The Breuil-Schneider conjecture, a survey

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Abstract This note is a survey of the Breuil-Schneider conjecture, based on the authors 30 minute talk at the 13th conference of the Canadian Number Theory Association (CNTA XIII) held at Carleton University, June 16–20, 2014. We give an overview of the problem, and describe certain recent developments by the author and others.

Key words: p-adic Langlands, trace formula, invariant norms, Galois representations

1 Prelude: The Fontaine-Mazur conjecture

Let $\tilde{F}$ be a number field[1], with absolute Galois group $\Gamma_{\tilde{F}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\tilde{F})$. The p-adic representations of $\Gamma_{\tilde{F}}$, which arise naturally, come from algebraic geometry. More precisely, if $X_{/\tilde{F}}$ is a (smooth projective) variety, one would look at irreducible constituents $r : \Gamma_{\tilde{F}} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_p)$ of the cohomology $H^{\bullet}_{\text{ét}}(X \times_{\tilde{F}} \bar{\mathbb{Q}}, \bar{\mathbb{Q}}_p(t))$. It is now known that such r are "*geometric*", which by definition means the following:

- $r_w := r|_{\Gamma_{\tilde{F}_w}}$ is unramified[2] at all but finitely many places $w \nmid p$.
- r_w is potentially semistable at all places $w|p$.

The Fontaine-Mazur conjecture (Conjecture 1 in [FM]) is the converse: These local conditions guarantee that r occurs in the cohomology of some X, up to Tate twist.

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[1] This is the notation we will eventually use in Section 5 below, to distinguish it from $F/\mathbb{Q}_p$.

[2] Here we tacitly fix algebraic closures $\bar{F} \hookrightarrow \bar{F}_w$ extending $F \hookrightarrow F_w$, in order to identify the decomposition group $\Gamma_{F_w} = \mathrm{Gal}(\bar{F}_w/F_w)$ with a subgroup of Γ_F. We say r_w is *unramified* if its restriction to the inertia group $I_{F_w} = \mathrm{Gal}(\bar{F}_w/F_w^{nr})$ is trivial.

In recent years, Emerton and Kisin have made impressive progress on this for odd two-dimensional representations of $\Gamma_{\mathbb{Q}}$ – as a result of the proof of Serre's conjecture [KW], by Khare and Wintenberger! To give the flavor, Emerton shows the following result (see Theorem 1.2.4 in [Eme]).

Theorem 1.1 *(Emerton) Let $r : \Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ be an irreducible odd representation, $p > 2$. Assume $\bar{r}|_{\Gamma_{\mathbb{Q}(\zeta_p)}}$ is irreducible, and that $\bar{r}|_{\Gamma_{\mathbb{Q}_p}}$ is "generic". If r is geometric, with distinct Hodge-Tate weights, then (up to a Tate twist) $r = r_f$ for some cuspidal eigenform f of weight $k \geq 2$.*

Remark 1. We write $\bar{r}$ for the reduction of r modulo p. Let us add a few details. By the compactness of $\Gamma_{\mathbb{Q}}$, the image of r lies in $\mathrm{GL}_2(\mathscr{O}_E)$ for some finite extension $E/\mathbb{Q}_p$. Composition with $\mathscr{O}_E \twoheadrightarrow \kappa_E = \mathscr{O}_E/\varpi_E\mathscr{O}_E$ yields the "naive" reduction mod p into $\mathrm{GL}_2(k_E) \subset \mathrm{GL}_2(\bar{\mathbb{F}}_p)$. The Brauer-Nesbitt principle tells us its *semi-simplification* is independent of the choice of stable $\mathscr{O}_E$-lattice in E^2. This semisimple representation is what we denote by $\bar{r}$. Of course, in Theorem 1.1 $\bar{r}$ is irreducible, so it's given by any of its naive reductions. The notion of $\bar{r}|_{\Gamma_{\mathbb{Q}_p}}$ being "generic" is a bit ad hoc. Here we take it to mean that $\bar{r}|_{\Gamma_{\mathbb{Q}_p}} \nsim \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & * \\ & \bar{\varepsilon}_{cyc} \end{pmatrix}$; even after twisting by a character, and * may or may not be zero. (Here $\bar{\varepsilon}_{cyc}$ is the mod p cyclotomic character.)

Since there is a motive attached to f, by Scholl and others [Scho], one knows how to carve out $r = r_f$ in the cohomology of some (Kuga-Sato) variety $X_{/\mathbb{Q}}$.

Very broadly speaking, the proof of 1.1 has two steps. The **first** step is essentially Serre's conjecture, which combined with "big" $R = \mathbb{T}$ theorems of Böckle [Boc] tells us that at least r comes from a *p-adic* modular form; we say r is *pro*-modular (closely related to the modularity of $\bar{r}$). The **second** step is to establish *classical* modularity of r, using that $r|_{\Gamma_{\mathbb{Q}_p}}$ is potentially semistable of regular weight. This is where the p-adic local Langlands correspondence $\Pi(\cdot)$ for $\mathrm{GL}_2(\mathbb{Q}_p)$ comes in. It goes from two-dimensional p-adic representation of $\Gamma_{\mathbb{Q}_p}$ to unitary p-adic Banach space representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, and its "inverse" has a nice clean description via Colmez's so-called Montreal functor. We're applying $\Pi(\cdot)$ to the restriction $r|_{\Gamma_{\mathbb{Q}_p}}$. Emerton shows in [Eme] that it satisfies local-global compatibility, which basically says that

$$\Pi(r|_{\Gamma_{\mathbb{Q}_p}}) \hookrightarrow \mathrm{Hom}_{\Gamma_{\mathbb{Q}}}(r, \hat{H}^1),$$

where $\hat{H}^1$ is a huge Banach-space of p-adic modular forms (the completed cohomology of the tower of modular curves). The condition at p ensures that $\Pi(r|_{\Gamma_{\mathbb{Q}_p}})$ has so-called "locally algebraic" vectors – much more on these in the next section! Hence so does $\mathrm{Hom}_{\Gamma_{\mathbb{Q}}}(r, \hat{H}^1)$. But the locally algebraic vectors in $\hat{H}^1$ have a description in terms of classical modular forms, and this essentially finishes the proof.

This is meant to motivate the search for a p-adic local Langlands correspondence for $\mathrm{GL}_n(F)$, for finite extensions $F/\mathbb{Q}_p$, and other groups. The Breuil-Schneider conjecture is one of the initial steps towards a precise formulation of what one is looking for.

2 Motivation: p-adic local Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$

The *classical* local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ is a bijection between certain two-dimensional Weil-Deligne representations of $W_{\mathbb{Q}_p}$ and irreducible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Here the coefficient field E is unspecified. One can take the complex numbers $\mathbb{C}$, the ℓ-adic numbers $\bar{\mathbb{Q}}_\ell$ for $\ell \neq p$, or even $\bar{\mathbb{Q}}_p$ – which we eventually will. The topology of E plays no role in the correspondence. A natural source of Weil-Deligne representations are Galois representations. This is where the topology of E plays a huge role. Thus, for $\ell \neq p$, with any continuous representation $\rho : \Gamma_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_\ell)$ one can associate a Weil-Deligne representation $\mathrm{WD}(\rho)$, and hence an irreducible smooth representation $\pi_{sm}(\rho)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\bar{\mathbb{Q}}_\ell$. Moreover, one can *recover* ρ from $\pi_{sm}(\rho)$. This completely fails for $\ell = p$, and this is what leads to the p-adic local Langlands correspondence. The classical local Langlands correspondence for $\mathrm{GL}(2)$ has been dealt with in detail in [BH]. For $\mathrm{GL}(n)$ there are several good expositions. We mention [Kud] and [Wed].

In what follows we will fix our coefficient field E, which we take to be a finite extension $E/\mathbb{Q}_p$. Suitably normalized, classical local Langlands is $\Gamma_{\mathbb{Q}_p}$-equivariant and can therefore be defined over E (as opposed to $\bar{\mathbb{Q}}_p$). If we restrict ourselves to potentially semistable representations $\rho : \Gamma_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$, a recipe of Fontaine associates a Weil-Deligne representation $\mathrm{WD}(\rho)$, and hence $\pi_{sm}(\rho)$ still makes sense. See [BrSc] and [Hu] for details on how this goes. However, $\rho \rightsquigarrow \pi_{sm}(\rho)$ is no longer reversible. In addition, we have the Hodge-Tate weights, which we will assume are distinct, $\mathrm{HT}(\rho) = \{w_1 < w_2\}$. These correspond to an irreducible algebraic representation $\pi_{alg}(\rho)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ – the one of highest weight $t \mapsto t_1^{w_2-1} t_2^{w_1}$ (relative to the upper-triangular Borel). More concretely,

$$\pi_{alg}(\rho) = \det{}^{w_1} \otimes_E \mathrm{Sym}^{w_2-w_1-1}(E^2).$$

Still, one cannot reconstruct ρ from $\pi_{sm}(\rho)$ and $\pi_{alg}(\rho)$. In a nutshell, the problem is the following: In p-adic Hodge theory, the potentially semistable ρ are classified by linear algebra data which includes a certain Hodge *filtration* – and this is lost in the process of constructing $\pi_{alg}(\rho)$. We only see its jumps.

The p-adic local Langlands correspondence takes *any* continuous representation $\rho : \Gamma_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ and attaches a Banach E-space $\Pi(\rho)$ with a unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-action. Consult [CDP] for the most recent developments (for small p). This map $\rho \rightsquigarrow \Pi(\rho)$ is reversible, and compatible with classical local Langlands in the following sense: When ρ is potentially semistable, with distinct Hodge-Tate weights,

$$\boxed{\Pi(\rho)^{alg} = \pi_{alg}(\rho) \otimes_E \pi_{sm}(\rho)}. \tag{1}$$

Furthermore, $\Pi(\rho)^{alg} = 0$ otherwise. Here the superscript alg indicates taking the locally algebraic vectors – an invariant subspace between the smooth vectors and the locally analytic vectors. Recall that a vector is smooth if some compact open subgroup acts trivially. Analogously, a vector is locally algebraic if some compact open subgroup acts polynomially. (A good go-to source for these notions, as well as local analyticity, is Breuil's ICM article [BrICM].)

When ρ is irreducible, $\Pi(\rho)$ is known to be topologically irreducible, and therefore the completion of $\pi_{alg}(\rho) \otimes_E \pi_{sm}(\rho)$ relative to a suitable $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant norm $\|\cdot\|$, which somehow "corresponds" to the lost filtration.

For $\mathrm{GL}_n(\mathbb{Q}_p)$, or even $\mathrm{GL}_2(F)$ for finite extensions $F/\mathbb{Q}_p$, the p-adic local Langlands correspondence remains elusive. The Breuil-Schneider conjecture is in some sense a "first approximation", which uses 1 as a guiding priniciple: The right-hand side can be defined for any potentially semistable representation $\rho : \Gamma_F \to \mathrm{GL}_n(E)$, with distinct Hodge-Tate weights, and the conjecture is that $\pi_{alg}(\rho) \otimes_E \pi_{sm}(\rho)$ admits at least one $\mathrm{GL}_n(F)$-invariant norm. The resulting completions should be closely related to the yet undefined $\Pi(\rho)$ – at least in the irreducible case.

Of course, ultimately one would want more than the mere *existence* of an invariant norm $\|\cdot\|$ on $\pi_{alg}(\rho) \otimes_E \pi_{sm}(\rho)$. In general it will *not* be unique (up to equivalence), but should correspond to the possible compatible Hodge filtrations. In the case of $\mathrm{GL}_2(\mathbb{Q}_p)$ this is exemplified by the connection to $\mathscr{L}$-invariants.

Example 1. Let $\mathscr{E}_{/\mathbb{Q}_p}$ be a semistable elliptic curve. That is, its reduction mod p has a nodal singularity (as opposed to a cusp). This can always be achieved by passing to a finite extension. By the theory of the Tate curve, any such $\mathscr{E}$ has p-adic uniformization; meaning there's a unique $q \in \bar{\mathbb{Q}}_p^\times$ with $|q| < 1$ such that $\mathscr{E}(\bar{\mathbb{Q}}_p) \simeq \bar{\mathbb{Q}}_p^\times / q^{\mathbb{Z}}$, respecting the $\Gamma_{\mathbb{Q}_p}$-action. The Galois action $\rho_{\mathscr{E},p}$ on the Tate module

$$V_p(\mathscr{E}) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(\mathscr{E}) \simeq \mathbb{Q}_p^{\oplus 2}, \quad T_p(\mathscr{E}) := \varprojlim_n \mathscr{E}[p^n] \simeq \mathbb{Z}_p^{\oplus 2},$$

can then be made explicit in terms of the parameter q – or rather the $\mathscr{L}$-invariant,

$$\mathscr{L} = \mathscr{L}_{\mathscr{E}} = \log(q)/\mathrm{ord}(q).$$

(Here log is the Iwasawa-log, with $\log(p) = 0$). This $\mathscr{L}$ is *encoded* in $\rho_{\mathscr{E},p}$ via p-adic Hodge theory; it can be read off from the Hodge-filtration (see [Ber] for the explicit relation). In this example, $\mathrm{HT}(\rho_{\mathscr{E},p}) = \{0,1\}$, so $\pi_{alg}(\rho_{\mathscr{E},p}) = 1$ and $\pi_{alg}(\rho_{\mathscr{E},p}) = St$; the Steinberg representation (up to an unramified quadratic twist). Recall that,

$$St = St_{\mathrm{GL}_2(\mathbb{Q}_p)} = \mathscr{C}^\infty(\mathbb{P}^1(\mathbb{Q}_p))/\{\text{constants}\}.$$

The sup-norm on St is certainly $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant, and "corresponds" to $\mathscr{L} = 0$ (that is, $q = p$). In [Bre], it is shown how any $\mathscr{L}$ defines an invariant norm $\|\cdot\|_{\mathscr{L}}$ on St, such that $\Pi(\rho_{\mathscr{E},p}) = St^\wedge$ (the completion). In fact, more generally, in loc. cit. Breuil defines a unitary Banach space representation $B(k,\mathscr{L})$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, for any $k \geq 2$ and $\mathscr{L} \in \bar{\mathbb{Q}}_p$. The dual $B(k,\mathscr{L})'$ is a certain space of $\log_{\mathscr{L}}$-rigid functions on the p-adic upper half-plane $\Omega = \mathbb{C}_p \backslash \mathbb{Q}_p$. Furthermore,

$$B(k,\mathscr{L})^{alg} = |\det|^{\frac{k-2}{2}} \cdot \mathrm{Sym}^{k-2} \otimes St.$$

(Breuil shows this for $k = 2$ and for $k > 2$ when $\mathscr{L}$ comes from a modular form; but it's now known in general due to subsequent work of Colmez and others). These Banach representations $B(k, \mathscr{L})$ correspond to the semistable *non*-crystalline representations $\rho : \Gamma_{\mathbb{Q}_p} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$, with weights $\{0, k-1\}$, under the p-adic local Langlands correspondence. It might be worth emphasizing that a very special feature for $\mathrm{GL}_2(\mathbb{Q}_p)$, is that for (irreducible) *crystalline* representations ρ, the invariant norm on $\pi_{alg}(\rho) \otimes_E \pi_{sm}(\rho)$ *is* in fact unique up to equivalence – this is extremely important in local-global compatibility; which explains how $\Pi(\rho)$ "sits" in completed cohomology when ρ arises from a p-adic modular form – see [Eme].

3 The Breuil-Schneider conjecture, the statement

Besides the coefficient field $E/\mathbb{Q}_p$, we now fix a base field $F/\mathbb{Q}_p$. We assume throughout that E is large enough to contain the Galois closure of F. Now suppose $\rho : \Gamma_F \to \mathrm{GL}_n(E)$ is a potentially semistable representation, with labeled Hodge-Tate weights $\mathrm{HT}_\tau(\rho) = \{w_{1,\tau} < \cdots < w_{n,\tau}\}$, for each embedding $\tau : F \hookrightarrow E$. As in Section 2, one defines a smooth E-representation $\pi_{sm}(\rho)$ of $\mathrm{GL}_n(F)$ via the classical local Langlands correspondence for GL_n plus Fontaine. There is a small wrinkle here, which we are tempted to gloss over: There is an issue with *genericity* – not in the sense of the Introduction, but in the (usual) sense of $\pi_{sm}(\rho)$ having a *Whittaker* model[3]. When $\pi_{sm}(\rho)$ is *non*-generic, one should really replace it by a generic (but reducible) principal series, of which it is quotient. For instance, in the GL_2-case, if ρ is the trivial representation one would take $\pi_{sm}(\rho)$ to be the non-split extension of 1 by Steinberg. For the most part we will sweep this under the rug, and usually assume ρ is generic (i.e., that $\pi_{sm}(\rho)$ is generic.)

For each τ, we let $\pi_{alg,\tau}(\rho)$ be the irreducible algebraic representation of $\mathrm{GL}_n(F)$ of highest weight $(w_{n,\tau} - (n-1), \ldots, w_{1,\tau})$ relative to the upper-triangular Borel. Then let $\pi_{alg}(\rho) = \otimes_{\tau \in \mathrm{Hom}(F,E)} \pi_{alg,\tau}(\rho)$, with $\mathrm{GL}_n(F)$ acting diagonally. The Breuil-Scneider recipe, from [BrSc], takes the tensor product, and gives a locally algebraic representation (again, with $\mathrm{GL}_n(F)$ acting diagonally):

Definition 3.1 $BS(\rho) := \pi_{alg}(\rho) \otimes_E \pi_{sm}(\rho)$.

Upon a closer look, this definition works in greater generality. It only depends on the data $\mathrm{WD}(\rho)$ and $\mathrm{HT}_\tau(\rho)$ associated with ρ. In other words, we could start with data $\mathscr{D} = (\mathrm{WD}, \mathrm{HT}_\tau)$, consisting of a Weil-Deligne representation WD, and tuples of integers $\mathrm{HT}_\tau = \{w_{1,\tau} < \cdots < w_{n,\tau}\}$, and define $\mathrm{BS}(\mathscr{D})$ as above. The main conjecture of [BrSc] is as follows (Conjecture 4.3 on p. 18 in loc. cit.).

Conjecture 3.2 *(Breuil, Schneider) The data $\mathscr{D}$ arises from a potentially semistable representation if and only if $\mathrm{BS}(\mathscr{D})$ admits a $\mathrm{GL}_n(F)$-invariant norm.*

[3] Meaning the representation $\pi_{sm}(\rho)$ embeds into $\mathrm{Ind}_{U_n}^{\mathrm{GL}_n}(\psi)$ for some additive character $\psi \neq 1$.

The "if" part of this conjecture is completely known [Hu], and is due to Y. Hu – who proved a stronger result: First, WD arises from an étale (ϕ,N)-module D in a standard way (see 2.2 in [Hu]). As part of his Orsay Ph.D. thesis, Hu translated the existence of an admissible filtration on D, with jumps HT_τ, into a purely group-theoretic condition known as the "Emerton condition" (in the vein of Theorem 1 in [FR], which deals with the crystalline case). We rephrase part of Theorem 1.2 (or the slightly more general Theorem 2.12, p. 12) in [Hu] in our notation:

Theorem 3.3 *(Hu) $\mathscr{D}$ comes from a potentially semistable representation if and only if the Emerton condition holds: For every parabolic P, with Haar modulus δ_P,*

$$|(\delta_P^{-1}\chi)(z)|_p \leq 1, \quad \forall z \in Z_M^+, \quad \forall \chi \hookrightarrow J_P(\mathrm{BS}(\mathscr{D})). \tag{1}$$

Here Z_M^+ is the contracting monoid in the center of the Levi $M = M_P$, and $J_P(\cdot)$ is Emerton's locally analytic Jacquet functor (introduced and studied in [EJ]).

Let us add a few words on the contracting monoid (see p. 33 in [EJ] for more details): Once and for all, fix a compact open subgroup $N_0 \subset N = N_P$, and let $M^+ = \{m \in M | mN_0m^{-1} \subset N_0\}$. Then $Z_M^+ = M^+ \cap Z_M$. In the case of GL_n, taking P to be the standard parabolic corresponding to the partition $n = n_1 + \cdots + n_r$, it's given by

$$Z_M^+ = \{\mathrm{diag}(t_1 \cdot I_{n_1}, \ldots, t_r \cdot I_{n_r}) : |t_1| \leq \cdots \leq |t_r|\}.$$

It may also be worth pointing out that since $\mathrm{BS}(\mathscr{D}) = \pi_{alg}(\mathscr{D}) \otimes_E \pi_{sm}(\mathscr{D})$ is locally algebraic, we strictly speaking don't need the locally analytic extension of the Jacquet functor here: By Proposition 4.3.6, p. 63, in [EJ],

$$J_P(\mathrm{BS}(\mathscr{D})) \simeq \pi_{alg}(\mathscr{D})^{N_P} \otimes \pi_{sm}(\mathscr{D})_{N_P}.$$

Here the N_P-invariants $\pi_{alg}(\mathscr{D})^{N_P}$ is an irreducible algebraic representation of M_P, and the N_P-coinvariants $\pi_{sm}(\mathscr{D})_{N_P}$ is the usual Jacquet module in smooth representation theory (up to a twist).

It is relatively easy to show that 1 is satisfied if $\mathrm{BS}(\mathscr{D})$ carries an invariant norm, which is how the condition arose in the first place. This takes care of the so-called "easy" direction of 3.2. The "only if" part is still open; the existence of a norm on $\mathrm{BS}(\rho)$. The purpose of this note is to report on recent progress in this direction. Note that asking for a norm amounts to asking for a (separated) *lattice*: Given $\|\cdot\|$, look at the unit ball Λ. Conversely, given Λ, look at its "gauge" $\|x\| = q_E^{-v_\Lambda(x)}$, where $v_\Lambda(x)$ is the largest v such that $x \in \varpi_E^v \Lambda$. Thus we are looking for integral structures in $\mathrm{BS}(\rho)$.

3.1 Forerunner: The crystalline case

For irreducible *crystalline* representations $\rho : \Gamma_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$, the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ has a very nice description: Berger and Breuil showed that $\mathrm{BS}(\rho)$ admits a *unique* invariant norm, up to equivalence, and $\Pi(\rho)$ is the completion. See [BB], or Theorem 2.3.2 on p. 8 of [Ber] (which is about the existence of a finite type lattice; all such are obviously commensurable).

In a precursor to [BrSc], Schneider and Teitelbaum tried to extend this picture to higher-dimensional crystalline representations $\rho : \Gamma_F \to \mathrm{GL}_n(E)$. Here $\pi_{sm}(\rho)$ is an unramified principal series representation, corresponding to a character of the spherical Hecke algebra, $\zeta : \mathscr{H}(G,K) \to E$, where we have introduced $G = \mathrm{GL}_n(F)$ and $K = \mathrm{GL}_n(\mathscr{O}_F)$. Alternatively, one may then think of $\pi_{sm}(\rho)$ as a "universal module" (as in works of Borel, Matsumoto, Serre, Lazarus, etc.)

$$\pi_{sm}(\rho) \simeq \mathscr{C}_c(G/K) \otimes_{\mathscr{H}(G,K),\zeta} E,$$

where $\mathscr{C}_c(G/K)$ denotes the space of compactly supported E-valued continuous functions G/K (i.e, *finitely* supported functions since G/K is discrete). This can be generalized to take into account the algebraic representation $\pi_{alg}(\rho)$, which is traditionally denoted ξ. It defines a variant $\mathscr{H}_\xi(G,K)$ of the spherical Hecke algebra, consisting of $\mathrm{End}(\xi)$-valued K-biequivariant functions on G. When $\xi = 1$ one recovers $\mathscr{H}(G,K)$. Since ξ is an irreducible representation of G (as opposed to just K), there is in fact a natural isomorphism $\mathscr{H}(G,K) \xrightarrow{\sim} \mathscr{H}_\xi(G,K)$ of E-algebras, which can be used to transfer ζ to an eigensystem $\zeta : \mathscr{H}_\xi(G,K) \to E$. See p. 639 in [ScTe]. Unwinding the definitions, one finds that

$$\mathrm{BS}(\rho) \simeq H_{\xi,\zeta} := \mathrm{ind}_K^G(\xi) \otimes_{\mathscr{H}_\xi(G,K),\zeta} E,$$

where $\mathrm{ind}_K^G = c - \mathrm{Ind}_K^G$ denotes the compact induction of the representation $\xi|_K$ (i.e., ξ-valued functions f on G such that $f(kg) = \xi(k)f(g)$, *and* such that f is compactly supported modulo K). This interpretation is useful, by virtue of $H_{\xi,\zeta}$ carrying a canonical G-invariant *seminorm*: Since K is compact, there is certainly a K-invariant norm $\|\cdot\|_\xi$ on ξ – and they are all equivalent since ξ is finite-dimensional. Thus $\mathrm{ind}_K^G(\xi)$ is endowed with the sup-norm, and finally $H_{\xi,\zeta}$ acquires the quotient *semi*-norm. Since we mod out by a subspace which may not be closed, a priori it could happen that $\|x\| = 0$ for some $x \neq 0$. A key construction in [ScTe], p. 671, see also [Sch], is the following.

Definition 3.4 *$B_{\xi,\zeta}$ is the Hausdorff completion of $H_{\xi,\zeta}$. That is, $(H_{\xi,\zeta}/\overline{\{0\}})^\wedge$.*

Thus $B_{\xi,\zeta}$ is a unitary Banach E-representation of G, which yields $\Pi(\rho)$ when $n = 2$ and $F = \mathbb{Q}_p$. Unfortunately, in most other cases, it is not even known to be non-trivial! It would be bizarre, but we cannot rule out that 0 is dense in $H_{\xi,\zeta}$. This led to the (weaker) predecessor of Conjecture 3.2:

Conjecture 3.5 *(Schneider, Teitelbaum) $B_{\xi,\zeta} \neq 0$.*

For comparison, this is Conjecture 5.2 in [Sch]. Note that if $\mathrm{BS}(\rho) \simeq H_{\xi,\zeta}$ admits a G-invariant *norm* $\|\cdot\|'$, then in fact the canonical semi-norm on $H_{\xi,\zeta}$ must be a norm (and consequently $H_{\xi,\zeta} \hookrightarrow B_{\xi,\zeta}$ is dense): By Frobenius reciprocity, or really its "opposite" – which holds for *compact* induction, we have

$$\mathrm{Hom}_G(\mathrm{ind}_K^G(\xi), H_{\xi,\zeta}) \simeq \mathrm{Hom}_K(\xi, H_{\xi,\zeta}|_K),$$

and all maps here are automatically continuous since $\dim(\xi) < \infty$, regardless of what topology we put on $H_{\xi,\zeta}$. In particular the canonical projection $\pi : \mathrm{ind}_K^G(\xi) \to H_{\xi,\zeta}$ is automatically continuous for the $\|\cdot\|'$-topology. Thus $\ker(\pi)$ is closed.

Although in general $B_{\xi,\zeta}$ is not even known to be nonzero, it is expected to be a huge inadmissible representation, with lots of (almost) quotients, for $n > 2$ or $F \neq \mathbb{Q}_p$. The expectation is that there should be a G-map, with dense image,

$$B_{\xi,\zeta} \stackrel{?}{\longrightarrow} \Pi(\rho),$$

for any irreducible crystalline ρ with $\pi_{alg} \leftrightarrow \xi$ and $\pi_{sm} \leftrightarrow \zeta$. There are scores of such representations in higher rank, corresponding to choices of admissible filtrations compatible with (ξ, ζ). The motto here is that $B_{\xi,\zeta}$ parametrizes the crystalline part of p-adic local Langlands.

There are other very important results in [ScTe], but it would take us too far afield to describe them in detail here. Let it suffice to say they develop a (crystalline) theory for general split reductive groups $G_{/F}$, and prove an analogue of 3.3 in that setup. See Theorem 5.5 in [Sch] for a nice summary.

3.2 Miscellaneous results

There are various partial results towards 3.2 scattered in the literature, all proved in a purely local setting. We give a quick (possibly incomplete) overview.

- For $\mathrm{GL}_2(F)$, *de Ieso* used compact induction in [dI] to produce a separated lattice in $\mathrm{BS}(\rho)$, under some technical p-smallness condition on the weight.
- For $\mathrm{GL}_2(F)$, *Assaf, Kazhdan, and de Shalit* used p-adic Fourier theory for the Kirillov model to get integral structures in two cases: (1) $\pi_{sm}(\rho)$ is unramifed principal series, and $\pi_{alg}(\rho)$ is a twist of Sym^n with $n < q_F$, and (2) $\pi_{sm}(\rho)$ is tamely ramified and $n = 0$. See [AKdS], and its prequel [KdS].
- For $\mathrm{GL}_2(F)$, *Vigneras* studied local systems on the tree in [Vig], and was able to show 3.2 in the Steinberg case (Proposition 0.9 in loc. cit.) and made progress for tamely ramified principal series (Proposition 0.10).
- For general split reductive groups, *Grosse-Klönne* looked at the universal module for the spherical Hecke algebra, and was able to show cases of 3.2 for unramifed principal series, again under some p-smallness condition on the Coxeter number (when $F = \mathbb{Q}_p$) plus other technical assumptions. See Theorem 9.1 in [GK].

4 The indecomposable case

When $\pi_{sm}(\rho)$ is supercuspidal (equivalently, $\mathrm{WD}(\rho)$ is absolutely irreducible), there are many ways to construct a norm on $\mathrm{BS}(\rho)$. One can write $\pi_{sm}(\rho)$ as a compactly induced representation, or take the sup-norm of matrix coefficients. (This case was already observed as Theorem 5.2 in [BrSc].)

Note that the central character of $\mathrm{BS}(\rho)$ always takes values in $\mathscr{O}_E^\times$. In fact this is equivalent to the Emerton condition 1 when $\mathrm{WD}(\rho)$ is indecomposable (equivalently, $\pi_{sm}(\rho)$ is generalized Steinberg) – it turns out the Jacquet modules of $\pi_{sm}(\rho)$ are simple in that case, if nonzero, and as a consequence one only has to check 1 for $P = G$, which is p-adic unitarity of the central character.

This leads to the following *special case* of 3.2, which was mentioned explicitly as Conjecture 5.5 in [BrSc].

Conjecture 4.1 *Let ρ be a potentially semistable representation, with distinct Hodge-Tate weights, such that $\mathrm{WD}(\rho)$ is indecomposable. Then $\mathrm{BS}(\rho)$ admits a $\mathrm{GL}_n(F)$-invariant norm.*

The author recently proved this conjecture. In fact, he proved the following stronger result, which generalizes 4.1 to arbitrary reductive groups [SoBS].

Theorem 4.2 *(S.) Let $G_{/\mathbb{Q}_p}$ be a connected reductive group. Let π_{alg} be an irreducible algebraic representation of $G(\mathbb{Q}_p)$, and let π_{sm} be an essentially discrete series representation of $G(\mathbb{Q}_p)$, both defined over E. Then $\pi_{alg} \otimes_E \pi_{sm}$ admits a $G(\mathbb{Q}_p)$-invariant norm if (and only if) its central character is unitary.*

One deduces 4.1 by taking $G = \mathrm{Res}_{F/\mathbb{Q}_p} \mathrm{GL}_n$. Indeed the generalized Steinberg representations of $\mathrm{GL}_n(F)$ are precisely the essentially discrete series representations – by which we mean some *twist* has L^2-matrix coefficients. Here lies a subtlety worth pointing out: We are working over a p-adic field E, not $\mathbb{C}$, so being L^2 only makes sense after choosing an embedding $E \hookrightarrow \mathbb{C}$. To get a good notion over E, we need to know the set of essentially discrete series is stable under $\mathrm{Aut}(\mathbb{C})$. For GL_n, this follows from the Bernstein-Zelevinsky classification (via segments of supercuspidals etc.) – see [Kud] for a nice overview. For a general G, it was shown by Clozel.

4.1 Sketch of the proof of Theorem 4.2

After fiddling around with the center, and *using* that the central character takes values in $\mathscr{O}_E^\times$, one may assume G is semisimple. In turn, decomposing G^{sc} into a product of simple groups, and fiddling around with the tensor product norm (see paragraph 17, Chapter IV in [ScNFA]), one may assume G is in fact *simple* and *simply connected*. We are to show $\pi_{alg} \otimes_E \pi_{sm}$ *always* admits a norm – there is no center, hence no constraint.

Step 1. We choose a *global* model $G_{/\mathbb{Q}}$ such that $G(\mathbb{R})$ is compact. This uses a result of Borel and Harder in Galois cohomology [BoHa], which shows that forms[4] (not necessarily inner forms) can be prescribed locally.

Step 2. We apply the trace formula for $G_{/\mathbb{Q}}$ – which is as simple as the trace formula gets: There is no contribution from Eisenstein series since G is anisotropic. Since $G(\mathbb{R})$ is compact, π_{alg} is discrete series. So is π_{sm}, by hypothesis. Therefore they admit *pseudo-coefficients* [ClDe] (functions on the group which behave like matrix-coefficients). By a now more-or-less standard argument, which goes back to Clozel [Clo] (which in turn builds on [DeGW]) – in much greater generality, one can "isolate" the contribution from π_{alg} and π_{sm} in the trace formula. The upshot being there exists *lots* of automorphic representations Π of $G(\mathbb{A})$ with $\Pi_\infty = \pi_{alg}$ and $\Pi_p = \pi_{sm}$ (under some fixed and suppressed embedding $E \hookrightarrow \mathbb{C}$). Here we use that G is simple to run the trace formula techniques.

Step 3. Let $\mathscr{A}_G$ be the space of automorphic forms on $G(\mathbb{A})$. Following [Gro], for each compact open subgroup $K \subset G(\mathbb{A}_f)$, consider the space of algebraic modular forms on G, of weight π_{alg} and level K. That is, $\mathrm{Hom}_{G(\mathbb{R})}(\pi_{alg}, \mathscr{A}_G^K)$ – which contains Π_f^K as a direct summand. Indeed,

$$\mathrm{Hom}_{G(\mathbb{R})}(\pi_{alg}, \mathscr{A}_G^K) \simeq \bigoplus\nolimits_{\pi:\pi_\infty \simeq \pi_{alg}} m_G(\pi) \cdot \pi_f^K.$$

Via $E \hookrightarrow \mathbb{C}$ this space gets identified with p-adic modular forms, $\mathrm{Hom}_K(\pi_{alg}, \mathscr{C}_G)$, where $\mathscr{C}_G$ is the space of all continuous functions $G(\mathbb{Q})\backslash G(\mathbb{A}_f) \to E$, and K acts via the projection to $G(\mathbb{Q}_p)$. Thus,

$$\pi_{alg} \otimes \Pi_f^K \subset \pi_{alg} \otimes \mathrm{Hom}_K(\pi_{alg}, \mathscr{C}_G) \hookrightarrow \mathscr{C}_G^{K^p}.$$

When varying the level K_p, these maps are compatible, and give rise to a map

$$(\pi_{alg} \otimes \Pi_p) \otimes \Pi_f^{K^p} \hookrightarrow \mathscr{C}_G^{K^p}.$$

Passing to the limit over shrinking K^p yields $\pi_{alg} \otimes \Pi_f = (\pi_{alg} \otimes \pi_{sm}) \otimes \Pi_f^p \hookrightarrow \mathscr{C}_G$. The quotient $G(\mathbb{Q})\backslash G(\mathbb{A}_f)$ is compact, so $\mathscr{C}_G$ endowed with the sup-norm is a Banach space on which $G(\mathbb{A}_f)$ acts unitarily via right-translations. Restricting the sup-norm yields a $G(\mathbb{A}_f)$-invariant norm on $\pi_{alg} \otimes \Pi_f$ – and by further restriction a $G(\mathbb{Q}_p)$-invariant norm on $\pi_{alg} \otimes \pi_{sm}$, which finishes the sketch of the proof.

Remark 2. To put things in perspective, $\mathscr{C}_G$ can be thought of as $\hat{H}_E^0$, a very degenerate case of Emerton's completed cohomology – developed in great generality in [Em06]. Let $H^0(K)_{\mathscr{O}_E}$ be the set of $\mathscr{O}_E$-valued functions on the finite set $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$. When $K \supset K'$, there is a natural pull-back map $H^0(K)_{\mathscr{O}_E} \to$

[4] Not to be confused with *automorphic* forms. For instance, the inner forms of GL(2) are the algebraic groups $D^\times$, for quaternion algebras D, whereas the outer forms of GL(2) are the unitary groups in two variables.

$H^0(K')_{\mathscr{O}_E}$. Fix a tame level K^p, and take full levels of the form $K = K_pK^p$, with $K_p \to 1$. The direct limit $\varinjlim_{K_p} H^0(K_pK^p)_{\mathscr{O}_E}$ is no longer ϖ_E-adically complete, so we consider its completion:

$$\hat{H}^0_{\mathscr{O}_E}(K^p) = (\varinjlim_{K_p} H^0(K_pK^p)_{\mathscr{O}_E})^{\wedge}, \quad \hat{H}^0_{\mathscr{O}_E} = \varinjlim_{K^p} \hat{H}^0_{\mathscr{O}_E}(K^p).$$

Thus $\hat{H}^0_{\mathscr{O}_E}$ is the unit ball in a Banach E-space $\hat{H}^0_E$, which can be identified with E-valued continuous functions on $G(\mathbb{Q})\backslash G(\mathbb{A}_f)$. Alternatively, $\hat{H}^0_{\mathscr{O}_E}(K^p)$ can be thought of as

$$\tilde{H}^0_{\mathscr{O}_E}(K^p) := \varinjlim_{K_p} \varprojlim_{s>0} H^0(K_pK^p)_{\mathscr{O}_E/\varpi_E^s\mathscr{O}_E}.$$

Indeed, the natural map $\hat{H}^0_{\mathscr{O}_E}(K^p) \to \tilde{H}^0_{\mathscr{O}_E}(K^p)$ is an isomorphism (see [Em06]).

5 The automorphic case: Local-global compatibility at p

The argument in Step 3 of 4.1 shows the following: If $G_{/\mathbb{Q}}$ is a *reductive* group, which is *compact* at infinity, and Π is an automorphic representation of $G(\mathbb{A})$ with $\Pi_\infty = \pi_{alg}$ and $\Pi_p = \pi_{sm}$, then $\pi_{alg} \otimes_E \pi_{sm}$ admits a $G(\mathbb{Q}_p)$-invariant norm.

We will apply this observation to definite unitary groups: Thus[5] $\tilde{F}/\tilde{F}^+$ is a CM extension, and B is a central simple $\tilde{F}$-algebra of dimension n^2 equipped with an anti-involution $\star$ of the 2nd kind (which means $\star|_{\tilde{F}} = c$). Introduce $U = U(B,\star)_{/\tilde{F}^+}$ and take its restriction of scalars, $G = \mathrm{Res}_{\tilde{F}^+/\mathbb{Q}}(U)$. We assume throughout that $G(\mathbb{R}) \simeq U(n)^{\mathrm{Hom}(\tilde{F}^+,\mathbb{R})}$ is compact, and that $G(\mathbb{Q}_p)$ can be identified with $\prod_{v|p} \mathrm{GL}_n(\tilde{F}_{\tilde{v}})$ upon a choice of divisor $\tilde{v}|v$, for each place $v|p$ of $\tilde{F}^+$.

One of the goals of the "Book Project" is to attach Galois representations to automorphic representations Π on $G(\mathbb{A})$ – roughly by a base change to $\mathrm{GL}_n(\mathbb{A}_{\tilde{F}})$, descent to an indefinite unitary group U' with a Shimura variety $Sh_{U'}$, and a detailed study of the cohomology of $Sh_{U'}$. The following result is now known, due to the collaborative efforts of many people.

Theorem 5.1 *("Book Project") Let Π be an automorphic representation of $G(\mathbb{A})$, with Π_∞ of weight $a = (a_\tau)_{\tau \in \mathrm{Hom}(\tilde{F}^+,\mathbb{R})}$. Fix an isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \bar{\mathbb{Q}}_p$. Then there exists a unique continuous semisimple Galois representation,*

$$r_{\Pi,\iota} : \Gamma_{\tilde{F}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\tilde{F}) \longrightarrow \mathrm{GL}_n(\bar{\mathbb{Q}}_p),$$

with the following properties:

(0)$r^\vee_{\Pi,\iota} \simeq r^c_{\Pi,\iota} \otimes \varepsilon^{n-1}_{cyc}$.

---

[5] We use the notation $\tilde{F}$ to distinguish it from our p-adic base field $F/\mathbb{Q}_p$.

(1)For **every** *finite place v of $\tilde{F}^+$, which splits in $\tilde{F}$, and every place $w|v$,*

$$\mathrm{WD}(r_{\Pi,\iota}|_{\Gamma_{\tilde{F}_w}})^{F-ss} \simeq rec_n(BC_{w|v}(\Pi_v) \otimes |\det|^{\frac{1-n}{2}}).$$

(Here $BC_{w|v}$ denotes "base change" from $U(\tilde{F}_v^+)$ to $\mathrm{GL}_n(\tilde{F}_w)$.)

(2)$r_{\Pi,\iota}|_{\Gamma_{\tilde{F}_w}}$ is potentially semistable for every $w|p$, with Hodge-Tate weights

$$\mathrm{HT}_\tau(r_{\Pi,\iota}|_{\Gamma_{\tilde{F}_w}}) = \{a_{\tau,j} + (n-j) : j = 1, \dots, n\},$$

for $\tau : \tilde{F}_w \hookrightarrow \bar{\mathbb{Q}}_p$. Here our convention is that $\mathrm{HT}_\tau(\varepsilon_{cyc}) = \{-1\}$.

Remark 3. In this Theorem, rec_n denotes the (classical) local Langlands correspondence for $\mathrm{GL}_n(\tilde{F}_w)$, normalized as in [HT]. It's a bijection between irreducible admissible representations of $\mathrm{GL}_n(\tilde{F}_w)$ (over $\mathbb{C} \simeq \bar{\mathbb{Q}}_p$) and Frobenius-semisimple n-dimensional Weil-Deligne representations for $\tilde{F}_w$. The latter are pairs (r,N) consisting of a (nilpotent) monodromy operator N and a *semisimple* representation r of the Weil group $W_{\tilde{F}_w}$, with open kernel. They naturally arise from potentially semistable p-adic representations of $\Gamma_{\tilde{F}_w}$. When $w \nmid p$, every continuous representation is potentially semistable, and (r,N) is given by a simple formula (see [Ta] for details). For $w|p$ the associated (r,N) is more complicated to define, and involves Fontaine's period ring B_{dR} (see [Fon] for details). The Weil-representation r may not be semisimple; the *Frobenius*-semisimplification is defined to be $(r,N)^{F-ss} = (r^{ss}, N)$.

What is important to us here is the "local-global compatibility" in part (1) at the places *v dividing p*. This was proved in [B+1] and [B+2] under a certain regularity assumption, which was later removed in [Car].

In [SoEV] the author pushed the ideas from 4.1 further, combining them with Theorem 5.1, part (1) above p. A bit of book-keeping yielded the result below.

Corollary 5.2 *(S.) Conjecture 3.2 holds for $\rho = r_{\Pi,\iota}|_{\Gamma_{\tilde{F}_w}}$, for any place $w|p$ of $\tilde{F}$.*

The implied norm on $\mathrm{BS}(\rho)$ comes from a $G(\mathbb{Q}_p)$-equivariant embedding

$$\bigotimes_{v|p} \mathrm{BS}(r_{\Pi,\iota}|_{\Gamma_{\tilde{F}_{\tilde{v}}}}) \hookrightarrow \mathscr{C}_G = \{G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \xrightarrow{cts.} \bar{\mathbb{Q}}_p\}, \tag{1}$$

obtained as in 4.1. Here the tensor product is viewed as a representation of $G(\mathbb{Q}_p)$ via the selection of places $\{\tilde{v}\}_{v|p}$. The speculation aired in [SoEV] is that the closure of the image of 1 ought to be "closely related" to $\hat{\otimes}_{v|p} \Pi(r_{\Pi,\iota}|_{\Gamma_{\tilde{F}_{\tilde{v}}}})$, where $\Pi(\cdot)$ should be the p-adic local Langlands correspondence for GL_n, if it exists. (At least if all the restrictions $r_{\Pi,\iota}|_{\Gamma_{\tilde{F}_{\tilde{v}}}}$ are irreducible.)

One can make this more precise: Given an automorphic representation Π as in Theorem 5.1, there is the associated Galois representation $r_{\Pi,\iota}$ of $\Gamma_{\tilde{F}}$, which we can restrict to the various decomposition groups at p – resulting in the collection $\{r_{\Pi,\iota}|_{\Gamma_{\tilde{F}_w}}\}_{w|p}$. On the other hand, the argument in section 4 ("Step 3") yields a *canonical* $G(\mathbb{A}_f)$-invariant norm on $(\pi_{alg} \otimes \pi_{sm}) \otimes \Pi_f^p$ – using the now known result that $m_G(\Pi) = 1$. For each choice of vector $0 \neq x \in \Pi_f^p$, this gives rise to a $G(\mathbb{Q}_p)$-invariant norm $\|\cdot\|_x$ on $\pi_{alg} \otimes \pi_{sm}$, by restriction. Since Π_f^p is irreducible, it is easy to see that all these $\|\cdot\|_x$ define the same topology. Thus we may also attach to Π (and ι) a unitary Banach space representation of $G(\mathbb{Q}_p)$, namely the completion $B_{\Pi,\iota} = (\pi_{alg} \otimes \pi_{sm})^\wedge$. It is natural to speculate that $B_{\Pi,\iota}$ only depends on $\{r_{\Pi,\iota}|_{\Gamma_{\tilde{F}_w}}\}_{w|p}$. That is, for two automorphic Π and Π' as above, it is natural to ask whether or not

$$r_{\Pi,\iota}|_{\Gamma_{\tilde{F}_w}} \simeq r_{\Pi',\iota}|_{\Gamma_{\tilde{F}_w}}, \forall w|p \overset{?}{\Longrightarrow} B_{\Pi,\iota} \simeq B_{\Pi',\iota},$$

and perhaps even conversely – under favorable circumstances. This seems to be a very hard question, which lies at the very heart of extending the p-adic local Langlands correspondence to GL_n.

Another speculation in [SoEV] was how these things vary across the eigenvariety $X = X_G$, of some fixed tame level K^p. On one hand, each point $x \in X(E)$ defines a pseudo-representation $t_x : \Gamma_{\tilde{F}} \to E$. On the other, one has the Hecke eigenspace $\mathscr{C}_{G,x}^{K^p}$, which is a Banach-Hecke module over E with a unitary $G(\mathbb{Q}_p)$-action. This way X parametrizes a bijection $t_x \leftrightarrow \mathscr{C}_{G,x}^{K^p}$, which we like to think of as some sort of p-adic "global" Langlands correspondence.

6 Patching and p-adic local Langlands

Recently there has been spectacular progress on Conjecture 3.2 in the principal series case, which is the deepest, by joint work of Caraiani, Emerton, Gee, Geraghty, Paškūnas, and Shin. In the preprint [P], using global methods, they construct a candidate $\Pi(\rho)$ for a p-adic local Langlands correspondence for $\mathrm{GL}_n(F)$, and are able to say enough about it to prove new cases of 3.2. The following is their Theorem 5.3 (a less precise version of which is Theorem B in their introduction).

Theorem 6.1 *(Caraiani, Emerton, Gee, Geraghty, Paškūnas, and Shin) Suppose* $p \nmid 2n$. *Let* $\rho : \Gamma_F \to \mathrm{GL}_n(E)$ *be a generic potentially* **crystalline** *representation, of regular weight. If* ρ *lies on an automorphic component of the corresponding potentially crystalline deformation ring* $R_{\tilde{\mathfrak{p}}}^{\square}(\sigma)[1/p]$*, then* $\mathrm{BS}(\rho)$ *admits a nonzero unitary admissible Banach completion.*

The conclusion is a little stronger than just the existence of a norm on $\mathrm{BS}(\rho)$, in that it asserts admissibility. They refer to it as "folklore" that every regular de Rham representation ρ should lie on an automorphic component. Thus they reduce Conjecture 3.2 to standard expectations in the automorphy lifting world. It should be emphasized that this condition is much weaker than saying ρ comes from an automorphic form (as in 5.2). It roughly says that it lies on the same *component* as some $r_{\Pi,\iota}|_{\Gamma_{\tilde{F}_{\tilde{\mathfrak{p}}}}}$ (identifying $\tilde{F}_{\tilde{\mathfrak{p}}} \simeq F$) – in turn more restrictive than saying the reduction $\bar{\rho}$ is automorphic. In some situations one can check this hypothesis, employing the available automorphy lifting theorems. See Corollary 5.4 on "potentially diagonalizable" ρ in [**?**], and their Corollary 5.5 – which we quote:

Corollary 6.2 *(—) Suppose $p > 2$. Let $\rho : \Gamma_F \to \mathrm{GL}_n(E)$ be a generic potentially semistable representation, of regular weight. Further, assume that* **either**

(1)$n = 2$, and ρ is potentially Barsotti-Tate, **or**
(2)$F/\mathbb{Q}_p$ is unramified, ρ is crystalline, $p \neq n$, and the Hodge-Tate weights of ρ are in the "extended Fontaine-Laffaille range" (meaning the difference of any two weights in $\mathrm{HT}_\tau(\rho)$ is at most $p-1$).

Then $\mathrm{BS}(\rho)$ admits a nonzero unitary admissible Banach completion.

As hinted at already, they actually construct a candidate $\Pi(\rho)$ for p-adic local Langlands and verify (under the given hypotheses) that $\mathrm{BS}(\rho) \hookrightarrow \Pi(\rho)$. Moreover, they express optimism in [P], Remark 5.8, that $\mathrm{BS}(\rho)$ coincides with the space of locally algebraic vectors $\Pi(\rho)^{alg}$ – evidence that $\Pi(\rho)$ is "purely local" although it is defined via choices of auxiliary global data. Convincing evidence for *Shimura curves* is provided by [EGS].

The definition of $\Pi(\rho)$ in [P] is based on their "hypothetical formulation" of p-adic local Langlands in Section 6.1: Given a representation $\bar{\rho} : \Gamma_F \to \mathrm{GL}_n(k_E)$ with $\mathrm{End}_{\Gamma_F}(\bar{\rho}) = k_E$ (for simplicity) there should be a finitely generated $R_{\bar{\rho}}[[\mathrm{GL}_n(\mathscr{O})]]$-module L_∞, which is $\mathscr{O}_E$-torsion free, and such that the $\mathrm{GL}_n(\mathscr{O})$-action can be *promoted* to an $R_{\bar{\rho}}$-linear action of the full $\mathrm{GL}_n(F)$. Then p-adic local Langlands $\rho \rightsquigarrow B(\rho)$ should arise from specializing L_∞ as follows. Say $\rho : \Gamma_F \to \mathrm{GL}_n(\mathscr{O}_E)$ is a lift of $\bar{\rho}$, corresponding to a point $x \in (\mathrm{Spec} R_{\bar{\rho}})(\mathscr{O}_E)$. Then,

$$B(\rho) = (L_\infty \otimes_{R_{\bar{\rho}},x} \mathscr{O}_E)^d[1/p],$$

where d denotes taking the (Schikhof) dual[6] $\mathrm{Hom}^{cts.}_{\mathscr{O}_E}(-,\mathscr{O}_E)$. When $\mathrm{End}_{\Gamma_F}(\bar{\rho}) \neq k_E$ one would replace $R_{\bar{\rho}}$ by the universal framed deformation ring $R^{\square}_{\bar{\rho}}$, and L_∞ should be endowed with additional structure (see the footnote on p. 45 of [P]).

The construction in [P] starts off from a $\bar{\rho} : \Gamma_F \to \mathrm{GL}_n(k_E)$, which admits a "potentially diagonalizable" lift $\rho_{p.d.}$ to $\mathscr{O}_E$. This assumptions allows them to pass to a global setup (cf. Section 5 above), and suitably extend $\bar{\rho}$ to a representation $\Gamma_{\tilde{F}} \to \mathrm{GL}_n(k_E)$. Carrying out a delicate variant of the Taylor-Wiles-Kisin patching method for modular forms on G, but letting the weight and the p-part of the level vary, they construct an $R_\infty[[\mathrm{GL}_n(\mathscr{O})]]$-module M_∞ (defined on p. 15 in loc. cit.), where R_∞ is a multivariate power series ring over some local deformation ring R^{loc} (rather, a tensor product of such). Proposition 2.8 in loc. cit. establishes the key fact that the $\mathrm{GL}_n(\mathscr{O})$-action on M_∞ extends naturally to a $\mathrm{GL}_n(F)$-action.

Now, if $\rho : \Gamma_F \to \mathrm{GL}_n(\mathscr{O}_E)$ is a lift of $\bar{\rho}$, the definition of $\Pi(\rho)$ proceeds in the following steps: Under the identification $F \simeq \tilde{F}_{\tilde{\mathfrak{p}}}$ our lift ρ corresponds to a

[6] An anti-equivalence between compact $\mathscr{O}_E$-modules and ϖ_E-adically *complete* $\mathscr{O}_E$-modules.

point $x : R^{\square}_{\tilde{\mathfrak{p}}} \to \mathscr{O}_E$, which we extend to a point $x' : R^{loc} \to \mathscr{O}_E$, using the given lift $\rho_{p.d.}$ at the places in S_p away from $\mathfrak{p}$. We extend x' further, and arbitrarily, to a homomorphism $y : R_\infty \to \mathscr{O}_E$. Recall that $R_\infty \simeq R^{loc}[[x_1,\dots,x_N]]$. Then,

$$\Pi(\rho) := (M_\infty \otimes_{R_\infty,y} \mathscr{O}_E)^d[1/p].$$

(See Section 2.10, p. 19 in [P].) It should be stressed that $\Pi(\rho)$ is *not* known to be independent of the "globalization", nor the choice of extension y. Furthermore, in general it is not known whether $\Pi(\rho) \neq 0$.

In Section 6.2 of [P] they speculate that $M_\infty \overset{?}{=} R_\infty \otimes_{R_{\tilde{\mathfrak{p}}}} L_\infty$.

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