

Divisible Motives and Tate's Conjecture

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This article gives an analog of Taylor's trick in the context of motives for absolute Hodge cycles. The aforementioned trick, first sketched by Taylor in a letter to Clozel, gives two conditions ensuring that an mn -dimensional ℓ -adic Galois representation is divisible by m . We give a detailed proof of this result, and extend it to motives by using Tannakian duality.

1 Introduction

Taylor's trick originates in the following basic question: For a number field F , let $\rho : \text{Gal}(\bar{F}/F) \rightarrow \text{GL}_{mn}(\bar{\mathbb{Q}}_\ell)$ be a continuous semisimple representation, unramified almost everywhere. Which conditions ensure that $\rho \simeq \tilde{\rho}^{\oplus m}$ for some n -dimensional $\tilde{\rho}$? In a letter to Clozel in 1991, Taylor showed that it is enough to assume (a) for unramified v , the eigenvalues of $\rho(\text{Frob}_v)$ have multiplicity at least m , and (b) for some $v|\ell$, and some $\tau : F_v \hookrightarrow \bar{\mathbb{Q}}_\ell$, each Hodge–Tate number has multiplicity m . See Proposition 1 below. The proof uses the Sen operator, introduced in [6], whose eigenspaces give the Hodge–Tate decomposition.

This scenario occurs naturally when one wants to associate Galois representations with algebraic automorphic representations Π on $\text{GL}_n(\mathbb{A}_F)$. In the setup of [3], one takes a CM field $F = EF^+$, where E is imaginary quadratic, F^+ is totally real and imposes further restrictions on Π . Namely, Π_∞ should be cohomological (equivalently, regular

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algebraic), $\Pi^\vee \simeq \Pi^c$, and Π_v is square integrable for some finite place v . (This last condition has since been removed in [8].) By the work of Clozel and Labesse, one can descend such a Π to a suitable unitary similitude group G/\mathbb{Q} such that $G(\mathbb{A}_E) = \mathrm{GL}_n(\mathbb{A}_F) \times \mathbb{A}_E^\times$. That is, there is an automorphic representation π of $G(\mathbb{A}_\mathbb{Q})$ such that $\mathrm{BC}_{E/\mathbb{Q}}(\pi) = \Pi \otimes \psi$ for some Hecke character ψ of E . Now, for each sufficiently small compact open subgroup U inside $G(\mathbb{A}_{\mathbb{Q},f})$, there is a smooth projective Shimura variety X_U over F , of PEL type. The weight of Π_∞ defines an irreducible algebraic representation ξ of $G \times_{\mathbb{Q}} \bar{\mathbb{Q}}_\ell$, which in turn defines a $\bar{\mathbb{Q}}_\ell$ -sheaf \mathcal{L}_ξ on X_U . Then

$$H^i(X, \mathcal{L}_\xi) = \lim_{\rightarrow U} H^i(X_U \times_F \bar{F}, \mathcal{L}_\xi) \simeq \bigoplus_{\pi_f} \pi_f^* \pi_f \otimes R_{\xi,t}^i(\pi_f)$$

is an admissible $G(\mathbb{A}_{\mathbb{Q},f})$ -module over $\bar{\mathbb{Q}}_\ell$. Here $t : \bar{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$ is a fixed isomorphism. One can show that our fixed π above is tempered, so that $R_{\xi,t}^i(\pi_f) = 0$ unless $i = \dim X_U = n - 1$. One would hope that $R_{\xi,t}^{n-1}(\pi_f)$ is the desired Galois representation $R_i(\Pi)$, but that is simply not always true. A computation shows,

$$\dim_{\bar{\mathbb{Q}}_\ell} R_{\xi,t}^{n-1}(\pi_f) = m_G \cdot n, \quad m_G = \tau(G) \cdot \#\ker^1(\mathbb{Q}, G),$$

where $\tau(G) \in \{1, 2\}$ is the Tamagawa number (Is it one in this case?). Hard work in [3] then verifies that $R_{\xi,t}^{n-1}(\pi_f)^{ss}$ does indeed satisfy conditions (a) and (b) mentioned earlier, after twisting by ψ , and therefore, by Taylor's trick (that is, Proposition 1 given later),

$$R_{\xi,t}^{n-1}(\pi_f)^{ss} \simeq R_i(\Pi)^{\oplus m_G} \otimes R_i(\psi).$$

A natural question then arises: whether the $R_i(\Pi)$ are the ℓ -adic realizations of a motive? We are cautiously optimistic that one can show $R_{\xi,t}^{n-1}(\pi_f)$ comes from a motive, cut out of the cohomology of a self-product of the universal abelian scheme \mathcal{A}_U over X_U (for suitable U), when one imposes certain special hypotheses on F and π . To remove these special hypotheses, a key input would then be a motivic analog of the usual patching lemma, a direction we plan to pursue in the near future. The goal of this paper is to provide a version of Taylor's trick in the motivic setting. Unfortunately, in addition to analogs of (a) and (b), we have to make a rather strong assumption on the Galois image (implying that $\mathrm{Gal}(\bar{F}/F)$ acts semisimply, for example), which, however, is expected to be vacuous, as predicted by the Tate conjecture.

As motivation, no pun intended, let us mention that the motivic patching lemma would also yield motives attached to certain automorphic representations Π on $\mathrm{GL}_n(\mathbb{A}_{F^+})$. This, in turn, would explain that the oddness (When n is odd, and $R_i(\Pi)$ is irreducible; this was proved by Taylor in [9] by giving another geometric realization of complex conjugation. The same method should work when n is even and some component is a discrete series.) of the book project Galois representations $R_i(\Pi)$. Here, following Gross, by *odd* we mean that $R_i(\Pi)(c_{\bar{\sigma}})$ is conjugate to $\phi_{\Pi_{\sigma}}(j)$ for every embedding $\sigma : F^+ \hookrightarrow \mathbb{C}$ (and every extension $\bar{\sigma}$ to \bar{F}). Here $\phi_{\Pi_{\sigma}}$ is the Langlands parameter, and $j \in \mathbb{W}_{\mathbb{R}}$ such that $j^2 = -1$. When Π_{σ} is cohomological, it is an isobaric sum of discrete series (and a character if n is odd); so in that case, odd simply means that $\mathrm{tr} R_i(\Pi)(c_{\bar{\sigma}}) \in \{0, \pm 1\}$

We now formulate the main result of this paper. First, we must establish some notation. Throughout, we fix a number field F and a subfield $L \subset \bar{\mathbb{Q}}$, possibly of infinite degree over \mathbb{Q} . Let $M = M^+(t)$ be a pure motive over F , for absolute Hodge cycles, with coefficients in L . Here $M^+ = (h(X), \epsilon)$ is an effective motive. That is, X is a smooth projective variety defined over F , which is not necessarily connected, but for simplicity we assume it is of pure dimension, so that

$$\epsilon = \epsilon^2 \in \mathrm{End} h(X) = C_{\mathrm{AH}}^d(X \times X), \quad d = \dim(X),$$

where C_{AH}^d is the finite-dimensional \mathbb{Q} -space of absolute Hodge cycles of codimension d . We refer to [1, 4] for the definitions. A succinct survey is given in [5]. We let r be the rank of M (over L), and let w be its weight. The motive M has various realizations, related by comparison isomorphisms:

- (1) *Betti*: For each embedding $\sigma : F \hookrightarrow \mathbb{C}$, one has an r -dimensional vector space M_{σ} over L , endowed with a Hodge decomposition $M_{\sigma} \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{i+j=w} M_{\sigma}^{i,j}$ into $L \otimes_{\mathbb{Q}} \mathbb{C}$ -submodules $M_{\sigma}^{i,j}$. Moreover, complex conjugation on the complex variety ${}_{\sigma}X = X \times_{F, \sigma} \mathbb{C}$ defines involutive L -linear isomorphisms $\mathrm{Frob}_{\sigma} : M_{\sigma} \rightarrow M_{c\sigma}$ such that $\mathrm{Frob}_{\sigma} \otimes 1$ identifies $M_{\sigma}^{i,j}$ with $M_{c\sigma}^{j,i}$. Thus, when σ is real, we get an infinite Frobenius Frob_{σ} on M_{σ} .
- (2) *de Rham*: One has a free $L \otimes_{\mathbb{Q}} F$ -module M_{DR} , of rank r , endowed with a decreasing filtration $\mathrm{Fil}^i(M_{\mathrm{DR}})$ by $L \otimes_{\mathbb{Q}} F$ -submodules, which are not necessarily free. The Hodge filtration is exhaustive and separating.
- (3) *λ -adic*: For each finite place λ of L , one has an r -dimensional vector space M_{λ} over the completion L_{λ} , endowed with a continuous action $\rho_{M, \lambda}$ of the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$, which we denote by Γ_F in what follows.

- (4) *Complex comparison isomorphism*: For each embedding $\sigma : F \hookrightarrow \mathbb{C}$ as mentioned already, one has a natural comparison isomorphism of $L \otimes_{\mathbb{Q}} \mathbb{C}$ -modules,

$$I_{\infty, \sigma} : M_{\sigma} \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow M_{DR} \otimes_{F, \sigma} \mathbb{C},$$

which identifies $\bigoplus_{i \geq i'} M_{\sigma}^{i, w-i}$ with $\text{Fil}^{i'}(M_{DR}) \otimes_{F, \sigma} \mathbb{C}$, for all i' .

- (5) *λ -adic comparison isomorphism*: For each extension $\tilde{\sigma} : \tilde{F} \hookrightarrow \mathbb{C}$ of the embedding $\sigma : F \hookrightarrow \mathbb{C}$, one has a natural comparison isomorphism

$$I_{\lambda, \tilde{\sigma}} : M_{\sigma} \otimes_L L_{\lambda} \rightarrow M_{\lambda}.$$

They are compatible with the Galois action, in the sense that $I_{\lambda, \tilde{\sigma}} = \rho_{M, \lambda}(\gamma) \circ I_{\lambda, \tilde{\sigma}\gamma}$ for all $\gamma \in \Gamma_F$. Furthermore, when σ is a real embedding, and $c_{\tilde{\sigma}} \in \Gamma_F$ is the complex conjugation, $I_{\lambda, \tilde{\sigma}} \circ (\text{Frob}_{\sigma} \otimes 1) = \rho_{M, \lambda}(c_{\tilde{\sigma}}) \circ I_{\lambda, \tilde{\sigma}}$.

Our main result in this paper is the following motivic analog of Taylor's trick.

Theorem 1. Let M be a motive over F , with coefficients in a number field L , of weight w and rank mn . Assume M satisfies the following three hypotheses.

- (i) The image $\rho_{M, \lambda}(\Gamma_F)$ is Zariski dense in $\mathcal{G}_{F, M}(\sigma) \times_{\mathbb{Q}} \tilde{\mathbb{Q}}_{\ell}$, for some embedding $\sigma : F \hookrightarrow \mathbb{C}$, where $\mathcal{G}_{F, M}(\sigma)$ is the corresponding reductive quotient of the motivic Galois group. (In particular, Γ_F acts semisimply.)
- (ii) For all but finitely many v (and some finite place λ), each eigenvalue of $\rho_{M, \lambda}(\text{Frob}_v)$ on $M_{\lambda} \otimes_{L_{\lambda}} \tilde{\mathbb{Q}}_{\ell}$ has algebraic multiplicity at least m .
- (iii) There exists an embedding $\sigma : F \hookrightarrow \mathbb{C}$ such that $\dim_{\mathbb{C}} M_{\sigma}^{i, j} = m[L : \mathbb{Q}]$ for each of the n distinct Hodge types (i, j) .

Then there is a motive \tilde{M} over F , with coefficients in a finite extension \tilde{L} of L , of weight w and rank n (over \tilde{L}), unique up to isomorphism, such that

$$M_{\tilde{L}} = M \otimes_L \tilde{L} \simeq m \cdot \tilde{M} = \tilde{M}^{\oplus m}.$$

Moreover, \tilde{M} is σ -regular. That is, $\dim_{\mathbb{C}} \tilde{M}_{\sigma}^{i, j} = [L : \mathbb{Q}]$ for Hodge types (i, j) . □

Here, up to isomorphism, $M \otimes_L \tilde{L}$ can be described as follows: Pick an L -basis for \tilde{L} . Look at the direct sum $M \oplus \dots \oplus M$, each summand corresponding to a basis element.

It has an obvious \tilde{L} -structure via the map $\tilde{L} \hookrightarrow \text{End}_L(\tilde{L})$. We leave it to the reader to work out the various realizations of $M \otimes_L \tilde{L}$.

We stress that condition (i) is expected to hold *always*, as a consequence of the Tate conjecture, see [7, question 3.2 on p. 379]. However, it may be difficult to check in practice for a given M . One would proceed by showing that the image $\rho_{M,\lambda}(\Gamma_F)$ is open in $\mathcal{G}_{F,M}(\sigma, \bar{\mathbb{Q}}_\ell)$, and that it intersects every connected component, see [7, p. 386]. In fact, we can get by with something slightly weaker. In Theorem 1, one can replace (i) and (ii) with the following condition: For every semisimple $g \in \mathcal{G}_{F,M}(\sigma, \bar{\mathbb{Q}}_\ell)$, every eigenvalue of g on $M_\lambda \otimes_{L_\lambda} \bar{\mathbb{Q}}_\ell$ has multiplicity at least m .

2 A Result in Representation Theory

Throughout this section, G denotes a fixed reductive algebraic group defined over an algebraically closed field k of characteristic zero. We take G° to be its identity component, which is a normal subgroup. The (finite) component group G/G° is denoted $\pi_0(G)$. We let \mathfrak{g} be the Lie k -algebra of G° . Then, our goal in this section is to prove the following lemma on multiplicity removal for algebraic representations of G .

Lemma 1. Let $r : G \rightarrow \text{GL}(W)$ be an algebraic representation of G over k , of dimension mn . Assume it satisfies the following two hypotheses.

- (1) For every semisimple $g \in G$, each eigenvalue of $r(g)$ on W has multiplicity at least m . (In particular, $r(g)$ has at most n distinct eigenvalues.)
- (2) There exists a semisimple $X \in \mathfrak{g} \otimes_k K$, for some algebraically closed field K/k , such that each eigenvalue of $dr(X)$ on $W \otimes_k K$ has multiplicity m . (In particular, $dr(X)$ has precisely n distinct eigenvalues.)

Then there is an algebraic k -representation $\tilde{r} : G \rightarrow \text{GL}(\tilde{W})$, of dimension n , uniquely determined up to isomorphism, such that

$$r \simeq m \cdot \tilde{r} = \tilde{r}^{\oplus m}.$$

Moreover, \tilde{r} has precisely n distinct weights for any maximal torus. □

Proof. We first fix a Borel pair (B, T) in G° , and let N be its normalizer in G ,

$$N = N_G(B, T) = N_G(B) \cap N_G(T).$$

This is an algebraic subgroup of G , containing T as a normal subgroup, and N/T is naturally identified with $\pi_0(G)$. Indeed, it is easy to see that

$$G = NG^\circ, \quad T = N \cap G^\circ,$$

since G° acts transitively on the set of Borel pairs, and $N_{G^\circ}(T) = T$. We choose T such that the semisimple element X in (2) belongs to its Lie algebra $\mathfrak{t} \otimes_k K$.

For each algebraic character $\lambda \in X(T)$, we let W_λ denote the associated weight space. The set of weights \mathcal{W} consists of those λ for which $W_\lambda \neq 0$. By simultaneous diagonalization, $W = \bigoplus_{\lambda \in \mathcal{W}} W_\lambda$. On the other hand, W is a completely reducible G° -module, since G is reductive, so we may also decompose W into a direct sum of simple G° -submodules. The latter are classified by their highest weights, relative to B . For each B -dominant weight $\lambda \in X(T)_+$, we let $V(\lambda)$ be the usual simple Weyl module of highest weight λ (recall that k has characteristic zero). Then $W[\lambda]$ is defined to be the sum of all simple G° -submodules of W of highest weight λ . By Frobenius reciprocity, we have the isotypic decomposition

$$W = \bigoplus_{\lambda \in \mathcal{W}_+} W[\lambda], \quad W[\lambda] \simeq \dim_k W_\lambda^U \cdot V(\lambda), \quad \mathcal{W}_+ = \mathcal{W} \cap X(T)_+.$$

Here U is the unipotent radical of B , and W_λ^U is the subspace of U -invariants.

Step 1: $\#\mathcal{W} = n$ and $\dim_k W_\lambda = m$, for all $\lambda \in \mathcal{W}$.

We first observe that the inequality $\#\mathcal{W} \leq n$ holds: Choose a $t \in T$ such that the $\lambda(t)$ are distinct, for all weights λ . Then $r(t)$ has $\#\mathcal{W}$ distinct eigenvalues on W . By (1) it has at most n distinct eigenvalues, proving the inequality. To get the equality, note that the eigenvalues of $dr(X)$ on $W \otimes_k K$ are of the form $d\lambda(X)$, for $\lambda \in \mathcal{W}$. By (2), these must be distinct, and $\#\mathcal{W} = n$. Finally, since the $d\lambda(X)$ are all distinct, W_λ is the $d\lambda(X)$ -eigenspace of $dr(X)$, which is m -dimensional by (2).

Step 2: $\dim_k W_\lambda^U = m$, for all $\lambda \in \mathcal{W}_+$ such that $W[\lambda] \neq 0$.

We first prove the equality when $\lambda \in \mathcal{W}_+$ is maximal. Pick $t \in T$ as in Step 1. Then $\lambda(t)$ is an eigenvalue of $r(t)$, with multiplicity $\dim_k W_\lambda^U$ in $W[\lambda]$. Moreover, $\lambda(t)$ is not an eigenvalue on $W[\mu]$ for any $\mu \neq \lambda$, by maximality. Therefore $\dim_k W_\lambda^U \geq m$, according to (1). Next, we consider an arbitrary $\lambda \in \mathcal{W}_+$, occurring in W , and show that $\dim_k W_\lambda^U = m$, assuming this is known for all $\mu > \lambda$. As mentioned earlier, $d\lambda(X)$ is an eigenvalue of $dr(X)$, with multiplicity $\dim_k W_\lambda^U$ in $W[\lambda] \otimes_k K$. If this is less than m , then $d\lambda(X)$ must be an eigenvalue on $W[\mu] \otimes_k K$, for some $\mu > \lambda$. By induction, the multiplicity of $d\lambda(X)$ is at

least m in the latter space. However, by (2) the multiplicity of the eigenvalue $d\lambda(X)$ in $W \otimes_k K$ equals m . This is a contradiction.

Thus, at least as a G° -module, W is of the form $m \cdot \tilde{W}$, where $\tilde{W} \simeq \bigoplus V(\lambda)$, the sum extending over all $\lambda \in \mathcal{W}_+$ such that $W[\lambda] \neq 0$. To achieve the analog for the G -action, we need to understand how G permutes the simple G° -submodules of W . Recall that N/T acts on $X(T)$, by letting $g\lambda(t) = \lambda(g^{-1}tg)$. Since N normalizes B , the subsets $X(T)_+$ and \mathcal{W} , and hence \mathcal{W}_+ , are invariant. In fact, as is easily checked, $r(g)W_\lambda = W_{g\lambda}$ and $r(g)W[\lambda] = W[g\lambda]$. Thus we introduce

$$W\{\lambda\} = \bigoplus_{g \in N/N_\lambda} W[g\lambda].$$

Here N_λ is the stabilizer of λ in N ; so we are summing over the N -orbit of λ . Then $W\{\lambda\}$ is the smallest G -invariant subspace of W containing $W[\lambda]$, and

$$W = \bigoplus_{\lambda \in N \setminus \mathcal{W}_+} W\{\lambda\}.$$

It is therefore enough to show that each $W\{\lambda\}$ is divisible by m .

Step 3: $W\{\lambda\} \simeq \text{Ind}_{G_\lambda}^G W[\lambda]$, where $G_\lambda = N_\lambda G^\circ$.

First observe that there is a natural bijection $N/N_\lambda \simeq G/G_\lambda$. Therefore,

$$W\{\lambda\} = \bigoplus_{g \in G/G_\lambda} r(g)W[\lambda].$$

Comparing dimensions, we need only embed the induced representation into $W\{\lambda\}$. This is done by mapping a function f to the tuple of all $r(g)f(g^{-1})$.

We are now reduced to showing that each $W[\lambda]$ is divisible by m , when viewed as a G_λ -representation. This follows once we show that each copy of $V(\lambda)$ in W is N_λ -invariant, and that all copies are isomorphic as G_λ -modules. We will deduce this from the following key step:

Step 4: N_λ acts on W_λ by a *character*, for all $\lambda \in \mathcal{W}$.

Fix some $g \in N_\lambda$. Since a power of g lies in T , it is a semisimple element (Jordan decomposition). We assume $r(g)$ has at least two distinct eigenvalues on W_λ , and obtain a contradiction. To do that, we consider the action of $r(gt)$ on W , for varying $t \in T$. It preserves the decomposition $W = W_\lambda \oplus W'_\lambda$, where we introduce $W'_\lambda = \bigoplus_{\mu \neq \lambda} W_\mu$. Let Λ be an eigenvalue of $r(g)$ on W_λ . Then $\lambda(t)\Lambda$ is an eigenvalue of $r(gt)$ on W_λ . The key point is

to show, for suitable $t \in T$,

$$W'_\lambda \cap \ker(r(gt) - \lambda(t)\Lambda \cdot \text{Id}_W) = 0.$$

Indeed, if we know this, the whole eigenspace $\ker(r(gt) - \lambda(t)\Lambda \cdot \text{Id}_W)$ is contained in W'_λ . By (1) the eigenspace is at least m -dimensional. Having two eigenvalues Λ , then forces the lower bound $\dim_k W'_\lambda \geq 2m$, contradicting Step 1. It remains to show that the earlier-mentioned intersection is trivial. For this purpose, we partition \mathcal{W} into $\langle g \rangle$ -orbits $\mathcal{O}_1, \dots, \mathcal{O}_s$. In this list, $\mathcal{O}_1 = \{\lambda\}$. Correspondingly, we let $W_i = \bigoplus_{\mu \in \mathcal{O}_i} W_\mu$, so that $W'_\lambda = \bigoplus_{i>1} W_i$ is a decomposition of W'_λ into $\langle g \rangle$ -invariant subspaces. We need to show that $\lambda(t)\Lambda$ is not an eigenvalue of $r(gt)|_{W_i}$ for any $i > 1$. Again, for suitable $t \in T$. Suppose it is an eigenvalue on W_i , and pick an eigenvector $w = \sum_{\mu \in \mathcal{O}_i} w_\mu$. Comparing components, we see that:

$$\mu(t)^j r(g^j)w_\mu = (\lambda(t)\Lambda)^j w_{g^j\mu}, \quad \forall \mu \in \mathcal{O}_i, \quad \forall j \in \mathbb{Z}_{\geq 0}.$$

In particular, we see that $w_\mu \neq 0$ for all $\mu \in \mathcal{O}_i$. Let $b = \text{ord}_{N/T}(gT)$, so that $g^b \in T$. Taking $j = b$ above, we deduce the following constraints,

$$\Lambda^b \lambda(t)^b = \mu(g^b)\mu(t)^b, \quad \forall \mu \in \mathcal{O}_i.$$

If this holds for all $t \in T$, we must have $\Lambda^b = \mu(g^b)$ and $\lambda = \mu \in \mathcal{O}_i$. But $i > 1$.

Step 5: Each simple G° -submodule $V \subset W[\lambda]$ is G_λ -invariant.

Fix some $g \in N_\lambda$, and a nonzero vector $v \in V_\lambda^U$. A straightforward computation shows that $r(g)v$ is a highest weight vector in $r(g)V$, which is again a simple G° -submodule of $W[\lambda]$. However, by Step 4, $r(g)v$ is a nonzero multiple of v , and hence the intersection $V \cap r(g)V$ is nonzero. Consequently, $r(g)V = V$.

Step 6: Any two simple G° -submodules of $W[\lambda]$ are isomorphic as G_λ -modules.

Let V and V' be two copies of $V(\lambda)$ in W , and pick highest weight vectors v and v' . Then there is a unique isomorphism of G° -modules $\phi: V \rightarrow V'$ such that $\phi(v) = v'$. We show that ϕ is N_λ -equivariant: Fix some $g \in N_\lambda$. Then $r(g) \circ \phi \circ r(g)^{-1}$ is an isomorphism $V \rightarrow V'$ of G° -modules, mapping v to v' by Step 4. By uniqueness, $\phi = r(g) \circ \phi \circ r(g)^{-1}$. In other words, ϕ is G_λ -equivariant.

We conclude that, as G -modules, $W \simeq m \cdot \tilde{W}$, where we may take the module

$$\tilde{W} = \bigoplus_{\lambda \in N \setminus \mathcal{W}_+} \text{Ind}_{G_\lambda}^G V.$$

Here V is any copy of $V(\lambda)$ in W . This shows the existence of the desired \tilde{r} .

Step 7: The representation \tilde{r} is uniquely determined up to isomorphism.

The character of \tilde{r} is uniquely determined, indeed $\text{tr}(r) = m\text{tr}(\tilde{r})$; so this follows from the Jacobson density theorem applied to the group algebra of G . ■

Remark. The earlier-mentioned proof was sketched in [3], as part of the proof of Lemma I.2.2 on p. 27. However, we feel that some steps in the latter proof are a bit imprecise. For example, there is no need to make the assumption that λ is a vertex of the convex hull of the set of weights. Moreover, the last inductive step (involving the complement Z) seems somewhat unjustified. □

3 Some Zariski Closed Subsets of GL_n

Let k be a fixed algebraically closed field. For positive integers $m, n \in \mathbb{Z}_{>0}$, we define $X_{m,n}$ to be the set of matrices $g \in \text{GL}_n(k)$ such that each eigenvalue of g has algebraic multiplicity at least m . Our goal in this section is the following.

Lemma 2. $X_{m,n}$ is a Zariski closed subset of GL_n . □

Proof. Let $p_g(x) = \det(xI - g)$ be the (monic) characteristic polynomial of g . Its coefficients are polynomials in the entries of g . Then $X_{m,n}$ is the set of g such that $p_g \in \mathcal{X}_{m,n}$, the set of degree n monic polynomials $p \in k[x]$ such that each root of p has multiplicity $\geq m$. The set of all degree n monic polynomials is naturally identified with k^n , and it is sufficient to show that $\mathcal{X}_{m,n} \subset k^n$ is Zariski closed. To do that, take an arbitrary monic polynomial $p \in k[x]$ of degree n . Then, a root $\alpha \in k$ has multiplicity at least m if and only if

$$p(\alpha) = p'(\alpha) = p''(\alpha) = \dots = p^{(m-1)}(\alpha) = 0.$$

Thus, any root of $p(x)$ is a root of $p^{(i)}(x)$ for all $0 \leq i < m$, but possibly of a different multiplicity. Since the multiplicities are at most n , we see that $p(x)$ divides $p^{(i)}(x)^n$ for all $0 \leq i < m$. The converse is obvious. We conclude that

$$\mathcal{X}_{m,n} = \{p(x) \text{ such that } p(x) \text{ divides } p^{(i)}(x)^n \text{ for all } 0 \leq i < m\} \subset k^n.$$

To see that this is Zariski closed, we argue as follows: By the division algorithm in $R[x]$, where R is the polynomial ring over k in the variables y_0, \dots, y_{n-1} and z_0, \dots, z_d , there

are unique polynomials $q, r \in R[x]$ such that $\deg_x(r) < n$ and

$$z_d x^d + \cdots + z_0 = q(x)(x^n + y_{n-1}x^{n-1} + \cdots + y_0) + r(x).$$

Thus, $x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in k[x]$ divides $b_d x^d + \cdots + b_0 \in k[x]$ if and only if all the coefficients of r vanish when substituting $y_i = a_i$ and $z_i = b_i$. This is a collection of polynomial conditions on the coefficients a_i and b_i . We are done since the coefficients of $p^{(i)}(x)^n$ are polynomials in the coefficients of $p(x)$. ■

Remark. This is probably well known to the experts. A brief outline of the earlier-mentioned argument is given in the first five lines of the proof of Lemma I.2.2 in [3]. Note that there is a typo there: The exponent should be an , not a . □

4 An Application to Galois Representations

We fix a number field F , an algebraic closure \bar{F} , and consider ℓ -adic representations of the Galois group $\Gamma_F = \text{Gal}(\bar{F}/F)$. That is, continuous representations

$$\rho : \Gamma_F \rightarrow \text{GL}(W),$$

where W is a finite-dimensional vector space over $\bar{\mathbb{Q}}_\ell$. For a finite place v , $\rho_v = \rho|_{\Gamma_{F_v}}$ is well-defined up to equivalence. We say ρ is *unramified* at v if ρ_v is trivial on the inertia group $I_{F_v} = \Gamma_{F_v}^{\text{ur}}$. Fix a lift Frob_v of the geometric Frobenius in Γ_{F_v} . Then $\rho(\text{Frob}_v)$ is a well-defined conjugacy class in $\text{GL}(W)$. When v is a finite place of F dividing ℓ , we say that ρ is *Hodge–Tate* at v when

$$D_{\text{HT}}(\rho_v) = (W \otimes_{\mathbb{Q}_\ell} B_{\text{HT}})^{\Gamma_{F_v}}$$

is a free $\bar{\mathbb{Q}}_\ell \otimes_{\mathbb{Q}_\ell} F_v$ -module of rank $\dim_{\bar{\mathbb{Q}}_\ell} W$. Here $B_{\text{HT}} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_{F_v}(i)$, where \mathbb{C}_{F_v} is the completion of \bar{F}_v . For each embedding $\tau : F_v \hookrightarrow \bar{\mathbb{Q}}_\ell$, we introduce the multiset $\text{HT}_\tau(\rho_v)$ of Hodge–Tate numbers, containing i with multiplicity

$$\dim_{\bar{\mathbb{Q}}_\ell} \text{gr}^i D_{\text{HT}, \tau}(\rho_v), \quad D_{\text{HT}, \tau}(\rho_v) = D_{\text{HT}}(\rho_v) \otimes_{\bar{\mathbb{Q}}_\ell \otimes_{\mathbb{Q}_\ell} F_v, 1 \otimes \tau} \bar{\mathbb{Q}}_\ell.$$

As an application of the results in the previous two sections, we are able to remove the multiplicity in Galois representations satisfying certain hypotheses. This is commonly

referred to as *Taylor's trick*, since it was first described in a letter from R. Taylor to Clozel in 1991, dated December 11.

Proposition 1. Let $\rho : \Gamma_F \rightarrow \mathrm{GL}(W)$ be a continuous semisimple representation on an mn -dimensional vector space W over $\bar{\mathbb{Q}}_\ell$. Assume ρ is unramified outside some finite set of places S , and satisfies the following two hypotheses.

- (a) For every $v \notin S$, each eigenvalue of $\rho(\mathrm{Frob}_v)$ on W has algebraic multiplicity at least m . (In particular, $\rho(\mathrm{Frob}_v)$ has at most n distinct eigenvalues.)
- (b) There exists a place v above ℓ such that ρ is Hodge–Tate at v , and for some embedding $\tau : F_v \hookrightarrow \bar{\mathbb{Q}}_\ell$ the multiset $HT_\tau(\rho_v)$ contains n distinct Hodge–Tate numbers, each occurring with multiplicity m .

Then there is a continuous semisimple representation $\tilde{\rho} : \Gamma_F \rightarrow \mathrm{GL}(\tilde{W})$, on an n -dimensional vector space \tilde{W} over $\bar{\mathbb{Q}}_\ell$, unique up to isomorphism, such that

$$\rho \simeq m \cdot \tilde{\rho} = \tilde{\rho}^{\oplus m}.$$

Moreover, $HT_\tau(\tilde{\rho}_v)$ is a set of n distinct integers (for the v and τ above). □

Proof. Take G to be the Zariski closure of $\rho(\Gamma_F)$ in $\mathrm{GL}(W)$, and consider the faithful algebraic representation $r : G \hookrightarrow \mathrm{GL}(W)$ through which ρ factors. Each simple Γ_F -summand of W is clearly G -invariant (look at a suitable Levi subgroup); so r is semisimple. Consequently, G is a reductive group, and Lemma 1 applies once we show r satisfies (1) and (2). By Chebotarev, $\Sigma = \{\mathrm{Frob}_v\}_{v \notin S}$ is a dense subset of Γ_F . Since ρ is continuous, and Zariski closed implies ℓ -adically closed, we infer that G is also the Zariski closure of $\rho(\Sigma)$. By assumption (a), in conjunction with Lemma 2, we deduce that for every $g \in G$, each eigenvalue of $r(g)$ has algebraic multiplicity at least m . This is stronger than (1). To check condition (2), we invoke the *Sen operator* as follows: First, by the Baire category theorem, there is a finite extension L/\mathbb{Q}_ℓ , inside $\bar{\mathbb{Q}}_\ell$, and a Γ_F -invariant L -structure W_L inside W . By enlarging L if necessary, we may (and will) assume that L contains $\tau(F_v)$ for every embedding $\tau : F_v \hookrightarrow \bar{\mathbb{Q}}_\ell$. As a consequence thereof, for any embedding $\mu : L \hookrightarrow \bar{F}_v$, its image $\mu(L)$ contains F_v (to see this, choose an extension of μ to an isomorphism $\tilde{\mu} : \bar{\mathbb{Q}}_\ell \hookrightarrow \bar{F}_v$. Then its inverse $\tilde{\mu}^{-1}$ maps F_v into L , done). For each such μ , it therefore makes sense to define $\tau_\mu = \mu^{-1}|_{F_v}$. We view W_L as a finite-dimensional vector space over \mathbb{Q}_ℓ , and we want to understand its Hodge–Tate decomposition. According to

the L -action,

$$W_L \otimes_{\mathbb{Q}_\ell} \mathbb{C}_{F_v} \simeq \bigoplus_{\mu: L \hookrightarrow \bar{F}_v} W_L \otimes_{L, \mu} \mathbb{C}_{F_v}.$$

For each fixed μ , a standard argument in p -adic Hodge theory shows that

$$W_L \otimes_{L, \mu} \mathbb{C}_{F_v} \simeq \bigoplus_{i \in HT_{\tau_\mu}(\rho_v)} \mathbb{C}_{F_v}(-i).$$

Of course, here we take into account the multiplicity of i in the multiset $HT_{\tau_\mu}(\rho_v)$. Following Sen, we look at the semisimple operator $\Phi \in \text{End}_{\mathbb{C}_{F_v}}(W_L \otimes_{\mathbb{Q}_\ell} \mathbb{C}_{F_v})$ acting by multiplication by i on each copy of $\mathbb{C}_{F_v}(i)$ in the Hodge–Tate decomposition. The main theorem of [6], Theorem 1 on p. 164, says that the Lie algebra \mathfrak{h} of $\rho(\Gamma_{F_v})$ is the smallest \mathbb{Q}_ℓ -subspace of $\text{End}_{\mathbb{Q}_\ell}(W_L)$ such that $\mathfrak{h} \otimes_{\mathbb{Q}_\ell} \mathbb{C}_{F_v}$ contains Φ . We denote by \mathfrak{g} the $\bar{\mathbb{Q}}_\ell$ -Lie algebra of G , and let \mathfrak{g}_L be the L -subalgebra, defined by W_L , such that $\mathfrak{g} = \mathfrak{g}_L \otimes_L \bar{\mathbb{Q}}_\ell$. Viewed as a \mathbb{Q}_ℓ -Lie algebra, \mathfrak{g}_L contains \mathfrak{h} . In particular, by Sen’s theorem, Φ belongs to

$$\mathfrak{g}_L \otimes_{\mathbb{Q}_\ell} \mathbb{C}_{F_v} = \bigoplus_{\mu: L \hookrightarrow \bar{F}_v} \mathfrak{g}_L \otimes_{L, \mu} \mathbb{C}_{F_v}.$$

Then Φ corresponds to a tuple (Φ_μ) , where Φ_μ acts by multiplication by i on each copy of $\mathbb{C}_{F_v}(i)$ in the Hodge–Tate decomposition of $W_L \otimes_{L, \mu} \mathbb{C}_{F_v}$. By assumption (b), Φ_μ has n distinct eigenvalues, of multiplicity m , if we choose μ such that $\tau = \tau_\mu$ (such exists, extend τ to an isomorphism $\tilde{\tau}: \bar{F}_v \hookrightarrow \bar{\mathbb{Q}}_\ell$, and check that the embedding $\mu = \tilde{\tau}^{-1}|_L$ works). Pick any extension of μ to an embedding $\tilde{\mu}: \bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}_{F_v}$. We take this to be the extension in condition (2),

$$X = \Phi_\mu \in \mathfrak{g} \otimes_{\bar{\mathbb{Q}}_\ell, \tilde{\mu}} \mathbb{C}_{F_v} \hookrightarrow \text{End}_{\mathbb{C}_{F_v}}(W \otimes_{\bar{\mathbb{Q}}_\ell, \tilde{\mu}} \mathbb{C}_{F_v}).$$

Finally, from Lemma 1 we get an algebraic representation $\tilde{r}: G \rightarrow \text{GL}(\tilde{W})$, of dimension n over $\bar{\mathbb{Q}}_\ell$, such that $r \simeq \tilde{r}^{\oplus m}$. Then define $\tilde{\rho} = \tilde{r} \circ \rho_G$, where ρ_G is the map $\Gamma_F \rightarrow G$ such that $\rho = r \circ \rho_G$. Then $\tilde{\rho}$ is continuous, semisimple (since G is reductive), and it is easily checked from the definitions that $\rho \simeq \tilde{\rho}^{\oplus m}$. \blacksquare

Remark. This Proposition is a key ingredient in the proof of Theorem C in [3]. Indeed, what we did above is essentially just the proof of Proposition VII.1.8 on p. 226 in [3]. In their setup, ρ is cut out of the cohomology of some compact unitary Shimura variety. Condition (a), about the unramified places, is then a result of Kottwitz (Corollary V.6.3 on p. 193), and condition (b) follows from their Corollary VI.2.8 on p. 208. \square

5 Proof of the Main Theorem

In this section, we give the proof of Theorem 1 from the introduction.

The motivic Galois group. This will be a crucial gadget in what follows. We briefly review its definition and some of its basic properties. Take $\mathcal{M}_F(L)$ to be the true category of (pure) motives over F , for absolute Hodge cycles, with coefficients in L . See [1] or [5] for its precise definition. The word *true* refers to the fact that one has to modify the commutativity constraints by a sign (after passing to a pseudo-abelian envelope, and then inverting the Lefschetz object). When $L = \mathbb{Q}$ we simply write \mathcal{M}_F . The sign change makes \mathcal{M}_F into a neutral Tannakian category over \mathbb{Q} . That is, a rigid abelian \mathbb{Q} -linear tensor category, for which there exists an exact faithful \mathbb{Q} -linear tensor functor $\omega : \mathcal{M}_F \rightarrow \text{Vec}_{\mathbb{Q}}$. Such an ω is called a fiber functor. For example, for each complex embedding $\sigma : F \hookrightarrow \mathbb{C}$, we have a corresponding fiber functor $\omega_\sigma : M \mapsto M_\sigma$. Tannakian duality identifies \mathcal{M}_F with a category of representations,

$$\omega_\sigma : \mathcal{M}_F \xrightarrow{\sim} \text{Rep}_{\mathbb{Q}} \mathcal{G}_F(\sigma), \quad \mathcal{G}_F(\sigma) = \underline{\text{Aut}}^{\otimes}(\omega_\sigma).$$

The motivic Galois group $\mathcal{G}_F(\sigma)$ is an affine group scheme over \mathbb{Q} . Its name stems from the fact that, analogously, $\text{Rep}_{\mathbb{Q}} \Gamma_F$ is equivalent to the category of Artin motives (those coming from zero-dimensional varieties). For each extension $\tilde{\sigma} : \bar{F} \hookrightarrow \mathbb{C}$ of σ , we may also identify $\mathcal{M}_{\bar{F}}$ with the category of representations of $\mathcal{G}_{\bar{F}}(\tilde{\sigma})$. The latter is a *connected* pro-reductive group over \mathbb{Q} ,

$$1 \longrightarrow \mathcal{G}_{\bar{F}}(\tilde{\sigma}) \longrightarrow \mathcal{G}_F(\sigma) \longrightarrow \Gamma_F \longrightarrow 1$$

is exact. In particular, $\mathcal{G}_{\bar{F}}(\tilde{\sigma})$ is the identity component of $\mathcal{G}_F(\sigma)$. Finally, the same holds with coefficients L . For example, $\mathcal{M}_F(L)$ is equivalent to the category of representations of $\mathcal{G}_F(\sigma) \times_{\mathbb{Q}} L$ on finite-dimensional L -vector spaces.

Reductive quotients. For each $M \in \text{Ob}(\mathcal{M}_F)$, we introduce $\mathcal{M}_{F,M}$, the smallest full Tannakian subcategory of \mathcal{M}_F containing M . Objects in this subcategory are said to be dominated by M . As mentioned earlier, by Tannakian duality, we have

$$\omega_\sigma : \mathcal{M}_{F,M} \xrightarrow{\sim} \text{Rep}_{\mathbb{Q}} \mathcal{G}_{F,M}(\sigma), \quad \mathcal{G}_{F,M}(\sigma) = \underline{\text{Aut}}^{\otimes}(\omega_\sigma|_{\mathcal{M}_{F,M}}).$$

Here $\mathcal{G}_{F,M}(\sigma)$ is a reductive algebraic group over \mathbb{Q} , and relative to dominance,

$$\mathcal{G}_F(\sigma) = \lim_{\leftarrow M} \mathcal{G}_{F,M}(\sigma).$$

Moreover, by definition, there is a faithful representation $\mathcal{G}_{F,M}(\sigma) \hookrightarrow \mathrm{GL}(M_\sigma)$. One can describe the image explicitly as follows: We use $\mathbf{1}$ to denote the trivial motive $h^0(\mathrm{Spec}F)$, and for nonnegative integers r, s we let $T^{r,s}(M)$ be the tensor product of r copies of M and s copies of M^\vee . A tensor $t \in T^{r,s}(M_\sigma)$ is said to be *invariant* if it comes from a morphism $\mathbf{1} \rightarrow T^{r,s}(M)$ of motives. We use the notation $T_{\mathrm{inv}}^{r,s}(M_\sigma)$ for the subspace of such t . It coincides with the space of $\mathcal{G}_F(\sigma)$ -invariants in $T^{r,s}(M_\sigma)$. Then, by Deligne et al. [1, Proposition I.3.1(c)],

$$\mathcal{G}_{F,M}(\sigma, R) \simeq \{g \in \mathrm{GL}(M_\sigma \otimes_{\mathbb{Q}} R) : g(t \otimes 1) = t \otimes 1, \forall t \in \cup_{r,s} T_{\mathrm{inv}}^{r,s}(M_\sigma)\},$$

for all commutative \mathbb{Q} -algebras R . Taking $R = \mathbb{Q}_\ell$, and composing with the ℓ -adic comparison isomorphism $I_{\ell, \tilde{\sigma}}$ for some $\tilde{\sigma}$, we identify $\mathcal{G}_{F,M}(\sigma) \times_{\mathbb{Q}} \mathbb{Q}_\ell$ with the subgroup of $\mathrm{GL}(M_\ell)$ fixing all $I_{\ell, \tilde{\sigma}}(t \otimes 1)$. However, these tensors are clearly fixed by Γ_F since they come from a morphism of motives $\mathbf{1} \rightarrow T^{r,s}(M)$. Thus,

$$\rho_{M,\ell}(\Gamma_F) \subset \mathcal{G}_{F,M}(\sigma, \mathbb{Q}_\ell) \subset \mathrm{GL}(M_\ell).$$

Note that these subgroups do *not* depend on the choice of $\tilde{\sigma}$, by the Galois-compatibility of the $I_{\ell, \tilde{\sigma}}$. We take G_M to be the Zariski closure of $\rho_{M,\ell}(\Gamma_F)$ inside $\mathrm{GL}(M_\ell \otimes_{\mathbb{Q}_\ell} \bar{\mathbb{Q}}_\ell)$. We just saw that $\mathcal{G}_{F,M}(\sigma) \times_{\mathbb{Q}} \bar{\mathbb{Q}}_\ell$ contains G_M . It is expected that the two are equal, see [7, question 3.2 on p. 379]. Similarly, if we take $R = \mathbb{C}$ above, we identify $\mathcal{G}_{F,M}(\sigma) \times_{\mathbb{Q}} \mathbb{C}$ with a subgroup of $\mathrm{GL}(M_\sigma \otimes_{\mathbb{Q}} \mathbb{C})$. As usual, the Hodge decomposition defines a group homomorphism

$$h_{M,\sigma} : \mathbb{C}^* \times \mathbb{C}^* \rightarrow \mathrm{GL}(M_\sigma \otimes_{\mathbb{Q}} \mathbb{C})$$

by letting $h_{M,\sigma}(z, z')$ act by multiplication by $z^i z'^j$ on $M_\sigma^{i,j}$. The image of $h_{M,\sigma}$ is a (connected) torus. By definition, the Mumford–Tate group $MT(M_\sigma)$ is the smallest \mathbb{Q} -subgroup of $\mathrm{GL}(M_\sigma)$ such that the complexification contains the image of $h_{M,\sigma}$. Since invariant tensors must lie in the bidegree $(0, 0)$ component, it is easily checked that $\mathcal{G}_{F,M}^\circ(\sigma) \times_{\mathbb{Q}} \mathbb{C}$ contains $MT(M_\sigma)$. Conjecturally they are equal, see [7, question 3.4 on p. 380]. We leave it to the reader to make the appropriate adjustments when we allow motives with coefficients.

p-adic Hodge theory. Let M be a motive as in the introduction, and let us fix a complex embedding $\sigma : F \hookrightarrow \mathbb{C}$. Once and for all, we fix an isomorphism $\iota : \bar{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$. Then σ corresponds to a pair (v, τ) , where v is a place of F dividing ℓ , and $\tau : F_v \hookrightarrow \bar{\mathbb{Q}}_\ell$. The correspondence is determined by the equality $\sigma = \iota \circ \tau_F$. We consider the action of Γ_{F_v} on the space $M_\lambda \otimes_{L_\lambda} \bar{\mathbb{Q}}_\ell$, and we wish to relate the Hodge–Tate decomposition to the Hodge components $M_\sigma^{i,j}$. Since

$$M_\lambda \otimes_{L_\lambda} \bar{\mathbb{Q}}_\ell \simeq \epsilon H_{\text{et}}^{w+2t}(X \times_F \bar{F}, \bar{\mathbb{Q}}_\ell(t)),$$

the main result from [2] shows that

$$L \otimes_{\mathbb{Q}} D_{\text{DR},\tau}(M_\lambda \otimes_{L_\lambda} \bar{\mathbb{Q}}_\ell) \simeq M_{\text{DR}} \otimes_{F,\tau} \bar{\mathbb{Q}}_\ell \simeq (M_\sigma \otimes_{\mathbb{Q}} \mathbb{C}) \otimes_{\mathbb{C},\iota^{-1}} \bar{\mathbb{Q}}_\ell.$$

By comparing the graded pieces, we see that

$$L \otimes_{\mathbb{Q}} \text{gr}^i D_{\text{HT},\tau}(M_\lambda \otimes_{L_\lambda} \bar{\mathbb{Q}}_\ell) \simeq M_\sigma^{i,w-i} \otimes_{\mathbb{C},\iota^{-1}} \bar{\mathbb{Q}}_\ell.$$

In particular,

$$[L : \mathbb{Q}] \cdot \dim_{\bar{\mathbb{Q}}_\ell} \text{gr}^i D_{\text{HT},\tau}(M_\lambda \otimes_{L_\lambda} \bar{\mathbb{Q}}_\ell) = \dim_{\mathbb{C}} M_\sigma^{i,w-i},$$

so that condition (iii) in Theorem 1 implies condition (b) in Proposition 1.

Representations of the group G_M . Recall that G_M is defined to be the Zariski closure of $\rho_{M,\lambda}(\Gamma_F)$ inside $\text{GL}(M_\lambda \otimes_{L_\lambda} \bar{\mathbb{Q}}_\ell)$. By the semisimplicity assumption in (i), which should be vacuous, G_M is reductive, and we may proceed as in the proof of Proposition 1. As in that proof, let $r : G_M \hookrightarrow \text{GL}(M_\lambda \otimes_{L_\lambda} \bar{\mathbb{Q}}_\ell)$ be the tautological representation. Then the Sen operator trick, combined with Lemma 1, yields an algebraic representation $\tilde{r} : G_M \rightarrow \text{GL}(\tilde{W})$, of dimension n over $\bar{\mathbb{Q}}_\ell$, such that $r \simeq \tilde{r}^{\oplus m}$. At this point, we have to invoke the full thrust of condition (i), namely that G_M is all of $\mathcal{G}_{F,M}(\sigma) \times_{\mathbb{Q}} \bar{\mathbb{Q}}_\ell$, and view \tilde{r} as a representation of the latter. It can be defined over a number field \tilde{L} , which we may enlarge so that it contains L , and so that the isomorphism $r \simeq \tilde{r}^{\oplus m}$ is defined over \tilde{L} . Pulling back to the motivic Galois group, \tilde{r} corresponds to a motive \tilde{M} in $\mathcal{M}_F(\tilde{L})$ by Tannakian duality, and \tilde{M} has the desired properties.

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