Level-raising for $\mathbf{GSp}(4)$

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Abstract

This paper is a summary of the authors thesis [Sor], based on a talk given at the 9th autumn workshop on number theory held in November 2006 in Hakuba, Japan. We produce congruences between certain Saito-Kurokawa forms and certain stable automorphic forms on GSp(4), by raising the level. We then discuss an application to the Bloch-Kato conjecture for classical modular forms.

1 Ribet's Theorem for $\mathbf{GL}(2)$

We first explain the classical level-raising result of Ribet [Ri1]. For simplicity, take f be a normalized eigenform for the group $\Gamma_0(N)$ of weight two. Thus

$$f(z)=\sum_{n=1}^{\infty}a_n(f)e^{2\pi iz},\quad \text{with}\quad a_1(f)=1,$$

and the Fourier coeffcients $a_n(f)$ are the eigenvalues of the Hecke operators T_n. Suppose λ is a finite place of $\bar{\mathbb{Q}}$, dividing ℓ, such that f is not congruent to an Eisenstein series modulo λ. Moreover, suppose $q\nmid N\ell$ is a prime satsifying:

$$a_q(f)^2\equiv(1+q)^2 \bmod \lambda.$$

Then Ribet proved the existence of a weight two q-new eigenform $\tilde{f}$ for $\Gamma_0(Nq)$, which is congruent to f modulo λ. He deduced this from a lemma of Ihara [Iha] saying that the natural map, obtained from the two degeneracy maps,

$$H^1(X_0(N),\mathbb{F}_\ell)^{\oplus 2}\to H^1(X_0(Nq),\mathbb{F}_\ell)$$

is injective, where $X_0(N)$ is the modular curve over $\mathbb{C}$ of level N.

To give an adelic interpretation of Ribet's theorem, let Π be the cuspidal automorphic representation of $\mathrm{PGL}(2,\mathbb{A})$ associated with f. The local component Π_p is then unramified for all primes $p\nmid N$, and we have the relation:

$$a_p(f)=p^{1/2}(\alpha_p+\alpha_p^{-1}),\quad \text{where}\quad \mathfrak{t}_{\Pi_p}=\begin{pmatrix}\alpha_p & \\ & \alpha_p^{-1}\end{pmatrix}\in\hat{T}/W$$

is the Satake parameter. In particular, by factoring the Hecke polynomial at q, we see that the level-raising condition above is equivalent to the following:

$$\mathfrak{t}_{\Pi_q\otimes|\det|^{-1/2}}\equiv\pm\begin{pmatrix}1 & \\ & q\end{pmatrix} \bmod \lambda.$$

Both sides of this congruence are viewed modulo W. Then the conclusion is the existence of a cuspidal automorphic representation $\tilde{\Pi} \equiv \Pi \bmod \lambda$, with the same infinity component as Π, and with the local component $\tilde{\Pi}_q$ being an unramified quadratic twist of the Steinberg representation. That is, for some $\xi \in \{\mathbf{1}, \xi_0\}$,

$$\tilde{\Pi}_q = \xi \mathrm{St}_{\mathrm{GL}(2)}.$$

Here ξ_0 denotes the non-trivial unramified quadratic character of $\mathbb{Q}_q^*$.

2 Level-raising for Saito-Kurokawa Forms

In this paper we present a level-raising result for $\mathrm{GSp}(4)_{\mathbb{Q}}$, which applies to certain Saito-Kurokawa forms. The complete proof is given in [Sor], and here we will only describe the main ideas. We consider the simplest case:

$$f \in S_4(\Gamma_0(N)) \text{ newform}, \quad \epsilon_f = -1.$$

Here the level N is assumed to be square-free. For example, there exists such a newform f when $N = 13$. The sign ϵ_f is given by the parity of the order of vanishing at $s = 2$ of the L-function. Indeed, we have the functional equation,

$$\Lambda(s, f) = (2\pi)^{-s}\Gamma(s)L(s, f) = \epsilon_f N^{2-s}\Lambda(4 - s).$$

With this notation, we have the following result from [Sor]:

Theorem A. *Let λ above ℓ be a finite place of $\bar{\mathbb{Q}}$ such that f is non-Eisenstein modulo λ. Suppose $q \nmid N\ell$ is a prime satisfying the following two conditions:*

- $q^n \not\equiv 1$ *(mod ℓ) for $n = 1, \ldots, 4$,*
- $\bar{\rho}_{f,\lambda}(Frob_q)$ *has a fixed vector.*

Then there exists a weight three holomorphic cuspidal automorphic representation Π of $PGSp(4, \mathbb{A})$, unramified outside Nq, with the following properties:

- *The Galois representation $\rho_{\Pi,\lambda}$ is irreducible,*
- $\bar{\rho}_{\Pi,\lambda} \simeq \bar{\rho}_{f,\lambda} \oplus \bar{\chi}_\ell \oplus \bar{\chi}_\ell^2$,
- $\Pi_q = Ind_Q(\xi St_{GL(2)})$ *(hence tempered, generic, and ramified),*
- Π_p *has vectors fixed by the paramodular group, for all $p|N$.*

Moreover, there exists a positive density of primes q satisfying the above.

We choose geometric conventions throughout, and so χ_ℓ denotes the inverse cyclotomic character. The Galois representation $\rho_{\Pi,\lambda}$ was constructed by Weissauer in [We1]. Its irreducibility in the above situation is essentially a result of

Ramakrishnan [Ram]. The component Π_q is irreducibly induced from the Klingen parabolic Q, with Levi subgroup GL(2). It is of type IIIa in the tables of Schmidt [Sch]. Recall that the p-adic group GSp(4) has two maximal compact subgroups up to conjugacy. One is the hyperspecial group of $\mathbb{Z}_p$-points, and the other is the paramodular group stabilizing a non-special vertex in the building.

It is our firm belief that this theorem has an analogue for forms of arbitrary level N, and all weights greater than or equal to 4. However, we have not worked out the details. Unfortunately, the weight two case is not within reach by our methods. So there is no immediate application to elliptic curves.

3 An Application

We continue with the setup from the previous section. In general, the Bloch-Kato conjecture [BK] predicts a relation between the order of vanishing of the L-function of a motive and the size of its Selmer group. We expect that

$$\mathrm{ord}_{s=2}L(s,f) \stackrel{?}{=} \dim_{\mathbb{Q}_\ell} H^1_f(\mathbb{Q}, \rho_{f,\lambda}(2))$$

in our special case. Now, by our assumption, the L-function vanishes to an odd order. Therefore, we expect that the Selmer group on the right is non-trivial:

$$H^1_f(\mathbb{Q}, \rho_{f,\lambda}(2)) \neq 0.$$

This was proved by Skinner and Urban in [SU], for arbitrary level and weight, under the assumption that f is ordinary at ℓ. This means that the Fourier coefficient $a_\ell(f)$ is an ℓ-adic unit. We do not assume f to be ordinary at ℓ.

Recall, if (ρ, V) is an ℓ-adic Galois representation, we define its Selmer group to be the subgroup of Galois cohomology cut out by certain local conditions:

$$H^1_f(\mathbb{Q}, V) = \ker\{H^1(\mathbb{Q}, V) \to \prod H^1(\mathbb{Q}_p, V)/H^1_f(\mathbb{Q}_p, V)\}.$$

For $p \neq \ell$ we require our cohomology classes to be unramified. That is, we let

$$H^1_f(\mathbb{Q}_p, V) = \ker\{H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p^{\mathrm{nr}}, V)\}.$$

When $p = \ell$ we consider Fontaine's ring B_{cris} and the subgroup defined by:

$$H^1_f(\mathbb{Q}_\ell, V) = \ker\{H^1(\mathbb{Q}_\ell, V) \to H^1(\mathbb{Q}_\ell, B_{\mathrm{cris}} \otimes V)\}.$$

Analogously, one can define Selmer groups of mod ℓ representations as in [Rub].

Also recall, when (ρ, V) is an ℓ-adic Galois representation as above and $p \neq \ell$, that by a theorem of Grothendieck there is a nilpotent operator N such that

$$\rho(\sigma) = \exp(t_\ell(\sigma)N)$$

holds for all σ in an open subgroup of inertia I_p. Here $t_\ell : I_p \to \mathbb{Z}_\ell$ is a surjective homomorphism, and N is called the monodromy operator of ρ at p. Our application of Theorem A relies on the following conjecture from [SU], which would follow from local-global compatibility of the Langlands correspondence:

Conjecture. *Let* Π *be as in Theorem A. If* Π_p *is spherical for the paramodular group, then the monodromy operator of* $\rho_{\Pi,\lambda}$ *at* p *has rank at most one.*

This conjecture is supported by the calculations in [Sch]. Assuming it holds, we prove that the Selmer group of the mod ℓ representation $\bar{\rho}_{f,\lambda}(2)$ is non-trivial:

Theorem B. *Conjecture* $\Longrightarrow H^1_f(\mathbb{Q}, \bar{\rho}_{f,\lambda}(2)) \neq 0$.

The proof is given in [Sor]. We detect a non-trivial cohomology class by finding a suitable torsion subquotient X of $\rho_{\Pi,\lambda}$ sitting in a non-split extension

$$0 \to \bar{\chi}_\ell \to X \to \bar{\rho}_{f,\lambda} \to 0,$$

having good reduction everywhere. This means that X is Fontaine-Laffaille at the prime ℓ, and that the sequence remains exact after taking I_p-invariants for p different from ℓ. For this we need to rule out certain other extensions. We need the above conjecture to prove these extensions have good reduction everywhere, and then we appeal to the work of Kato [Kat] to show they cannot exist.

The analogue of Theorem B for $\rho_{f,\lambda}(2)$, without the bar, would follow if the ℓ-part of its Tate-Shafarevich is trivial. This is expected to be true.

Once we extend Theorem A to forms of higher weight $2k$, we expect to have unconditional applications when the level is $N = 1$. Here the sign condition is equivalent to k being odd, and if so, Theorem B should hold unconditionally.

4 Proof of Theorem A

We let τ denote the cuspidal automorphic representation of $\mathrm{PGL}(2, \mathbb{A})$ associated with f. By the sign condition, there is some prime r such that τ_r is the actual Steinberg representation $\mathrm{St}_{\mathrm{GL}(2)}$, not its unramified quadratic twist. Then, we let $G_{/\mathbb{Q}}$ be the inner form of $\mathrm{GSp}(4)$ having the following properties:

- $G(\mathbb{R})$ is compact modulo its center,
- $G(\mathbb{Q}_r)$ has rank one modulo its center,
- G splits over $\mathbb{Q}_p$ for all primes $p \neq r$.

The existence of such an inner form follows from results on the cohomology of reductive groups [Ko1]. However, we can construct G explicitly as the similitude spin group of some definite quadratic form in five variables over $\mathbb{Q}$. Alternatively, one can think of G as the unitary similitude group of D^2 (equipped with the natural hermitian form represented by the identity), where D is the quaternion algebra over $\mathbb{Q}$ with ramification locus $\{\infty, r\}$. Both viewpoints are useful.

4.1 Lifting to an Inner Form

We explain how to lift the form f to an automorphic representation π of $G(\mathbb{A})$. By thinking of $\mathrm{PGL}(2)$ as being split $\mathrm{SO}(3)$ to utilize the θ-correspondence, each local component τ_p determines a Waldspurger packet

$$A_{\tau_p} = \{\sigma^{\pm}_{\tau_p}\}$$

of representations of p-adic $\widetilde{\mathrm{SL}}(2)$. When τ_p is a principal series, this packet is a singleton. On the other hand, when τ_p is a discrete series, the packet has two members. Basically the $+$ member is the θ-lift of τ_p, whereas the $-$ member is the θ-lift of its Jacquet-Langlands transfer. Then we consider the global packet,

$$A_\tau = \{\sigma = \otimes \sigma_{\tau_p}^{\epsilon_p} \text{ with } \epsilon_p = \pm \text{ and } \epsilon_p = + \text{ for almost all } p\},$$

of representations of adelic $\widetilde{\mathrm{SL}}(2)$. By a famous theorem of Waldspurger [Wal],

$$\sigma = \otimes \sigma_{\tau_p}^{\epsilon_p} \text{ is automorphic} \Leftrightarrow \epsilon(1/2, \tau) = \prod \epsilon_p.$$

By our assumption on the root number, A_τ contains the automorphic member

$$\sigma = \sigma_{\tau_\infty}^+ \otimes \sigma_{\tau_r}^- \otimes_{p \neq r} \sigma_{\tau_p}^+.$$

Thinking of G^{ad} as anisotropic SO(5), we then use θ-series to lift σ to the inner form. We are certainly not in the stable range, so we need to make sure that

$$\pi = \theta(\sigma) \neq 0.$$

However, this follows from work of Rallis [Ral]. Indeed, the L-values occurring in his inner product formula are nonzero, so to see that $\theta(\sigma)$ does not vanish we just have to check this locally. The local lifts can be computed explicitly. First, one finds that π_∞ is the trivial representation $\mathbf{1}$. For primes $p \neq r$ one has

$$\pi_p = L(\nu^{1/2}\tau_p \rtimes \nu^{-1/2})$$

in the notation of [ST]. This means we induce from the Siegel parabolic, and look at the unique irreducible quotient. Finally, π_r is in fact irreducibly induced:

$$\pi_r = \nu^{1/2}\mathbf{1}_{D_r^*} \rtimes \nu^{-1/2},$$

where we induce from the Siegel parabolic in $G_{/\mathbb{Q}_r}$. Here we use τ_r is Steinberg.

4.2 Raising the Level

This step is the heart of the proof. Since $G^{\mathrm{ad}}(\mathbb{R})$ is compact, and π is trivial at infinity, the Hecke module of K-invariants is in the space of automorphic forms

$$\mathcal{A}_K = \{f : G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K \to \mathbb{C}\}$$

for every compact open subgroup K. This space has a natural lattice, simply given by the $\mathbb{Z}$-valued functions, and therefore the combinatorics of modular forms mod ℓ is quite simple. Adapting ideas of Bellaiche [Bel], and Clozel [Clo], this allows us to prove a general level-raising result for the inner form G applicable to all its automorphic forms π. Indeed, suppose $q \nmid N\ell$ is a prime such that

$$\mathbf{t}_{\pi_q \otimes |c|^{-3/2}} \equiv \begin{pmatrix} 1 & & & \\ & q & & \\ & & q^2 & \\ & & & q^3 \end{pmatrix} \bmod \lambda$$

and $q^n \not\equiv 1 \bmod \ell$ for $n = 1, \ldots, 4$. Then there exists $\tilde{\pi} \equiv \pi \bmod \lambda$, an automorphic representation of $G(\mathbb{A})$, with local component $\tilde{\pi}_q$ determined by:

$$\tilde{\pi}_q \text{ is of type } \begin{cases} IIIa \\ IIa \end{cases} \quad \text{when } \pi_q \text{ is of type } \begin{cases} IIb \\ IIIb \end{cases}.$$

In the remaining case where π_q is generic, one can choose $\tilde{\pi}_q$ to be of type IIa or IIIa. Here we use the notation from [Sch]. The proof of this result uses work of Lazarus [Laz] on the constituents of universal modules mod ℓ. We note that the two types IIb and IIIb are the typical unramified local components of CAP representations. The representations of type IIa are of the form $\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$, induced from the Siegel parabolic, while those of type IIIa are $\chi \rtimes \sigma \mathrm{St}_{\mathrm{GL}(2)}$, induced from the Klingen parabolic. Both are ramified, generic, Klingen- and Siegel-spherical. Moreover, a representation of type IIIa is tempered if and only if it is unitary. In the generic case one can choose between the types IIa and IIIa depending on the application one has in mind. Representations of type IIa are expected to transfer to the inner form over $\mathbb{Q}_q$, while those of type IIIa cannot occur in endoscopic lifts. We will indicate the proof of this below.

We apply this result to our Saito-Kurokawa lifting π from the previous section. The unramified local components are all of type IIb. Since π is a lift of σ, it follows that q satisfies the level-raising condition precisely when Frob_q has a fixed vector on $\bar{\rho}_{f,\lambda}$. To satisfy the banality condition on q simultaneously, we appeal to a result of Ribet [Ri2] on the image of the Galois representation.

4.3 Transfer to GSp(4)

We now have an automorphic representation $\tilde{\pi} \equiv \pi$ of $G(\mathbb{A})$ with local component $\tilde{\pi}_q$ of type IIIa, and we want to transfer it back to GSp(4). In other words, we want to establish a special case of the Jacquet-Langlands correspondence in this situation. For this, we use the stable trace formula developed by Arthur [Art].

For simplicity, we let G' denote the group GSp(4) in this section. It is well known that it admits a unique non-trivial elliptic endoscopic group. Namely,

$$H = (\mathrm{GL}(2) \times \mathrm{GL}(2))/\mathrm{GL}(1).$$

Its representations correspond to pairs of representations of GL(2) with the same central character. In this case, Hales proved the fundamental lemma [Ha2] and the transfer conjecture [Ha1]. We consider the invariant distribution

$$I_{\mathrm{disc}}^{G'}(f') = \sum_{\Pi} a_{\mathrm{disc}}^{G'}(\Pi) \mathrm{tr}\Pi(f')$$

given by the terms occurring discretely in the trace formula. It is unstable. However, if we subtract certain endoscopic error terms, it becomes stable:

$$S_{\mathrm{disc}}^{G'}(f') = I_{\mathrm{disc}}^{G'}(f') - \frac{1}{4} I_{\mathrm{disc}}^{H}(f'^H)$$

is stable. Here, if $f' = \otimes f'_p$ is a pure tensor, we take $f'^H = \otimes f'^H_p$ to be a matching function. Now we turn our attention to G. It is anisotropic modulo its center, so the trace formula takes its simplest form. All terms occur discretely,

$$I^G_{\mathrm{disc}}(f) = \sum_\pi a^G_{\mathrm{disc}}(\pi)\mathrm{tr}\pi(f).$$

Here $a^G_{\mathrm{disc}}(\pi)$ is always the multiplicity of π. Again, this distribution is unstable, but it can be rewritten in terms of stable distribution on the endsocopic groups:

$$I^G_{\mathrm{disc}}(f) = S^{G'}_{\mathrm{disc}}(f^{G'}) + \frac{1}{4}I^H_{\mathrm{disc}}(f^H).$$

This was first proved by Kottwitz and Langlands, but it is also a very special case of the aforementioned work of Arthur. We mention that the transfer conjecture for G follows from the computations in [Ha1], combined with the general results of [LS]. This last formula allows us to compare the automorphic spectra. By linear independence of characters, we can restrict ourselves to automorphic representations with fixed components $\tilde{\pi}_p$ away from the ramification locus

$$S = \{\infty, r\}.$$

Now, since $\tilde{\pi}_q$ is of type IIIa, it turns out that all the endoscopic terms vanish. We will explain why below. Therefore, we get the following spectral identity:

$$\sum\nolimits_{\pi_S} a^G_{\mathrm{disc}}(\pi_S \otimes \tilde{\pi}^S)\mathrm{tr}\pi_S(f_S) = \sum\nolimits_{\Pi_S} a^{G'}_{\mathrm{disc}}(\Pi_S \otimes \tilde{\pi}^S)\mathrm{tr}\Pi_S(f'_S)$$

for any pair of matching functions. Again, using linear independence of characters, we get the existence of an automorphic $\tilde{\Pi}$ occurring on the right with

$$\tilde{\Pi}^S = \tilde{\pi}^S.$$

This $\tilde{\Pi}$ is our functorial transfer of $\tilde{\pi}$ to the group GSp(4). By Shelstad [-], we can even choose $\tilde{\Pi}_\infty$ to be a holomorphic (or generic) discrete series.

It remains to explain why type IIIa representations cannot occur in the endoscopic character identities of Weissauer [We2]. We think of H as being the neutral component $\mathrm{GSO}(2,2)$ over $\mathbb{Q}_q$. Then its non-trivial inner form is

$$\breve{H} = (D^* \times D^*)/\mathrm{GL}(1) \simeq \mathrm{GSO}(4),$$

where D here denotes the division quaternion algebra over $\mathbb{Q}_q$. Let us recall the following result of Roberts [Rob]. If ρ is a representation of H, or its inner form $\breve{H}$, there is a unique extension to the full orthogonal similitude group ρ^+ occurring in the θ-correspondence with GSp(4). When ρ is a discrete series representation of H, and $\breve{\rho}$ is associated with ρ under the Jacquet-Langlands correspondence,

$$\mathrm{tr}\rho(f^H) = \mathrm{tr}\theta(\rho^+)(f) - \mathrm{tr}\theta(\breve{\rho}^+)(f)$$

by Weissauer [We2]. When ρ is a principal series, only $\theta(\rho^+)$ contributes. Consequently, to show that the endoscopic error terms above vanish, it suffices to show that type IIIa representations do not occur in the θ-correspondence:

$$\theta_{\mathrm{GO}(2,2)}(IIIa) = 0, \quad \text{and} \quad \theta_{\mathrm{GO}(4)}(IIIa) = 0.$$

Using Kudla's filtration of the Jacquet modules of the Weil representation [Kud], after restriction to Sp(4), this reduces to the well-known fact that $\mathrm{St}_{\mathrm{SL}(2)}$ does not occur in the θ-correspondence with $\mathrm{O}(1,1)$.

4.4 A Matching Result

The last step is to make sure that we can arrange for $\tilde{\Pi}_r$ to have nonzero vectors fixed by the paramodular group. Let us briefly review the Bruhat-Tits theory of $G(\mathbb{Q}_r)$. The building is an inhomogeneous tree. Therefore, the group has two maximal compact subgroups up to conjugacy (necessarily special). One is contained in the Siegel parahoric over the unramified quadratic extension $\mathbb{Q}_{r^2}$, whereas the other is contained in the paramodular group over $\mathbb{Q}_{r^2}$. The latter is referred to as the paramodular group in $G(\mathbb{Q}_r)$, and we denote it by K. It is not hard to see that π_r, and hence $\tilde{\pi}_r$, are both spherical for K. If we take

$$f_r = \mathbb{I}_K$$

to be the characteristic function of K, the left-hand-side of the trace formula is positive. We prove that we can choose a matching function f_r' on GSp(4) which is bi-invariant under the paramodular group. More precisely, we can take

$$f_r' = \mathbb{I}_{K'\eta K'},$$

where K' momentarily denotes the paramodular group in GSp(4) and η is a certain Atkin-Lehner element. It satisfies $\eta^2 = r$, so the corresponding Atkin-Lehner operator $\mathbb{I}_{K'\eta K'}$ has eigenvalues ± 1 on the K'-invariants of a representation with trivial central character. Once we prove the matching formula

$$\mathbb{I}_K' = \mathbb{I}_{K'\eta K'}$$

it follows that there is some $\tilde{\Pi}$ on the right-hand-side of the trace formula such that $\tilde{\Pi}_r$ has a positive Atkin-Lehner trace. In particular, $\tilde{\Pi}_r$ must be spherical for K' as required. We can also show that $\tilde{\Pi}_r$ is ramified. If it is tempered, as we suspect, it follows from the tables in [Sch] that it must be of type IIa.

To prove the matching formula, we compare both sides with twisted orbital integrals over the splitting field $\mathbb{Q}_{r^2}$. This can be done by work of Kottwitz [Ko2], after minor modifications. After fixing an inner twisting, the comparison of these twisted orbital integrals is a simple computation.

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