should be the maximal closed invariant subspace of $\Pi(\rho)$ whose constituents are (subquotients of) continuous principal series. The construction of $\Pi(\rho)^{ord}$ is based on the belief that there should be an analogue F of Colmez's Montreal functor such that $L^\otimes \circ \rho = F(\Pi(\rho))$, where $L^\otimes$ is the tensor product of all the fundamental algebraic representations of $\hat{G}$. In a recent preprint [10], Breuil gives strong evidence for this: In Corollary 9.8 of loc. cit. he defines an exact contravariant functor F from representations built from principal series to étale (ϕ, Γ)-modules, which enjoys all the desired properties. In fact, his construction is more general, and involves projective *limits* of étale (ϕ, Γ)-modules.

Based on the case of $\mathrm{GL}_2(\mathbb{Q}_p)$, one would hope that $\Pi(\rho)^{ord}$ contains nonzero locally algebraic vectors precisely when ρ is potentially semistable[2] and regular, in a suitable sense. For a general group G, and a general ρ, we take *regular* to mean that $\alpha^\vee \circ \hat{\chi}_\rho|_J \neq 1$ for all roots α and all open subgroups $J \subset I_{\mathbb{Q}_p}$. For $G = \mathrm{GL}(n)$, and ρ de Rham, this means the Hodge-Tate weights are distinct. For a general group G, we are not quite able to show that $\Pi(\rho)^{ord,alg} \neq 0$ implies ρ is potentially semistable. Instead we obtain an a priori weaker conclusion involving a certain Weyl element $w_\rho \in W$, which we now define: Assume ρ is regular and $\chi_\rho = \chi_{\rho,alg} \cdot \chi_{\rho,sm}$ is locally algebraic. By regularity, $\chi_{\rho,alg} \in X(T)$ is "off the walls", that is $\langle \chi_{\rho,alg}, \alpha^\vee \rangle \neq 0$ for all roots α. Since W permutes the Weyl chambers in a simply transitive manner there is a unique $w_\rho \in W$ such that $\chi^+_{\rho,alg} := w_\rho^{-1}(\chi_{\rho,alg}) \in X_+(T)$ is dominant. That is, for which $\langle \chi^+_{\rho,alg}, \alpha^\vee \rangle \geq 0$ (and hence >0) for all positive roots α. The overall theme of our paper is to decide when w_ρ in fact lies in the subset $W_{C_\rho} \subset W$, which parametrizes the indecomposable summands of $\Pi(\rho)^{ord}$. We refer the reader to Sect. 2 below for the precise definitions of C_ρ and W_{C_ρ}. Here let it suffice to say that C_ρ somehow measures the indecomposability of ρ, and if one conjugates ρ by an element $w \in W_{C_\rho}$ it stays $\hat{B}$-valued. The following *preliminary* result (proved in Sect. 3) shows how the condition $w_\rho \in W_{C_\rho}$ relates to ρ being de Rham.

Theorem 1.2. *Let $\rho : \Gamma_{\mathbb{Q}_p} \to \hat{B}(E)$ be a good, generic, regular representation as above. If ρ is potentially semistable, then χ_ρ is locally algebraic and $w_\rho \in W_{C_\rho}$. The converse is true for $GL(n)$.*

We do not know if the converse is true for an arbitrary group G. The problem is that a dominant cocharacter of $\hat{T}$ may not remain dominant upon composition with a faithful representation $r : \hat{G} \hookrightarrow \mathrm{GL}(n)$. Theorem 1.2 can be thought of as a G-version of the fact from p-adic Hodge theory that there are no non-split *de*

[2] Recall that, by definition, $\rho : \Gamma_{\mathbb{Q}_p} \to \hat{G}(E)$ is potentially semistable if $r \circ \rho$ is potentially semistable for every algebraic representation $r : \hat{G} \to \mathrm{GL}(n)$. See Definition 1.1.1 in [31].

Rham extensions $\begin{pmatrix} 1 & * \\ & \epsilon^h \end{pmatrix}$ with $h > 0$. A good reference for this is the discussion on p. 122 in [2]. See also the table in Example 3.9 in [5], which is reproduced in "Appendix B" at the very end of this paper for easy access.

Although not directly related to the current paper, we want to point out an interesting similarity between Theorem 1.2 and some results in [18]. In [18] (throughout this paragraph, all notations and references point to [18]), a key result (Prop. 4.1.3) is that a certain Weyl group element σ falls in W_C for certain closed subset of R^+. There, C is defined (see 4.1.2) from a certain Kisin module. But by Prop. 4.2.2 (see also the proof of Thm 5.0.4 in loc. cit.), the Weyl group element σ actually falls in $W_{C_{\bar{\rho}}}$.

Our *main* result computes $\Pi(\rho)^{ord,alg}$ in terms of the classical local Langlands correspondence, à la the Breuil–Schneider recipe in the GL(n)-case, introduced in [11] (see also its predecessor [29]).

Theorem 1.3. *Suppose $\rho : \Gamma_{\mathbb{Q}_p} \to \hat{B}(E)$ is good, residually*[3] *generic, and regular. Let V be an irreducible algebraic representation of $G(\mathbb{Q}_p)$, defined over E, of highest weight $\lambda \in X_+(T)$ relative to B. Then the following holds.*

(1) *If $\Pi(\rho)^{ord,V-alg} \neq 0$, then χ_ρ is locally algebraic, with $w_\rho \in W_{C_\rho}$, and the dominant Weyl conjugate of the algebraic part is $\chi^+_{\rho,alg} = \lambda + \theta$.*
(2) *If χ_ρ is locally algebraic, with $w_\rho \in W_{C_\rho}$, and $\chi^+_{\rho,alg} = \lambda + \theta$, then*

$$\Pi(\rho)^{ord,alg} = \Pi(\rho)^{ord,V-alg} = V \otimes_E \left(Ind_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_{\rho,sm} \cdot |\theta|^{-1}\right)^{\mathcal{C}^\infty}.$$

In particular (2) applies to potentially semistable ρ (by Theorem 1.2).

(The superscript $\mathcal{C}^\infty$ means we take the unnormalized smooth parabolic induction).

The GL(n)-case of Theorem 1.3 is probably known to the experts. In fact, Theorem 8.9 (together with Lemma 8.2) in [9] computes the locally algebraic vectors in the *socle* $\mathrm{soc}_{\mathrm{GL}_n(\mathbb{Q}_p)} \Pi(\rho)^{ord}$. Our goal is twofold: (a) To show that the locally algebraic vectors all lie in the socle, and (b) to extend this to an arbitrary group G as above —which requires a bit of delicate Lie-theoretic book-keeping.

In the last Chapter we give a quick application of our main result to the p-adic local-global compatibility conjecture in [4]. Their Theorem 4.4.8 shows that $\Pi(\rho)^{ord}$ of a modular representation $\rho := r|_{\Gamma_{\mathbb{Q}_p}}$ *injects* into a space of p-adic modular forms, assuming that $\bar{r}$ is *totally indecomposable* at the places above p (meaning C_ρ consists of all positive roots, in other words $W_{C_\rho} = \{1\}$). We prove a slight extension (Corollary 5.2) of this result in which we allow $\bar{r}$ to be arbitrarily decomposable (arbitrary C_ρ), but the trade-off is we only get a nonzero map into p-adic modular forms, as opposed to an injection (unless $W_{C_\rho} = \{1\}$ of course). The proof follows that from [4] almost ad verbatim, but makes critical use of the fact from Theorem 1.2 that $w_\rho \in W_{C_\rho}$.

[3] This can be slightly weakened. Under Assumption 2.3 below, generic is enough.

2. Review of the Breuil–Herzig construction

2.1. Group-theoretic notation

We adopt the notation and setup from [4], which we briefly recall. We fix a split connected reductive group $G_{/\mathbb{Z}_p}$, endowed with a choice of split maximal torus T, and a Borel subgroup $B \supset T$, whose opposite is denoted B^-. Let $(X(T), R, X^\vee(T), R^\vee)$ be the corresponding root datum, and let $S \subset R^+$ be the simple roots associated with B. For each $\alpha \in S$, we have the simple reflection $s_\alpha \in \mathrm{Aut} X(T)$ given by $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$. They generate the Weyl group W. The dual based root datum determines (up to inner automorphisms) a dual triple $(\hat{G}, \hat{B}, \hat{T})$, which we view as being defined (and split) over $\mathcal{O}_E$, for a fixed finite extension $E/\mathbb{Q}_p$. Similarly, we have simple reflections $s_{\alpha^\vee} \in \mathrm{Aut} X(\hat{T})$, for $\alpha \in S$, which generate the Weyl group $\hat{W} \simeq W$ (via the duality, $s_{\alpha^\vee} \leftrightarrow s_\alpha$).

Assumption 2.1. The center Z_G is connected, and the derived group G^{der} is simply connected (equivalently, $Z_{\hat{G}}$ is connected—see Proposition 2.1.1 in [4]).

Thus, both G and $\hat{G}$ admit fundamental weights, $\{\lambda_\alpha\}_{\alpha \in S}$ and $\{\lambda_{\alpha^\vee}\}_{\alpha \in S}$ respectively. For example, $\lambda_\alpha \in X(T)$ is a (necessarily dominant) weight such that $\langle \lambda_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}$ for $\beta \in S$. This determines λ_α modulo $X^0(T) \simeq X(G)$. Once and for all, we choose a twisting element θ (in the sense of [3]). That is, a $\theta \in X(T)$ with $\langle \theta, \alpha^\vee \rangle = 1$ for all $\alpha \in S$. For example, $\theta = \sum_{\alpha \in S} \lambda_\alpha$—everything defined modulo $X^0(T)$.

Let $U \subset B$ be the unipotent radical. For each $\alpha \in R$, we have an associated root subgroup $U_\alpha \simeq \mathbb{G}_a$. Moreover, $U_\alpha \subset B$ precisely when $\alpha \in R^+$. A subset $C \subset R$ is closed if $\forall \alpha, \beta \in C$, $\alpha + \beta \in R$ implies $\alpha + \beta \in C$. Thus C is the set of roots of U_C, the group generated by $\{U_\alpha\}_{\alpha \in C}$. We let $B_C = TU_C$. With C, we associate the following subset of the Weyl group (see Lemma 2.3.6 in [4]),

$$W_C = \{w \in W : w^{-1}(C) \subset R^+\} = \{w \in W : \dot{w}^{-1} B_C \dot{w} \subset B\}.$$

(Here $\dot{w} \in N_G(T)$ is any representative of w). Analogous notation and terminology will be used for the dual group $\hat{G}$ below.

2.2. Galois-theoretic notation

Our starting point is a continuous homomorphism ("representation") taking values in $\hat{B}$,

$$\rho : \Gamma_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \hat{G}(E), \quad \rho(\Gamma_{\mathbb{Q}_p}) \subset \hat{B}(E).$$

These are called ordinary in [4], but this can be somewhat misleading since this usually pertains to semistable representations. Pink has a notion of quasi-ordinary [26], which seems more appropriate. We prefer to just call ρ triangular. Regardless, we let $C_\rho \subset R^{+\vee}$ be the smallest closed subset such that $\rho(\Gamma_{\mathbb{Q}_p}) \subset \hat{B}_{C_\rho}(E)$. By composition with the projection $\hat{B} \twoheadrightarrow \hat{T}$, we obtain the "semisimplification",

$$\hat{\chi}_\rho : \Gamma_{\mathbb{Q}_p} \longrightarrow \hat{B}(E) \twoheadrightarrow \hat{T}(E).$$

This is continuous, and therefore takes values in the maximal compact subgroup $\hat{T}(\mathcal{O}_E)$. Via local class field theory, we attach a continuous character $\chi_\rho : T(\mathbb{Q}_p) \to \mathcal{O}_E^\times$ as follows: First recall that, for any $\mathbb{Z}_p$-algebra A,

$$T(A) = \mathrm{Hom}(X(T), A^\times) = \mathrm{Hom}(X(T), \mathbb{Z}) \otimes_{\mathbb{Z}} A^\times = X(\hat{T}) \otimes_{\mathbb{Z}} A^\times.$$

Taking $A = \mathbb{Q}_p$, and composing with the Artin map $\mathbb{Q}_p^\times \hookrightarrow \Gamma_{\mathbb{Q}_p}^{ab}$, gives rise to χ_ρ,

$$T(\mathbb{Q}_p) = X(\hat{T}) \otimes_{\mathbb{Z}} \mathbb{Q}_p^\times \hookrightarrow X(\hat{T}) \otimes_{\mathbb{Z}} \Gamma_{\mathbb{Q}_p}^{ab} \xrightarrow{1\otimes\hat{\chi}_\rho} X(\hat{T}) \otimes_{\mathbb{Z}} \hat{T}(\mathcal{O}_E) \longrightarrow \mathcal{O}_E^\times,$$

where the last map is just evaluation. By construction, χ_ρ is continuous.

2.3. *The construction*

We say ρ is *good* if $C_\rho \subset C_{b\rho b^{-1}}$, for all $b \in \hat{B}(E)$. By 3.2.3 in [4], our representation ρ admits a good $\hat{B}(E)$-conjugate if $\alpha^\vee \circ \hat{\chi}_\rho \neq 1$, for all $\alpha \in R^+$. We will always assume something stronger. Namely, that ρ is *generic* (3.3.1 in loc. cit.), which means $\alpha^\vee \circ \hat{\chi}_\rho \notin \{1, \epsilon^{\pm 1}\}$, for all $\alpha \in R$. (Here ϵ is the p-adic cyclotomic character). Thus, from now on, ρ is good and generic. For such ρ, Breuil and Herzig define a corresponding E-Banach space representation $\Pi(\rho)^{ord}$ of $G(\mathbb{Q}_p)$, which is expected to be the largest sub-representation of a conjectural p-adic local Langlands correspondence $\Pi(\rho)$ built out of continuous principal series. The definition of $\Pi(\rho)^{ord}$ is based on the belief[4] that there should be a "Colmez" functor F such that

$$F(\Pi(\rho)) \overset{?}{=} L^\otimes \circ \rho, \quad L^\otimes = \otimes_{\alpha\in S} L(\lambda_{\alpha^\vee}).$$

The submodule structure of $\Pi(\rho)^{ord}$ should therefore reflect that of the "ordinary" part $(L^\otimes|_{\hat{B}_{C_\rho}})^{ord}$, worked out in Theorem 2.4.1 of [4]. Thus, it decomposes as

$$\Pi(\rho)^{ord} = \oplus_{w\in W_{C_\rho}} \Pi(\rho)_{C_\rho, w},$$

where $\Pi(\rho)_{C_\rho,w}$ is a certain direct limit $\varinjlim_I \Pi(\rho)_I$, with $I \subset w(S^\vee) \cap C_\rho$ ranging over (the finitely many) subsets of pairwise orthogonal roots. Here $\Pi(\rho)_I$ is a parabolically induced representation,

$$\Pi(\rho)_I = \left(\mathrm{Ind}_{P_J(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \tilde{\Pi}(\rho)_I\right)^{\mathcal{C}^0},$$

[4] Now strongly supported by the construction in [10]!

induced from the parabolic $P_J \supset B^-$ with Levi $G_J \simeq T'_J \times \mathrm{GL}_2^J$ (see 3.1.4 in loc. cit.). Here J is the corresponding set of simple roots, $J = w^{-1}(I)^\vee \subset S$. Finally, $\tilde{\Pi}(\rho)_I$ is a representation of $G_J(\mathbb{Q}_p)$, characterized by its socle filtration (3.3.3 in loc. cit.)—essentially it's built out of the p-adic local Langlands correspondence for the copies of $\mathrm{GL}_2(\mathbb{Q}_p)$ sitting in $G_J(\mathbb{Q}_p)$ (a copy for each simple root $\beta \in J$).

We will not need the details of the construction in this paper. It is expected that the summand $\Pi(\rho)_{C_\rho,w}$ can be characterized by its socle filtration. More precisely, one expects the following, which is Conjecture 3.5.1, p. 40, in [4].

Conjecture 2.2. *There is a unique admissible unitary continuous representation $\Pi(\rho)_{C_\rho,w}$ of $G(\mathbb{Q}_p)$ over E with socle filtration $Fil_j\Pi(\rho)_{C_\rho,w}$ such that $\forall j \geq 0$,*

$$\begin{aligned} Gr_j\Pi(\rho)_{C_\rho,w} &= Fil_j\Pi(\rho)_{C_\rho,w}/Fil_{j-1}\Pi(\rho)_{C_\rho,w} \\ &\simeq \bigoplus_{I\subset w(S^\vee)\cap C_\rho, |I|=j} \left(Ind_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\left(\left(\prod_{\alpha\in I^\vee} s_\alpha\right)w\right)^{-1}(\chi_\rho)\cdot\left(\epsilon^{-1}\circ\theta\right)\right)^{\mathcal{C}^0}. \end{aligned}$$

Here $I \subset w(S^\vee) \cap C_\rho$ runs over subsets of pairwise **orthogonal** *roots with $|I| = j$.*

Breuil and Herzig also speculate that $\Pi(\rho)_{C_\rho,w}$ should be characterized by the following weaker conditions:

- $\mathrm{soc}_{G(\mathbb{Q}_p)}\Pi(\rho)_{C_\rho,w} = \mathrm{Fil}_0\Pi(\rho)_{C_\rho,w} \simeq \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} w^{-1}(\chi_\rho)\cdot(\epsilon^{-1}\circ\theta)\right)^{\mathcal{C}^0}$.
- All constituents of $\Pi(\rho)_{C_\rho,w}$ occur with multiplicity one, and have the form

$$\left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\left(\left(\prod_{\alpha\in I^\vee} s_\alpha\right)w\right)^{-1}(\chi_\rho)\cdot(\epsilon^{-1}\circ\theta)\right)^{\mathcal{C}^0}$$

for varying subsets $I \subset w(S^\vee) \cap C_\rho$ of pairwise orthogonal roots.

In a recently updated preprint [21], J. Hauseux proves Breuil–Herzig's Conjecture 2.2—indeed the stronger version from the last paragraph—under the assumption that the principal series involved are topologically irreducible. See his Theorem 4.1.4 and its Corollary 4.3.3 in [21]. This irreducibility assumption should follow from our standing genericity assumption on χ_ρ (according to the "folklore" Conjecture 3.1.2 in [4]. See also [28], 2.5).

As mentioned on p. 40 in [4], irreducibility is known if $\bar{\chi}_\rho \circ \alpha^\vee \neq \omega$ for all $\alpha \in w(S)$, with $w \in W_\rho$ (a subset of the Weyl group comprising all $(\prod_{\alpha\in I^\vee} s_\alpha)w$, see (14) on p. 30 in loc. cit., it contains W_{C_ρ}—but it may be bigger). Here $\omega = \bar{\epsilon}$ denotes the mod p cyclotomic character, viewed as a character $\mathbb{Q}_p^\times \to k_E^\times$. Indeed, if this condition holds, all the continuous principal series above are topologically irreducible due to independent works of Ollivier ($G = \mathrm{GL}_n$) and Abe (G split) on the mod p situation (see Theorem 4 in [24], and Theorem 1.3 in [1] respectively). We will impose this "weak mod p genericity" condition on $\bar{\rho}$ in what follows:

Assumption 2.3. $\bar{\chi}_\rho \circ \alpha^\vee \neq \omega$ for all $\alpha \in w(S)$, with $w \in W_\rho$.

3. The dominant Weyl-conjugate. Proof of Theorem 1.2

In this section we prove Theorem 1.2 from the introduction. We briefly remind the reader of the setup of that result. We consider a good and generic representation $\rho : \Gamma_{\mathbb{Q}_p} \to \hat{B}(E)$, which we moreover assume is *regular*, as defined in the introduction. That is, $\alpha^\vee \circ \hat{\chi}_\rho \neq 1$ on arbitrarily small open subgroups of inertia $I_{\mathbb{Q}_p}$ (where α can be any root). Our goal is to show the implication

$$\rho \text{ potentially semistable} \Longrightarrow \chi_\rho \text{ locally algebraic and } w_\rho \in W_{C_\rho}, \tag{3.1}$$

where $w_\rho \in W$ is the "dominizer" of $\chi_{\rho,alg}$: The unique Weyl element $w \in W$ such that $w^{-1}(\chi_{\rho,alg}) \in X_+(T)$. This is well-defined only when ρ is regular and χ_ρ is locally algebraic. The fact that χ_ρ must be locally algebraic if ρ is de Rham is clear (χ_ρ locally algebraic $\Leftrightarrow$ $\hat{\chi}_\rho$ potentially crystalline $\Leftarrow$ ρ de Rham). The point is to show $w_\rho \in W_{C_\rho}$.

Remark 3.2. In the GL(2)-case, this is the fact that a *de Rham* representation $\begin{pmatrix} \epsilon^{h_1} & * \\ & \epsilon^{h_2} \end{pmatrix}$ must split if $h_1 < h_2$. We will reduce the general case (for any G) to this special case.

We now show the implication (3.1) by induction on the length of w_ρ.

Proof. We proceed by induction on the length $\ell(w_\rho)$. If the length is zero, $w_\rho = 1$, and we're done since $C_\rho \subset R^{+\vee}$. Assume the Proposition holds for all ρ' (satisfying the given hypotheses) of length $\ell(w_{\rho'}) < \ell(w_\rho)$. To show that $w_\rho \in W_{C_\rho}$, once and for all we pick a simple root $\beta \in S$ such that $\langle \chi_{\rho,alg}, \beta^\vee \rangle < 0$. (Such a β exists since $\ell(w_\rho) > 0$. That is, $\chi_{\rho,alg}$ is not dominant). View $\beta^\vee$ as an element of $S^\vee \subset X(\hat{T})$, and look at the simple reflection $s_{\beta^\vee} \in \hat{W}$, which corresponds to $s_\beta \in W$ under the identification of Weyl groups $W \simeq \hat{W}$. Pick a representative $s := \dot{s}_{\beta^\vee} \in N_{\hat{G}}(\hat{T})$, and introduce the conjugate $\rho' := s\rho s^{-1}$. A priori this is just a continuous homomorphism $\rho' : \Gamma_{\mathbb{Q}_p} \to \hat{G}(E)$. We claim it satisfies all the criteria in Theorem 1.2, plus the induction hypothesis. More precisely, in a sequence of steps below we will show that ρ' is triangular representation with $\hat{\chi}_{\rho'} = s_{\beta^\vee}(\hat{\chi}_\rho)$ (Step 2). Moreover, ρ' is good, generic, and potentially semistable (Step 3) with $w_{\rho'} = s_{\beta^\vee} w_\rho$ (Step 4) of smaller length than w_ρ.

Step 1. *C_ρ does not contain $\beta^\vee$.*

To see this, let $\hat{P} \supset \hat{B}$ be the standard parabolic associated with $\{\beta^\vee\}$. It is generated by $\hat{B}$ and s, and has a Levi decomposition $\hat{P} = \hat{M}\hat{N}$, where $\hat{M} \simeq \hat{G}_{\beta^\vee}$ in the notation of [4] (see p. 24). By their Lemma 3.1.4, there is an isomorphism $\hat{M} \simeq \hat{T}' \times \mathrm{GL}(2)$ for a subtorus $\hat{T}' \subset \hat{T}$, central in $\hat{M}$. (This uses Assumption 2.1). We can arrange that, under this isomorphism, the Borel pair $(\hat{B} \cap \hat{M}, \hat{T})$ corresponds to the upper-triangular Borel pair in GL(2)—times $\hat{T}'$. In particular, the unique simple root $\beta^\vee$ of $\hat{M}$ corresponds to $1 \otimes \alpha_{\mathrm{GL}(2)}$, where $\alpha_{\mathrm{GL}(2)}$ denotes the simple root of GL(2). Now consider the map $\hat{B} \to \mathrm{GL}(2)$ defined as the composition

$$\hat{B} \hookrightarrow \hat{P} \twoheadrightarrow \hat{M} \simeq \hat{T}' \times \mathrm{GL}(2) \twoheadrightarrow \mathrm{GL}(2).$$

It maps $\hat{T}$ onto the diagonal torus in GL(2), annihilates all the root groups $\hat{U}_{\alpha^\vee}$ for positive roots $\alpha \neq \beta$, and maps $\hat{U}_{\beta^\vee}$ isomorphically to $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$. The resulting two-dimensional representation $\rho_\beta : \Gamma_{\mathbb{Q}_p} \to \mathrm{GL}(2)$, obtained by composing with ρ, is of the form $\rho_\beta = \begin{pmatrix} \chi_1 & * \\ & \chi_2 \end{pmatrix}$ for two potentially crystalline characters $\chi_i : \Gamma_{\mathbb{Q}_p} \to E^\times$. Note that ρ_β *is* potentially semistable since ρ is assumed to be. Furthermore, by the way we set things up,

$$\chi_1/\chi_2 = \alpha_{\mathrm{GL}(2)} \circ \hat{\chi}_{\rho_\beta} = \beta^\vee \circ \hat{\chi}_\rho.$$

On an open subgroup of $I_{\mathbb{Q}_p}$, this character becomes $\epsilon^{\langle \chi_{\rho,alg}, \beta^\vee \rangle}$, which is a *negative* power of the cyclotomic character. By 3.2, we infer that ρ_β must split. This allows us to find a $b \in \hat{B}(E)$ such that $(b\rho b^{-1})_\beta = \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix}$. In particular, $\beta^\vee \notin C_{b\rho b^{-1}}$. Since ρ is assumed to be good, $C_\rho \subset C_{b\rho b^{-1}}$, and we can conclude that $\beta^\vee \notin C_\rho$.

Step 2. *The representation ρ' is triangular.*

We show that $\rho' = s\rho s^{-1}$ maps $\Gamma_{\mathbb{Q}_p}$ into $\hat{B}(E)$. Choose arbitrarily an ordering of the roots in C_ρ. As is well-known, the multiplication map gives an isomorphism of varieties,

$$m : \prod_{\alpha^\vee \in C_\rho} \hat{U}_{\alpha^\vee} \xrightarrow{\sim} \hat{U}_{C_\rho}.$$

(See (11) on p. 25 in [4]). Let $\sigma \in \Gamma_{\mathbb{Q}_p}$, and find the Jordan decomposition $\rho(\sigma) = tu$, where $t = \hat{\chi}_\rho(\sigma) \in \hat{T}(E)$, and $u = \prod u_{\alpha^\vee} \in \hat{U}_{C_\rho}(E)$. Upon conjugation by s, we get $\rho'(\sigma) = t'u'$, where $t' = s\hat{\chi}_\rho(\sigma)s^{-1}$, and $u' = \prod s u_{\alpha^\vee} s^{-1}$. It is a general fact that $\dot{w}\hat{U}_{\alpha^\vee}\dot{w}^{-1} = \hat{U}_{w(\alpha^\vee)}$ (which we note is used in the proof of 3.2.5 in loc. cit.). Consequently, $s u_{\alpha^\vee} s^{-1} \in \hat{U}_{s_{\beta^\vee}(\alpha^\vee)}$. From basic Lie theory, we know that $s_{\beta^\vee}$ permutes $R^{+\vee} - \{\beta^\vee\}$, and sends $\beta^\vee \mapsto -\beta^\vee$. Therefore, to check that $u' \in \hat{U}(E)$, we just have to verify that $\beta^\vee$ does not contribute to the product $u = \prod u_{\alpha^\vee}$. In other words, that $\beta^\vee \notin C_\rho$. This is the preceding Step 1.

Step 3. *ρ' is good, generic, and potentially semistable.*

During Step 2 we found that $\hat{\chi}_{\rho'} = s_{\beta^\vee}(\hat{\chi}_\rho)$. Therefore, $\alpha^\vee \circ \hat{\chi}_{\rho'} = s_{\beta^\vee}(\alpha^\vee) \circ \hat{\chi}_\rho$, for all $\alpha \in R$, which is not among $\{1, \epsilon^{\pm 1}\}$. That is, ρ' is generic (and clearly potentially semistable, being a conjugate of ρ). ρ' is good by Lemma 3.2.5 in [4]: We just have to check that $s_{\beta^\vee} \in W_{C_\rho}$. I.e., that $s_{\beta^\vee}(C_\rho) \subset R^{+\vee}$. Again, this follows from Step 1, since $s_{\beta^\vee}$ permutes $R^{+\vee} - \{\beta^\vee\}$, which contains C_ρ.

Step 4. *Invoking the induction hypothesis.*

From the relation $\chi_{\rho',alg} = s_\beta(\chi_{\rho,alg})$, we find that $w_{\rho'} = s_{\beta^\vee} w_\rho$, which has length $\ell(w_{\rho'}) = \ell(w_\rho) - 1$ since $w_\rho^{-1}(\beta^\vee)$ is a *negative* root:

$$\langle \chi^+_{\rho,alg}, w_\rho^{-1}(\beta^\vee) \rangle = \langle w_\rho^{-1}(\chi_{\rho,alg}), w_\rho^{-1}(\beta^\vee) \rangle = \langle \chi_{\rho,alg}, \beta^\vee \rangle < 0.$$

Applying the induction hypothesis to ρ', we infer that $w_{\rho'} \in W_{C_{\rho'}}$. The rest is completely formal, using that $C_{\rho'} = s_{\beta^\vee}(C_\rho)$ (as follows from the computation in Step 2; see also the proof of Lemma 3.3.5 in [4]). As a result, $W_{C_{\rho'}} = s_{\beta^\vee}(W_{C_\rho})$, and we conclude that $w_\rho \in W_{C_\rho}$, which finishes the proof. $\square$

To finish the proof of Theorem 1.2 we have to observe that the converse of (3.1) holds when $G = \mathrm{GL}(n)$. This follows from a result of Nekovar (and independently Perrin-Riou) stating that "ordinary implies semistable". We quote Propositions 1.24–1.28 in [23], nicely summarized as Lemma 3.1.4 in [19]:

Lemma 3.3. *Suppose $\rho : \Gamma_{\mathbb{Q}_p} \to \hat{B}(E)$ is ordinary of some integral weight $\lambda_1 \geq \cdots \geq \lambda_n$. That is,*

$$\rho \simeq \begin{pmatrix} \chi_1 & \cdots & * \\ & \ddots & \vdots \\ & & \chi_n \end{pmatrix},$$

where $\chi_i = \epsilon^{-(i-1)-\lambda_{n-i+1}}$ on an open subgroup of $I_{\mathbb{Q}_p}$. Then ρ is de Rham.

This shows the converse of (3.1) when $w_\rho = 1$: Indeed, if we know χ_ρ is locally algebraic and its algebraic part $\chi_{\rho,alg}$ is already dominant, then $\hat{\chi}_\rho = \mathrm{diag}(\chi_1, \ldots, \chi_n)$ where each χ_i is potentially crystalline, say $\chi_i = \epsilon^{h_i}$ on some open subgroup $J \subset I_{\mathbb{Q}_p}$, and $h_1 > h_2 > \cdots > h_n$ by dominance and regularity. Lemma 3.3 guarantees ρ is de Rham.

If we only know that χ_ρ is locally algebraic and $w_\rho \in W_{C_\rho}$, we look at the conjugate $\rho' := w_\rho^{-1} \rho w_\rho$ which *is* $\hat{B}$-valued. Furthermore, $\chi_{\rho'} = w_\rho^{-1}(\chi_\rho)$ is locally algebraic and has $w_{\rho'} = 1$. By the previous paragraph ρ' is de Rham. Then so is ρ.

Is there an analogue of Lemma 3.3 for general groups G? More precisely, we pose the following question:

Question 3.4. *Suppose $\rho : \Gamma_{\mathbb{Q}_p} \to \hat{B}(E)$ is a good, generic, and* **regular** *representation for which χ_ρ is locally algebraic and $\chi_{\rho,alg}$ is dominant. Is ρ then necessarily de Rham?*

By Lemma 3.3 above, this question has an affirmative answer when $\hat{G} = \mathrm{GL}_n$. One can also easily verify it for $\hat{G} = \mathrm{GSp}_{2n}$. But we do not know a general proof. The conundrum is to find a faithful representation $\hat{B} \hookrightarrow \mathrm{GL}_n$ such that 3.3 applies to the composition $\rho_{\mathrm{GL}_n} : \Gamma_{\mathbb{Q}_p} \to \mathrm{GL}_n$. The regularity condition we work with may be too weak to guarantee this.

4. Proof of the main result

In this section we prove our main Theorem 1.3 from the Introduction.

Proof. The heart of the proof is to show the implication below for any $w_0 \in W_{C_\rho}$.

$$\Pi(\rho)^{V-alg}_{C_\rho, w_0} \neq 0 \Longrightarrow \chi_\rho \text{ is locally algebraic and } \boxed{w_\rho = w_0}. \tag{4.1}$$

Let us fix such a w_0 once and for all, and for this proof simply write $\Pi := \Pi(\rho)_{C_\rho, w_0}$. Let $\mathrm{Fil}_j \Pi$ be its socle filtration, with graded pieces $\mathrm{Gr}_j \Pi$ as described in Conjecture 2.2 (recall we are assuming 2.3, so the conjecture holds). By convention,

$\mathrm{Fil}_{-1}\Pi := 0$. For the purposes of this proof, we will introduce the notation below for the principal series occurring in Π,

$$I_w := \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} w^{-1}(\chi_\rho)\cdot(\epsilon^{-1}\circ\theta)\right)^{\mathcal{C}^0},$$

for any $w \in W$. These are all topologically irreducible by Assumption 2.3. Thus,

$$\mathrm{Gr}_j\Pi \simeq \oplus_{w\in X_j} I_w,$$

for a certain subset $X_j \subset W$. Recall that X_j consists of all Weyl elements of the form $w = (\prod_{\alpha\in I^\vee} s_\alpha)w_0$, where I varies over subsets $I \subset w(S^\vee)\cap C_\rho$ of pairwise orthogonal roots with $|I| = j$. As $\Pi^{V-alg} \neq 0$, there is a unique $\nu \geq 0$ for which

$$(\mathrm{Fil}_{\nu-1}\Pi)^{V-alg} = 0, \qquad (\mathrm{Fil}_\nu\Pi)^{V-alg} \neq 0.$$

By left-exactness of $(-)^{V-alg}$ (cf. Lemma A.4), $(\mathrm{Gr}_\nu\Pi)^{V-alg} \neq 0$. In other words, $I_w^{V-alg} \neq 0$ for some $w \in X_\nu$. By Lemma A.1, part (1), we deduce that the inducing character $w^{-1}(\chi_\rho)\cdot(\epsilon^{-1}\circ\theta)$ must be locally algebraic, with algebraic part λ. Therefore χ_ρ itself must be *locally algebraic*, and $w^{-1}(\chi_{\rho,alg}) = \lambda+\theta$; which is dominant. Therefore we must have $w = w_\rho$. In particular, $w_\rho \in X_\nu$. If we can show $\nu = 0$, we are done proving (4.1) since $X_0 = \{w_0\}$.

Suppose, for the sake of contradiction, that $\nu > 0$. Then $\mathrm{Fil}_\nu\Pi/\mathrm{Fil}_{\nu-2}\Pi$ makes sense, and it sits in a short exact sequence

$$0 \longrightarrow \oplus_{w\in X_{\nu-1}} I_w \longrightarrow \mathrm{Fil}_\nu\Pi/\mathrm{Fil}_{\nu-2}\Pi \xrightarrow{\pi} \oplus_{w\in X_\nu} I_w \longrightarrow 0.$$

Consider the pull-back of $I_{w_\rho} \hookrightarrow \oplus_{w\in X_\nu} I_w$. That is $\mathcal{E} := \pi^{-1}(I_{w_\rho})$, which sits in an extension

$$0 \longrightarrow \oplus_{w\in X_{\nu-1}} I_w \xrightarrow{\iota} \mathcal{E} \xrightarrow{\pi} I_{w_\rho} \longrightarrow 0, \tag{4.2}$$

which is necessarily *non-split*: Otherwise I_{w_ρ} would embed into $\mathrm{soc}_G(\mathcal{E})$, and hence into $\mathrm{soc}_G(\Pi/\mathrm{Fil}_{\nu-2}\Pi) = \mathrm{Gr}_{\nu-1}\Pi$, which cannot happen ($\Pi$ has multiplicity one). Furthermore, $\mathcal{E}^{V-alg} \neq 0$: Clearly $(\mathrm{Fil}_\nu\Pi/\mathrm{Fil}_{\nu-2}\Pi)^{V-alg} \neq 0$ by definition of ν, and A.4, and there is a short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathrm{Fil}_\nu\Pi/\mathrm{Fil}_{\nu-2}\Pi \longrightarrow \oplus_{w\in X_\nu-\{w_\rho\}} I_w \longrightarrow 0.$$

All the I_w appearing here have no locally V-algebraic vectors by our argument above, and we conclude that $\mathcal{E}^{V-alg} \neq 0$.

Now consider the various push-outs of (4.2). That is, $\mathcal{E}_w := (\mathcal{E}\oplus I_w)/\Delta$ where $\Delta \simeq \oplus_{w\in X_{\nu-1}} I_w$ is embedded via $(-\iota, \mathrm{pr}_w)$. Thus we arrive at extensions

$$0 \longrightarrow I_w \longrightarrow \mathcal{E}_w \longrightarrow I_{w_\rho} \longrightarrow 0.$$

At least one of these is non-split, since (4.2) is non-split. Once and for all we pick a $w \in X_{\nu-1}$ for which $\mathcal{E}_w$ is non-split. Note that $\mathcal{E}_w^{V-alg} \neq 0$ since $\mathcal{E}^{V-alg} \neq 0$ and $\Delta^{V-alg} = 0$.

By the main result from Hauseux's thesis [20] (Théorème 1.1, part (i)) we firstly deduce that $w_\rho^{-1}(\chi_\rho) = s_\beta w^{-1}(\chi_\rho)$ for some simple root β (again, ruling out the equality $w_\rho^{-1}(\chi_\rho) = w^{-1}(\chi_\rho)$ by multiplicity one for Π). In fact we must have $w_\rho = ws_\beta$, since $(ws_\beta)^{-1}(\chi_{\rho,alg})$ is dominant. Secondly, by part (ii) of Théorème 1.1 there is a *unique* nonsplit extension in $\mathrm{Ext}^1_{G(\mathbb{Q}_p)}(I_{w_\rho}, I_w)$, up to equivalence, and it can be constructed by parabolic induction as done in section 3.3 of [4] (cf. the proof of Théorème 5.2.1 in [20]; or section 9 in [10]). The construction goes as follows: As in Sect. 2.3 we consider the parabolic $P_J \supset B^-$ associated with the singleton $J = \{\beta\}$. It has Levi factor $G_J \simeq T'_J \times \mathrm{GL}_2$. Then

$$\mathcal{E}_w \simeq \left(\mathrm{Ind}_{P_J(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} w^{-1}(\chi_\rho)\cdot(\epsilon^{-1}\circ\theta)|_{T'_J(\mathbb{Q}_p)} \otimes \mathcal{E}_J\right)^{\mathcal{C}^0},$$

where $\mathcal{E}_J$ is the unique non-split extension in $\mathrm{Ext}^1_{\mathrm{GL}_2}(\mathcal{I}_{w_\rho}, \mathcal{I}_w)$, where we have introduced the notation

$$\mathcal{I}_w := \left(\mathrm{Ind}_{\left(\begin{smallmatrix} * & \\ * & * \end{smallmatrix}\right)}^{\mathrm{GL}_2} w^{-1}(\chi_\rho)\cdot(\epsilon^{-1}\circ\theta)|_{T_{\mathrm{GL}_2}(\mathbb{Q}_p)}\right)^{\mathcal{C}^0}.$$

(Proposition B.2 in [4] summarizes the GL_2-theory nicely). If we write the inducing character as

$$w^{-1}(\chi_\rho)\cdot(\epsilon^{-1}\circ\theta)|_{T_{\mathrm{GL}_2}(\mathbb{Q}_p)} = \chi_1\epsilon^{-1}\otimes\chi_2,$$

then each χ_i is locally algebraic (since χ_ρ is) and $\mathcal{E}_J$ corresponds to the non-split Galois representation $\left(\begin{smallmatrix} \chi_1 & * \\ & \chi_2 \end{smallmatrix}\right)$ under the p-adic local Langlands correspondence, cf. Example A.5.

Now, recall that $\mathcal{E}_w^{alg} \neq 0$. By Lemma A.3 we see that $\mathcal{E}_J^{alg} \neq 0$. This in turn implies that $\left(\begin{smallmatrix} \chi_1 & * \\ & \chi_2 \end{smallmatrix}\right)$ must be de Rham (by "compatibility with classical local Langlands"; see the proof of 4.14 in [25] for details in the *reducible* case). However, $\chi_1\epsilon^{-1}\otimes\chi_2$ is *not* dominant ($\chi_2\epsilon^{-1}\otimes\chi_1$ is, since $w_\rho = ws_\beta$). This contradicts Example 3.9 in [5]: There are no such non-split de Rham representations.

This finishes the proof of (4.1). To deduce part (1) of Theorem 1.3 note that we have shown $\Pi(\rho)^{ord,V-alg} \neq 0$ implies χ_ρ is locally algebraic, with $\chi^+_{\rho,alg} = \lambda+\theta$, *and* that $w_\rho \in W_{C_\rho}$ [since $\Pi(\rho)^{V-alg}_{C_\rho,w_0} \neq 0$ for at least some $w_0 \in W_{C_\rho}$ which necessarily equals w_ρ by (4.1)]. To deduce part (2), the first equality $\Pi(\rho)^{ord,alg} = \Pi(\rho)^{ord,V-alg}$ follows immediately from (1) (since $\chi^+_{\rho,alg}$ encodes the highest weight λ). Moreover, by assumption χ_ρ is locally algebraic with $w_\rho \in W_{C_\rho}$. Therefore,

$$\Pi(\rho)^{ord,V-alg} \overset{(4.1)}{=} \Pi(\rho)^{V-alg}_{C_\rho,w_\rho} = (\mathrm{soc}_G\Pi(\rho)_{C_\rho,w_\rho})^{V-alg}.$$

The last equality follows from the fact that $(\mathrm{Gr}_j\Pi(\rho)_{C_\rho,w_\rho})^{V-alg} = 0$ for $j > 0$. Now use the computation in the Appendix; part (2) of A.1. The inducing character $w_\rho^{-1}(\chi_\rho)\cdot(\epsilon^{-1}\circ\theta)$ has smooth part $w_\rho^{-1}(\chi_{\rho,sm})\cdot|\theta|^{-1}$. However, for *smooth* parabolic induction, we have intertwining operators, which allows us to ignore w_ρ here. $\square$

5. An application to local-global compatibility

Chapter 4 of [4] presents various local-global compatibility conjectures on how the construction $\Pi(-)^{ord}$ relates to spaces of p-adic and mod p modular forms (see their p-adic Conjecture 4.2.2, and the mod p Conjecture 4.2.5). We employ the notation of loc. cit., without much comment, and focus on the p-adic case. Basically, one starts off with a Galois representation $r : \Gamma_F \to \mathrm{GL}_n(E)$, where F/F^+ is a CM field, which occurs in a space of p-adic modular forms $\hat{S}(U^p, E)$ on some definite unitary group $G_{/F^+}$ split above p. Assuming $r_w = r|_{\Gamma_{F_w}}$ is triangular and generic, for all places $w|p$ of F, it is expected that for some integer $d > 0$,

$$\left(\widehat{\bigotimes_{v|p}}\Pi(r_{\tilde{v}})^{ord} \otimes (\epsilon^{n-1} \circ \det)\right)^{\oplus d} \xrightarrow{\sim} \hat{S}(U^p, E)[\mathfrak{p}^\Sigma]^{ord}, \qquad (5.1)$$

as representations of $G(F^+ \otimes \mathbb{Q}_p)$. Here $\mathfrak{p}^\Sigma \subset \mathbb{T}^\Sigma$ is the prime ideal of the Hecke algebra associated with r, see p. 45 in [4]. Theorem 4.4.8 in loc. cit. gives a speck of evidence for 5.1, proving there is an injection under rather strong hypotheses: Notably, they assume r is *modular* (which is something one would like to *deduce* from 5.1, cf. Fontaine-Mazur), and they assume all the restrictions $\bar{r}_w = \bar{r}|_{\Gamma_{F_w}}$ are "*totally indecomposable*" for $w|p$—that is, $C_{\bar{r}_w} = R^{+\vee}$ is maximal—or in other words, $W_{C_{\bar{r}_w}} = 1$, so that $\Pi(r_{\tilde{v}})^{ord}$ has a simple socle. Theorem 1.2 lets us relax this last hypothesis, by essentially running the same argument, but instead of getting an injection we only get a nonzero map in general. As far as we know, this is the first result towards 5.1 with no restrictions on the sets $C_{\bar{r}_w}$.

Corollary 5.2. *Let $r : \Gamma_F \to GL_n(E)$ be a Galois representation which satisfies the following assumptions:*

(a) *r is modular.*
(b) *$\bar{r}|_{\Gamma_{F(\zeta_p)}}$ is absolutely irreducible.*
(c) *r_w is triangular and generic, for $w|p$.*
(d) *$\bar{r}_w$ is triangular and inertially generic, for $w|p$.*
(e) *$p > 2n + 2$ and $\zeta_p \notin F$.*

Fix U^p such that r is modular of (some weight and) level U_pU^p, for some U_p small enough. Then there is a $G(F^+ \otimes \mathbb{Q}_p)$-equivariant nonzero map,

$$\widehat{\bigotimes_{v|p}}\Pi(r_{\tilde{v}})^{ord} \otimes (\epsilon^{n-1} \circ \det) \longrightarrow \hat{S}(U^p, E)[\mathfrak{p}^\Sigma]^{ord},$$

in the category of admissible unitary continuous representations.

Remark 5.3. Since we are not saying anything about surjectivity, we may as well take $d = 1$ above. The conditions (a)–(e) can be traced back to the modularity lifting theorems used in [4]. The differences with their Theorem 4.4.8 is that our condition (d) is weaker; it does not require $C_{\bar{r}_w}$ to be maximal—but instead of an injection we only get a nonzero map.

Proof. Pick an automorphic representation π of $G(\mathbb{A}_{F^+})$, of level of the form U_pU^p, such that $r \simeq r_\pi$. In particular, r_w is potentially crystalline with distinct Hodge-Tate weights, for $w|p$—as follows from the ordinarity assumption (c). Let $\widetilde{BS}(r_w)$ be the locally algebraic representation of $\mathrm{GL}_n(F_w)$ attached to r_w by the Breuil–Schneider recipe, normalized as in [30]. Essentially by local-global compatibility at the places above p, due to Caraiani and others (see [6,7,12]), and base change from G to GL_n, we know that

$$\widetilde{BS}(r) := \bigotimes_{v|p} \widetilde{BS}(r_{\tilde{v}}) \hookrightarrow \hat{S}(U^p, E)[\mathfrak{p}^\Sigma]. \tag{5.4}$$

(See Chapter 3 of [30] for details on how this goes). On the other hand, since $r_{\tilde{v}}$ is ordinary, we can relate $\widetilde{BS}(r_{\tilde{v}})$ to one of the principal series occurring in the socle of $\Pi(r_{\tilde{v}})^{ord}$. The twist $\epsilon^{n-1} \circ \det$ arises in the distinction between $\widetilde{BS}(r_{\tilde{v}})$, and the original normalization $BS(r_{\tilde{v}})$ from [11], see Section 2.4 in [30], and also 4.2.1 in [4]. The latter is what sits in $\Pi(r_{\tilde{v}})^{ord}$. Indeed, identifying $F_{\tilde{v}} = \mathbb{Q}_p$,

$$BS(r_{\tilde{v}}) \simeq \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} w_{r_{\tilde{v}}}^{-1}(\chi_{r_{\tilde{v}}}) \cdot (\epsilon^{-1} \circ \theta)\right)^{alg},$$

Passing to the universal unitary completion (cf. Def. 1.1 in [16]), one gets an injection

$$\widehat{\bigotimes_{v|p}} \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} w_{r_{\tilde{v}}\otimes\epsilon^{n-1}}^{-1}(\chi_{r_{\tilde{v}}\otimes\epsilon^{n-1}}) \cdot (\epsilon^{-1} \circ \theta)\right)^{\mathcal{C}^0} \hookrightarrow \hat{S}(U^p, E)[\mathfrak{p}^\Sigma].$$

Let us elaborate a bit on this step: The locally algebraic representation $\widetilde{BS}(r_{\tilde{v}})$ is a finitely generated G-representation (equipped with its finest locally convex topology). Therefore, by [16, Prop. 1.17] it admits a universal unitary completion by completing it relative to any finite type G-invariant lattice (not necessarily separated a priori); such as the unit ball of the natural sup-norm on $\widetilde{BS}(r_{\tilde{v}})$ obtained by viewing it as the locally algebraic functions in the continuous principal series $I_v := (\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} w_{r_{\tilde{v}}\otimes\epsilon^{n-1}}^{-1}(\chi_{r_{\tilde{v}}\otimes\epsilon^{n-1}}) \cdot (\epsilon^{-1} \circ \theta))^{\mathcal{C}^0}$. We conclude that the latter *is* the universal unitary completion of $\widetilde{BS}(r_{\tilde{v}})$, and we get the analogous result for their completed tensor product $\widehat{\bigotimes}_{v|p} I_v$ by a similar argument. By the universal property ([16, Def. 1.1]) at least this yields a continuous G-map $\widehat{\bigotimes}_{v|p} I_v \to \hat{S}(U^p, E)[\mathfrak{p}^\Sigma]$ extending (5.4) and which is therefore nonzero. This implies injectivity since I_v is topologically irreducible under our standing hypotheses (more precisely Assumption 2.3, which is implied by (d) of Corollary 5.2). A similar type of argument was used in the proof of [4, Thm. 4.4.8].

Now to finish the proof we employ Corollary 4.3.11 in [4] for each $\rho = r_{\tilde{v}} \otimes \epsilon^{n-1}$, which applies by (a)–(e), and get that restriction to the socle induces an isomorphism

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Q}_p)}(\Pi(\rho)^{ord}, \hat{S}) \\ &\simeq \bigoplus_{w\in W_{C_\rho}} \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Q}_p)}\left(\left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} w^{-1}(\chi_\rho) \cdot (\epsilon^{-1} \circ \theta)\right)^{\mathcal{C}^0}, \hat{S}\right), \end{aligned}$$

where $\hat{S}$ is short-hand for $\hat{S}(U^p, E)[\mathfrak{p}^{\Sigma}]$. Finally we note that $w_\rho \in W_{C_\rho}$ by our Theorem 1.2. Therefore $\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Q}_p)}(\Pi(\rho)^{ord}, \hat{S}) \neq 0$ as desired. □

Appendix A: Continuous principal series

For convenience, we gather a few well-known facts about locally algebraic vectors in continuous principal series. For lack of a reference, we provide proofs. As in the main text, $G_{/\mathbb{Q}_p}$ denotes a split connected reductive group, and $B = TU$ a choice of Borel subgroup. Let $B^- = TU^-$ be the opposite Borel subgroup. Suppose we are given a unitary continuous character $\chi : T(\mathbb{Q}_p) \to \mathcal{O}_E^\times$, with values in a finite extension $E/\mathbb{Q}_p$. We inflate it to $B^-(\mathbb{Q}_p)$, by making it trivial on the unipotent radical, and look at the continuous parabolic induction,

$$I(\chi) = \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi\right)^{\mathcal{C}^0} = \{f : G(\mathbb{Q}_p) \xrightarrow{cts.} E,\ f(bg) = \chi(b) f(g)\}.$$

Endowed with the sup-norm, this becomes a p-adic Banach space on which $G(\mathbb{Q}_p)$ acts unitarily via right translations. It affords an admissible unitary continuous representation of $G(\mathbb{Q}_p)$—which conjecturally is topologically irreducible precisely when $\chi \circ \alpha^\vee \neq 1$ for all simple roots α. (Irreducibility is known if this condition holds mod p. This is due to Ollivier for GL_n, and Abe in general. See [1,24], resp.)

Lemma A.1. *Let V be an irreducible algebraic representation of $G(\mathbb{Q}_p)$, defined over E, of highest weight $\lambda \in X_+(T)$ relative to B. Then the following holds.*

(1) *If $I(\chi)^{V-alg} \neq 0$, then χ is locally algebraic, with algebraic part $\chi_{alg} = \lambda$.*
(2) *If $\chi = \chi_{alg} \cdot \chi_{sm}$ is locally algebraic, with $\chi_{alg} = \lambda$, then*

$$I(\chi)^{alg} = I(\chi)^{V-alg} = V \otimes_E \left(Ind_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_{sm}\right)^{\mathcal{C}^\infty}.$$

(Here the superscript $\mathcal{C}^\infty$ means we take the smooth induction of χ_{sm}).

Proof. Recall that the locally V-algebraic vectors in $I(\chi)$ are defined as the subspace

$$I(\chi)^{V-alg} = \varinjlim_K V \otimes_E \mathrm{Hom}_K(V, I(\chi)) \hookrightarrow I(\chi),$$

with K ranging over the compact open subgroups of $G(\mathbb{Q}_p)$. For each such K,

$$\mathrm{Hom}_K(V, I(\chi)) \simeq \bigoplus_{x \in B^-(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)/K} (V^\vee|_{B^-(\mathbb{Q}_p)} \otimes \chi)^{B^-(\mathbb{Q}_p) \cap xKx^{-1}}. \tag{A.2}$$

To show part (1), suppose this is nonzero. Pick an x such that the corresponding summand is nonzero. Thus $V^\vee$ contains a vector $v \neq 0$ on which $B^-(\mathbb{Q}_p) \cap xKx^{-1}$ acts via (the inflation of) the inverse character χ^{-1}. In particular, the unipotent elements $U^-(\mathbb{Q}_p) \cap xKx^{-1}$ must act trivially on v. Therefore, since this subgroup is Zariski dense in the full unipotent radical, we deduce that $v \in (V^\vee)^{U^-(\mathbb{Q}_p)}$ must be a highest weight vector (relative to the opposite Borel). Consequently, $B^-(\mathbb{Q}_p)$ acts on v via the highest weight $-\lambda$. We conclude that χ and λ agree on the open subgroup $T(\mathbb{Q}_p) \cap xKx^{-1}$, and therefore χ is locally algebraic with $\chi_{alg} = \lambda$.

For part (2), suppose $\chi = \lambda \cdot \chi_{sm}$. The first equality follows from part (1). For the second equality, we realize V as the algebraically induced representation $\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \lambda$, and show that the natural multiplication map

$$\begin{aligned} \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_{sm}\right)^K &\longrightarrow \mathrm{Hom}_K(V, I(\chi)) \\ f_{sm} &\mapsto (f_{alg} \mapsto f_{alg} \cdot f_{sm}) \end{aligned}$$

is an isomorphism for sufficiently small K. Tensoring with V, and taking $\varinjlim_K$ then yields the result. First off, the multiplication map is injective: For suppose $f_{alg} \cdot f_{sm} = 0$, for all f_{alg}. Let $f_+ \in V$ be a highest weight vector, and take $f_{alg} = g^{-1} \cdot f_+$, for varying $g \in G(\mathbb{Q}_p)$. Since $f_+(e) \neq 0$, this shows $f_{sm}(g) = 0$. To get bijectivity, we compare dimensions. Taking K small enough that $\chi_{sm} = 1$ on every subgroup $B^-(\mathbb{Q}_p) \cap xKx^{-1}$, as in the previous paragraph, the source is identified with $E^{\# B^-(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)/K}$. It remains to show the target has the same dimension. In turn, in A.2 we must show that each summand $(V^\vee|_{B^-(\mathbb{Q}_p)} \otimes \chi)^{B^-(\mathbb{Q}_p)\cap xKx^{-1}}$ is (at most) one-dimensional. We break up $V^\vee$ into weight spaces, as $\oplus_{\mu \in X(T)} V^\vee(\mu)$.

$$(V^\vee|_{B^-(\mathbb{Q}_p)} \otimes \chi)^{B^-(\mathbb{Q}_p)\cap xKx^{-1}} \hookrightarrow \bigoplus_{\mu \in X(T)} (V^\vee(\mu) \otimes \chi)^{T(\mathbb{Q}_p)\cap xKx^{-1}}.$$

Only the highest weight $-\lambda$ contributes. Indeed, if the μ-summand is nontrivial, we must have $\mu = -\lambda$ (since $\chi_{alg} = \lambda$ and χ_{sm} is trivial on every $T(\mathbb{Q}_p) \cap xKx^{-1}$). Finally, reminding ourselves that $\dim_E V^\vee(-\lambda) = 1$, concludes the proof. □

Part (1) of Lemma A.1 can be slightly generalized as follows: Let $P = MN \supset B$ be a parabolic subgroup of G, defined over $\mathbb{Q}_p$, and let X be a unitary Banach representation of $M(\mathbb{Q}_p)$ over E, inflated to $P(\mathbb{Q}_p)$. Let $I(X) := (\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} X)^{\mathcal{C}^0}$ be its continuous induction; a unitary Banach representation of $G(\mathbb{Q}_p)$ over E. We keep the algebraic representation V from Lemma A.1. The N-invariants V^N is the irreducible algebraic representation of M of highest weight λ relative to $B \cap M$.

Lemma A.3. $I(X)^{V-alg} \neq 0 \Longrightarrow X^{V^N-alg} \neq 0$.

Proof. The proof is an easy adaption of the ideas above. By assumption,

$$\mathrm{Hom}_K(V, I(X)) \simeq (V^\vee \otimes I(X))^K \simeq I(V^\vee|_{P(\mathbb{Q}_p)} \otimes X)^K$$

is nonzero for small enough compact open subgroups $K \subset G(\mathbb{Q}_p)$. Moreover,

$$I(V^\vee|_{P(\mathbb{Q}_p)} \otimes X)^K \simeq \bigoplus_{x \in P(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)/K} (V^\vee|_{P(\mathbb{Q}_p)} \otimes X)^{P(\mathbb{Q}_p)\cap xKx^{-1}}.$$

Pick an x such that $(V^\vee|_{P(\mathbb{Q}_p)} \otimes X)^{P(\mathbb{Q}_p)\cap xKx^{-1}} \neq 0$. Since $N(\mathbb{Q}_p)$ acts trivially on X, we infer that

$$0 \neq (V^\vee|_{P(\mathbb{Q}_p)} \otimes X)^{P(\mathbb{Q}_p)\cap xKx^{-1}} \subset \Big((V^\vee)^N \otimes X\Big)^{M(\mathbb{Q}_p)\cap xKx^{-1}}.$$

Finally, $(V^\vee)^N$ is the irreducible algebraic representation of M of highest weight $-\lambda$ relative to $B^- \cap M$. Consequently $(V^\vee)^N \simeq (V^N)^\vee$ and we conclude that $\mathrm{Hom}_{M(\mathbb{Q}_p)\cap xKx^{-1}}(V^N, X) \neq 0$. In particular $X^{V^N-alg} \neq 0$. □

In the main body of the text we have used the following basic observation repeatedly, often without mention.

Lemma A.4. *The functors* $(-)^{alg}$ *and* $(-)^{V-alg}$ *are left exact on the category of* $G(\mathbb{Q}_p)$*-representations over* E.

Proof. Say we are given an exact sequence of $G(\mathbb{Q}_p)$-representations over E,

$$0 \longrightarrow W' \longrightarrow W \longrightarrow W'' \longrightarrow 0.$$

Apply $\mathrm{Hom}_K(V, -)$, then tensor with V over E, and take the limit over K. This results in a long exact sequence,

$$0 \longrightarrow W'^{V-alg} \longrightarrow W^{V-alg} \longrightarrow W''^{V-alg} \longrightarrow \varinjlim_K V \otimes_E \mathrm{Ext}^1_K(V, W') \longrightarrow \cdots,$$

which proves left exactness of $(-)^{V-alg}$. Now use that $W^{alg} = \oplus_V W^{V-alg}$. □

A basic example showing $(-)^{alg}$ is not exact in general is the following.

Example A.5. Let $\rho = \begin{pmatrix} \chi_1 & * \\ & \chi_2 \end{pmatrix}$ be a non-split extension, where the $\chi_i : \Gamma_{\mathbb{Q}_p} \to E^\times$ are potentially crystalline characters; say $\chi_i = \epsilon^{h_i}$ on an open subgroup of $I_{\mathbb{Q}_p}$. Suppose $h_1 < h_2$. Such ρ exist by Example 3.9 of [5], but they are *not* de Rham. The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ associates a Banach representation $\Pi(\rho) = \Pi(\rho)^{ord}$, which sits in a non-split short exact sequence (where $G := \mathrm{GL}_2(\mathbb{Q}_p)$ and $\bar{B} := B^-(\mathbb{Q}_p)$ denotes the lower-triangular Borel)

$$0 \longrightarrow \mathrm{Ind}_{\bar{B}}^G(\chi_1\epsilon^{-1} \otimes \chi_2) \longrightarrow \Pi(\rho) \longrightarrow \mathrm{Ind}_{\bar{B}}^G(\chi_2\epsilon^{-1} \otimes \chi_1) \longrightarrow 0.$$

To be safe here, we make the assumption that $\chi_1\chi_2^{-1} \neq \epsilon^{\pm 1}$. Taking locally algebraic vectors then yields a long exact sequence

$$0 \longrightarrow \mathrm{Ind}_{\bar{B}}^G(\chi_1\epsilon^{-1} \otimes \chi_2)^{alg} \longrightarrow \Pi(\rho)^{alg} \longrightarrow \mathrm{Ind}_{\bar{B}}^G(\chi_2\epsilon^{-1} \otimes \chi_1)^{alg} \longrightarrow \cdots,$$

where $\cdots$ is necessarily nonzero: $\Pi(\rho)^{alg} = 0$ since ρ is not de Rham. (For details on the reducible case see the proof of Proposition 4.14 in [25]). However, the character $\chi_2\epsilon^{-1} \otimes \chi_1$ is locally algebraic with algebraic part $\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \mapsto t_1^{h_2-1}t_2^{h_1}$, which is *dominant* relative to the upper-triangular Borel B. Part (2) of Lemma A.1 shows that $\mathrm{Ind}_{\bar{B}}^G(\chi_2\epsilon^{-1} \otimes \chi_1)^{alg} \neq 0$.

Appendix B: Cohomology of $\mathbb{Q}_p(r)$

For the convenience of the reader, we here reproduce the table from Example 3.9 on page 359 in [5]. The notation H^1_* for various $* \in \{e, f, g\}$ is defined in (3.7.2) on page 353 of [5]. One has inclusions $H^1_e \subset H^1_f \subset H^1_g$. The numbers below give the $\mathbb{Q}_p$-dimensions of these spaces for $V = \mathbb{Q}_p(r)$.

r	$H^1_e(\Gamma_{\mathbb{Q}_p}, \mathbb{Q}_p(r))$	$H^1_f(\Gamma_{\mathbb{Q}_p}, \mathbb{Q}_p(r))$	$H^1_g(\Gamma_{\mathbb{Q}_p}, \mathbb{Q}_p(r))$	$H^1(\Gamma_{\mathbb{Q}_p}, \mathbb{Q}_p(r))$
$r < 0$	0	0	**0**	1
$r = 0$	0	1	1	2
$r = 1$	1	1	2	2
$r > 1$	1	1	1	1

Recall that $H^1_g(\Gamma_{\mathbb{Q}_p}, V)$ parametrizes the de Rham extensions in $\mathrm{Ext}^1_{\Gamma_{\mathbb{Q}_p}}(1, V)$, and $H^1_f(\Gamma_{\mathbb{Q}_p}, V)$ parametrizes the crystalline extensions. What is important for us in this paper is the bold entry: $H^1_g(\Gamma_{\mathbb{Q}_p}, \mathbb{Q}_p(r)) = 0$ for $r < 0$. This translates into the non-existence of non-split de Rham extensions $\begin{pmatrix} 1 & * \\ & \epsilon^h \end{pmatrix}$ with $h > 0$.

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