A Patching Lemma

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Abstract

In this note, we discuss how to patch up certain collections of Galois representations over solvable extensions. We give a fairly elaborate proof, based on ideas from [HT]. The patching lemma, referred to in the title, will be of importance in the construction of automorphic Galois representations [Har]. Descent from quadratic extensions was used by Blasius and Rogawski in the case of Hilbert modular forms [Bro].

1 Patching: Extensions of prime degree

We discuss a patching argument, used in various guises by other authors. For example, see Proposition 4.3.1 in [BRo], or section 4.3 in [BRa]. Here we will prove a variant of a Proposition in the *first* version (v.1) of [Har], which in turn was based on the discussion on p. 230-231 in [HT]. The proof in the first version of [Har] was somewhat brief, and somewhat imprecise at the end, so we decided to write up the more detailed proof below. In this section, we use Γ_F as shorthand notation for the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$. The setup is the following: We let $\mathcal{I}$ be a set of cyclic Galois extensions E, of a fixed number field F, of *prime* degree q_E. For every $E \in \mathcal{I}$ we assume we are given an n-dimensional continuous semisimple ℓ-adic Galois representation over E,

$$\rho_E : \Gamma_E \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell).$$

Here ℓ is a fixed prime. The family of representations $\{\rho_E\}$ is assumed to satisfy:

(a) Galois invariance: $\rho_E^\sigma \simeq \rho_E, \quad \forall \sigma \in \mathrm{Gal}(E/F)$,

(b) Compatibility: $\rho_E|_{\Gamma_{EE'}} \simeq \rho_{E'}|_{\Gamma_{EE'}}$,

for all E and E' in $\mathcal{I}$. These conditions are certainly necessary for the ρ_E to be of the form $\rho|_{\Gamma_E}$ for a representation ρ of Γ_F. What we will show, is that in fact (a) and (b) are also sufficient conditions if $\mathcal{I}$ is large enough. That is,

Definition 1. *For a finite set S of places of F, we say that $\mathcal{I}$ is S-general if and only if the following holds: For any finite place $v \notin S$, and any finite extension M of F, there is an $E \in \mathcal{I}$ linearly disjoint from M such that v splits completely in E. In this case, there will be infinitely many such E.*

Recall that since E is Galois over F, it is linearly disjoint from M precisely when $E \cap M = F$. Moreover, it is of prime degree, so this just means E is not contained in M. Hence, $\mathcal{I}$ being S-general is equivalent to: For $v \notin S$, there are infinitely many $E \in \mathcal{I}$ in which v splits. A slightly stronger condition is:

Definition 2. *We say that $\mathcal{I}$ is strongly S-general if and only if the following holds: For any finite set Σ of places of F, disjoint from S, there is an $E \in \mathcal{I}$ in which every $v \in \Sigma$ splits completely.*

To see that this is *stronger*, we use an argument from [Har, v.1]: Fix a finite place $v \notin S$, and a finite extension M of F. Clearly we may assume $M \neq F$ is Galois. Let $\{M_i\}$ be the subfields of M, Galois over F, with a simple Galois group. For each i we then choose a place v_i of F, not in S, which does *not* split completely in M_i. We take Σ to be $\{v, v_i\}$ in the above definition, and get an $E \in \mathcal{I}$ in which v and every v_i splits. If E was contained in M, it would be one of the M_i, but this contradicts the choice of v_i. Thus, E and M are disjoint.

Example. Let $\Sigma = \{p_i\}$ be a finite set of primes. As is well-known, for odd p_i,

$$p_i \text{ splits in } \mathbb{Q}(\sqrt{d}) \Longleftrightarrow p_i \nmid d \text{ and } \left(\frac{d}{p_i}\right) = 1.$$

Here d is any square-free integer. Moreover, 2 splits when $d \equiv 1 \bmod 8$. The set of *all* integers d satisfying the congruences $d \equiv 1 \bmod p_i$, for all i, form an arithmetic progression. By Dirichlet's Theorem, it contains infinitely many primes. Therefore, the following family of imaginary quadratic extensions

$$\mathcal{I} = \{\mathbb{Q}(\sqrt{-p}) \colon \text{almost all primes } p\}$$

is strongly $\varnothing$-general. This gives rise to a similar family of CM extensions of any given totally real field F, by taking the set of all the composite fields $F\mathcal{I}$.

The main result of this section, is a strengthening of a Proposition in [Har, v.1]:

Lemma 1. *Let $\mathcal{I}$ be an S-general set of extensions E over F, of prime degree q_E, and let ρ_E be a family of semisimple Galois representations satisfying the conditions (a) and (b) above. Then there is a continuous semisimple*

$$\rho : \Gamma_F \to GL_n(\bar{\mathbb{Q}}_\ell), \quad \rho|_{\Gamma_E} \simeq \rho_E,$$

for **all** *$E \in \mathcal{I}$. This determines the representation ρ uniquely up to isomorphism.*

Proof. The proof below is strongly influenced by the proof in [Har, v.1], and the proof of Theorem VII.1.9 in [HT]. We simply include more details and clarifications. The proof is quite long and technical, so we divide it into several steps. Before we construct ρ, we start off with noting that it is necessarily *unique*: Indeed, for any place $v \notin S$, we find an $E \in \mathcal{I}$ in which v splits. In particular, $E_w = F_v$ for all places w of E dividing v. Thus, all the restrictions $\rho|_{\Gamma_{F_v}}$ are uniquely determined. We conclude that ρ is unique, by the Cebotarev Density Theorem. For the construction of ρ, we first establish some *notation* used throughout the proof: We fix an arbitrary *base* point $E_0 \in \mathcal{I}$, and abbreviate

$$\rho_0 \stackrel{\text{df}}{=} \rho_{E_0}, \quad \Gamma_0 \stackrel{\text{df}}{=} \Gamma_{E_0}, \quad G_0 \stackrel{\text{df}}{=} \text{Gal}(E_0/F), \quad q_0 \stackrel{\text{df}}{=} q_{E_0}.$$

We let H denote the Zariski closure of $\rho_0(\Gamma_0)$ inside $\text{GL}_n(\bar{\mathbb{Q}}_\ell)$, and consider its identity component H°. Define M to be the finite Galois extension of E_0 with

$$\Gamma_M = \rho_0^{-1}(H^\circ), \quad \text{Gal}(M/E_0) = \pi_0(H).$$

Let T be the set of isomorphism classes of irreducible constituents of ρ_0, ignoring multiplicities. By property (a), the group G_0 acts on T from the right. We note that τ and τ^σ occur in ρ_0 with the same multiplicity. We want to describe the G_0-orbits on T. First, we have the set P of fixed points $\tau = \tau^\sigma$ for all σ. The set of non-trivial orbits is denoted by C. Note that any $c \in C$ has prime cardinality q_0. For each such c, we pick a representative $\tau_c \in T$, and let C_0 be the set of all these representatives $\{\tau_c\}$. Each $\tau \in C_0$ obviously has a trivial stabilizer in G_0.

Step 1: *The extensions of ρ_0 to Γ_F.*

Firstly, a standard argument shows that each $\tau \in P$ has an extension $\tilde{\tau}$ to Γ_F. This uses the divisibility of $\bar{\mathbb{Q}}_\ell^*$, in order to find a suitable intertwining operator $\tau \simeq \tau^\sigma$. All the other extensions are then obtained from $\tilde{\tau}$ as unique twists:

$$\tilde{\tau} \otimes \eta, \quad \eta \in \hat{G}_0.$$

Here $\hat{G}_0$ is the group of characters of G_0. Secondly, for a $\tau \in c$, we introduce

$$\tilde{\tau} \overset{\text{df}}{=} \text{Ind}_{\Gamma_0}^{\Gamma_F}(\tau).$$

Since τ is *not* Galois-invariant, this $\tilde{\tau}$ is irreducible. It depends only on the orbit c containing τ, and it is invariant under twisting by $\hat{G}_0$. It has restriction

$$\tilde{\tau}|_{\Gamma_0} \simeq \bigoplus\nolimits_{\sigma \in G_0} \tau^\sigma.$$

If we let m_τ denote the multiplicity with which $\tau \in T$ occurs in ρ_0, we get that

$$\{\bigoplus_{\tau \in C_0} m_\tau \cdot \tilde{\tau}\} \oplus \{\bigoplus_{\tau \in P} \bigoplus\nolimits_{\eta \in \hat{G}_0} m_{\tilde{\tau},\eta} \cdot (\tilde{\tau} \otimes \eta)\}$$

is an extension of ρ_0 to Γ_F for all choices of non-negative $m_{\tilde{\tau},\eta} \in \mathbb{Z}$ such that

$$\sum\nolimits_{\eta \in \hat{G}_0} m_{\tilde{\tau},\eta} = m_\tau$$

for every fixed $\tau \in P$.

Step 2: *$\rho_0(\Gamma_{NE_0})$ is dense in H, when N is linearly disjoint from M over F.*

To see this, let us momentarily denote the Zariski closure of $\rho_0(\Gamma_{NE_0})$ by H_N. N is a finite extension, so H_N has finite index in H. Consequently, we deduce that $H_N^\circ = H^\circ$. Now, NE_0 and M are linearly disjoint over E_0, and therefore

$$\Gamma_0 = \Gamma_{NE_0} \cdot \Gamma_M \Longrightarrow \rho_0(\Gamma_0) \subset \rho_0(\Gamma_{NE_0}) \cdot H^\circ \subset H_N.$$

Taking the closure, we obtain that $H_N = H$.

Step 3: *If N is a finite extension of F, linearly disjoint from M over F. Then:*

(1) $\tau|_{\Gamma_{NE_0}}$ is irreducible, for all $\tau \in T$.

(2) $\tilde{\tau}|_{\Gamma_N}$ is irreducible, for all $\tau \in P$.

(3) $\tau|_{\Gamma_{NE_0}} \simeq \tau'|_{\Gamma_{NE_0}} \Rightarrow \tau \simeq \tau'$, for all $\tau, \tau' \in T$.

(4) $\tilde{\tau}|_{\Gamma_N} \simeq (\tilde{\tau}' \otimes \eta)|_{\Gamma_N} \Rightarrow \tau \simeq \tau'$ and $\eta = 1$, for all $\tau, \tau' \in P$ and $\eta \in \hat{G}_0$.

Parts (1) and (3) follow immediately from Step 2, and obviously (1) implies (2). Also, part (3) immediately implies that $\tau \simeq \tau'$ in (4). Suppose $\eta \in \hat{G}_0$ satisfies:

$$\tilde{\tau}|_{\Gamma_N} \simeq \tilde{\tau}|_{\Gamma_N} \otimes \eta|_{\Gamma_N}$$

for some $\tau \in P$. In other words, $\eta|_{\Gamma_N}$ occurs in $\mathrm{End}_{\Gamma_{NE_0}}(\tilde{\tau}|_{\Gamma_N})$, which is trivial by part (1). So, η is trivial on Γ_N and on Γ_0. Hence, $\eta = 1$ by disjointness.

Step 4: *$\tilde{\tau}|_{\Gamma_N}$ is irreducible for all $\tau \in C_0$. That is, part (2) holds for all $\tau \in T$.*

Since N and E_0 are linearly disjoint over F, we see that $\Gamma_F = \Gamma_E \cdot \Gamma_0$. Hence,

$$\tilde{\tau}|_{\Gamma_N} = \mathrm{Ind}_{\Gamma_0}^{\Gamma_F}(\tau)|_{\Gamma_N} \simeq \mathrm{Ind}_{\Gamma_{NE_0}}^{\Gamma_N}(\tau|_{\Gamma_{NE_0}}),$$

by Mackey theory. Now, $\tau|_{\Gamma_{NE_0}}$ is irreducible and *not* Galois-invariant.

Step 5: *Suppose $E \in \mathcal{I}$ is linearly disjoint from M over F. Then, for a unique choice of non-negative integers $m_{\tilde{\tau},\eta,E}$ with η-sum m_τ, we have the formula:*

$$\rho_E \simeq \{\bigoplus_{\tau \in C_0} m_\tau \cdot \tilde{\tau}|_{\Gamma_E}\} \oplus \{\bigoplus_{\tau \in P} \bigoplus\nolimits_{\eta \in \hat{G}_0} m_{\tilde{\tau},\eta,E} \cdot (\tilde{\tau} \otimes \eta)|_{\Gamma_E}\}.$$

In particular, ρ_0 and ρ_E have a <u>common</u> extension to Γ_F.

The uniqueness of the $m_{\tilde{\tau},\eta,E}$ follows directly from part (4) in Step 3. Recall,

$$\rho_E|_{\Gamma_{EE_0}} \simeq \rho_0|_{\Gamma_{EE_0}} \simeq \{\bigoplus_{\tau \in C_0} m_\tau \cdot \bigoplus\nolimits_{\sigma \in G_0} \tau|^{\sigma}_{\Gamma_{EE_0}}\} \oplus \{\bigoplus_{\tau \in P} m_\tau \cdot \tau|_{\Gamma_{EE_0}}\},$$

by the compatibility condition (b). Here all the $\tau|^{\sigma}_{\Gamma_{EE_0}}$ are distinct by (3). First, let us pick an arbitrary $\tau \in P$. As representations of G_0, viewed as the Galois group of EE_0 over E by disjointness, we have

$$\mathrm{Hom}_{\Gamma_{EE_0}}(\tilde{\tau}|_{\Gamma_E}, \rho_E) \simeq \bigoplus\nolimits_{\eta \in \hat{G}_0} \dim_{\bar{\mathbb{Q}}_\ell} \mathrm{Hom}_{\Gamma_E}((\tilde{\tau} \otimes \eta)|_{\Gamma_E}, \rho_E) \cdot \eta.$$

The $\bar{\mathbb{Q}}_\ell$-dimension of the left-hand side clearly equals m_τ, and the right-hand side defines the partition $m_{\tilde{\tau},\eta,E}$ of m_τ. Next, let us pick an arbitrary $\tau \in C_0$. By the same argument, using that $\tilde{\tau}$ is invariant under twisting by $\hat{G}_0$, we get:

$$\mathrm{Hom}_{\Gamma_{EE_0}}(\tilde{\tau}|_{\Gamma_E}, \rho_E) \simeq \dim_{\bar{\mathbb{Q}}_\ell} \mathrm{Hom}_{\Gamma_E}(\tilde{\tau}|_{\Gamma_E}, \rho_E) \cdot \bigoplus\nolimits_{\eta \in \hat{G}_0} \eta.$$

Now the left-hand side obviously has dimension $m_\tau q_0$. We deduce that $\tilde{\tau}|_{\Gamma_E}$ occurs in ρ_E with multiplicity m_τ. Counting dimensions, we obtain the desired decomposition of ρ_E. Note that we have not used the Galois invariance of ρ_E. In fact, it is a *consequence* of the above argument, assuming $E \cap M = F$.

Step 6: *Fix an $E_1 \in \mathcal{I}$ disjoint from M over F. Introduce the representation*

$$\rho \stackrel{\mathrm{df}}{=} \{\bigoplus_{\tau \in C_0} m_\tau \cdot \tilde{\tau}\} \oplus \{\bigoplus_{\tau \in P} \bigoplus\nolimits_{\eta \in \hat{G}_0} m_{\tilde{\tau},\eta,E_1} \cdot (\tilde{\tau} \otimes \eta)\}.$$

Then $\rho|_{\Gamma_E} \simeq \rho_E$ for all extensions $E \in \mathcal{I}$ linearly disjoint from ME_1 over F.

By definition, and Step 5, we have that $\rho|_{\Gamma_{E_1}} \simeq \rho_{E_1}$. Take $E \in \mathcal{I}$ to be any extension, disjoint from ME_1 over F. We compare the decomposition of $\rho|_{\Gamma_E}$,

$$\rho|_{\Gamma_E} = \{\bigoplus_{\tau \in C_0} m_\tau \cdot \tilde{\tau}|_{\Gamma_E}\} \oplus \{\bigoplus_{\tau \in P} \bigoplus\nolimits_{\eta \in \hat{G}_0} m_{\tilde{\tau},\eta,E_1} \cdot (\tilde{\tau} \otimes \eta)|_{\Gamma_E}\},$$

to the decomposition of ρ_E in Step 5. We need to show the multiplicities match:

$$m_{\tilde{\tau},\eta,E} = m_{\tilde{\tau},\eta,E_1}, \quad \forall \tau \in P, \quad \forall \eta \in \hat{G}_0.$$

By property (b), for the pair $\{E, E_1\}$, we know that $\rho|_{\Gamma_E}$ and ρ_E become isomorphic after restriction to Γ_{EE_1}. Once we prove EE_1 is linearly disjoint from M over F, we are done by (2) and (4). The disjointness folllows immediately:

$$EE_1 \otimes_F M \simeq E \otimes_F E_1 \otimes_F M \simeq E \otimes_F ME_1 \simeq EE_1M.$$

Step 7: $\rho|_{\Gamma_E} \simeq \rho_E$ *for* **all** $E \in \mathcal{I}$.

By Step 6, we may assume $E \in \mathcal{I}$ is *contained* in ME_1. Now take an auxiliary extension $\mathcal{E} \in \mathcal{I}$ linearly disjoint from ME_1 over F. Consequently, using (b),

$$\rho|_{\Gamma_{\mathcal{E}}} \simeq \rho_{\mathcal{E}} \Rightarrow \rho|_{\Gamma_{E\mathcal{E}}} \simeq \rho_{\mathcal{E}}|_{\Gamma_{E\mathcal{E}}} \simeq \rho_E|_{\Gamma_{E\mathcal{E}}}.$$

Thus, $\rho|_{\Gamma_E}$ agrees with ρ_E when restricted to $\Gamma_{E\mathcal{E}}$. It suffices to show that the union of these subgroups $\Gamma_{E\mathcal{E}}$, as $\mathcal{E}$ varies, is dense in Γ_E. Again, we invoke the Cebotarev Density Theorem. Indeed, let w be a place of E, lying above $v \notin S$. It is then enough to find an $\mathcal{E} \in \mathcal{I}$, as above, such that w splits completely in $E\mathcal{E}$. Then Γ_{E_w} is contained in $\Gamma_{E\mathcal{E}}$. We *know*, by the S-generality of $\mathcal{I}$, that we can find an $\mathcal{E} \in \mathcal{I}$, not contained in ME_1, in which v splits completely. This $\mathcal{E}$ works: This follows from elementary splitting theory, as E and $\mathcal{E}$ are disjoint.

This finishes the proof of the patching lemma. $\square$

Remark. From the proof above, we infer the following concrete description of the patch-up representation ρ. First fix *any* $E_0 \in \mathcal{I}$, and let P be the set of Galois-invariant constituents τ of ρ_{E_0}. For each such τ, we fix an extension $\tilde{\tau}$ to F once and for all. Furthermore, let C_0 be a set of representatives for the non-trivial Galois orbits of constituents of ρ_{E_0}. Then ρ is of the following form

$$\rho \simeq \{\bigoplus_{\tau \in C_0} m_\tau \cdot \mathrm{Ind}_{\Gamma_0}^{\Gamma_F}(\tau)\} \oplus \{\bigoplus_{\tau \in P} \bigoplus\nolimits_{\eta \in \mathrm{Gal}(E_0/F)^\wedge} m_{\tilde{\tau},\eta} \cdot (\tilde{\tau} \otimes \eta)\}.$$

Here the $m_{\tilde{\tau},\eta}$ are *some* non-negative integers with η-sum m_τ, the multiplicity of τ in ρ_{E_0}. This fairly explicit description may be useful in deriving properties of ρ from those of ρ_{E_0}.

2 Patching: Solvable extensions

We finish with a discussion of the generalization of the patching lemma to *solvable* extensions. Thus, $\mathcal{I}$ now denotes a collection of solvable Galois extensions E over F, and we assume we are given Galois representations ρ_E, as above, satisfying (a) and (b). For any extension L over F, let us introduce

$$\mathcal{I}_L \stackrel{\mathrm{df}}{=} \{E \in \mathcal{I} : L \subset E\}.$$

Loosely speaking, we say that $\mathcal{I}$ is S-general if it is S-general in prime layers:

Definition 3. *For a finite set S of places of F, we say that $\mathcal{I}$ is S-general if and only if the following holds: For every L such that $\mathcal{I}_L \neq \varnothing$,*

$$\{\textit{prime degree extensions } K/L, \textit{ with } \mathcal{I}_K \neq \varnothing\}$$

is $S(L)$-general in the previous sense. $S(L)$ denotes the places of L above S.

From now on, we will make the additional hypothesis that all the extensions $E \in \mathcal{I}$ have uniformly *bounded heights*. That is, there is an integer $H_{\mathcal{I}}$ such that every index $[E : F]$ has at most $H_{\mathcal{I}}$ prime divisors (not necessarily distinct).

Lemma 2. *Assume the collection $\mathcal{I}$ has uniformly bounded heights. Then $\mathcal{I}$ is S-general if and only if the following condition holds for every L with $\mathcal{I}_L \neq \varnothing$: Given a finite place $w \notin S(L)$ and a finite extension M over L, there is an extension $E \in \mathcal{I}_L$ linearly disjoint from M over L, in which w splits completely.*

Proof. The *if* part follows immediately by unraveling the definitions. The *only if* part is proved by induction on the maximal height of the collection $\mathcal{I}_L$ over L, the height *one* case being the definition. Suppose $\mathcal{I}_L$ has maximal height H, and assume the lemma holds for smaller heights. Let w and M be as above. By S-generality, there is a prime degree extension K over L with $\mathcal{I}_K \neq \varnothing$, disjoint from M over L, in which w splits. Fix a place $\tilde{w}$ of K above w. Now, $\mathcal{I}_K$ clearly has maximal height *less* than H. By the induction hypothesis there is an $E \in \mathcal{I}_K$, disjoint from MK over K, in which $\tilde{w}$ splits. This E works. □

Under the above assumptions on $\mathcal{I}$, a given place $w \notin S(L)$ splits completely in infinitely many $E \in \mathcal{I}_L$, unless L belongs to $\mathcal{I}$. One has a stronger notion:

Definition 4. *We say that $\mathcal{I}$ is strongly S-general if and only if the following holds: For any L such that $\mathcal{I}_L \neq \varnothing$, and any finite set Σ of places of L disjoint from $S(L)$, there is an $E \in \mathcal{I}_L$ in which every $v \in \Sigma$ splits completely.*

As in the prime degree case, treated above, one shows that this is indeed a stronger condition. Our next goal is to prove the following generalization of the patching lemma to certain collections of solvable extensions:

Theorem 1. *Let $\mathcal{I}$ be an S-general collection of solvable Galois extensions E over F, with uniformly bounded heights, and let ρ_E be a family of n-dimensional continuous semisimple ℓ-adic Galois representations satisfying the conditions (a) and (b) above. Then there is a continuous semisimple representation*

$$\rho : \Gamma_F \to GL_n(\bar{\mathbb{Q}}_\ell), \quad \rho|_{\Gamma_E} \simeq \rho_E,$$

for **all** *$E \in \mathcal{I}$. This determines the representation ρ uniquely up to isomorphism.*

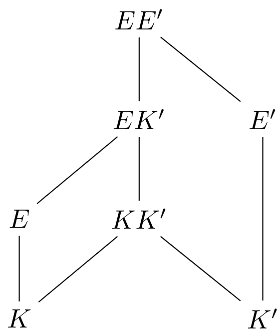
Proof. Uniqueness is proved by paraphrasing the argument in the prime degree situation. The existence of ρ is proved by induction on the maximal height of $\mathcal{I}$ over F, the height one case being the previous patching lemma. Suppose $\mathcal{I}$ has maximal height H, and assume the Theorem holds for smaller heights. Take an arbitrary prime degree extension K over F, with $\mathcal{I}_K \neq \varnothing$. Clearly $\mathcal{I}_K$ is an $S(K)$-general set of solvable Galois extensions of K, of maximal height strictly smaller than H. Moreover, the subfamily $\{\rho_E\}_{E \in \mathcal{I}_K}$ obviously satisfies (a) and (b). By induction, we find a continuous semisimple ℓ-adic representation

$$\rho_K : \Gamma_K \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell), \quad \rho_K|_{\Gamma_E} \simeq \rho_E,$$

for all $E \in \mathcal{I}_K$. We then wish to apply the prime degree patching lemma to the family $\{\rho_K\}$, as K varies over extensions as above. By definition, such K do form an S-general collection over F. It remains to show that $\{\rho_K\}$ satisfies (a) and (b). To check property (a), take any $\sigma \in \Gamma_F$, and note that ρ_K^σ agrees with ρ_K after restriction to Γ_E for an arbitrary extension $E \in \mathcal{I}_K$. The union of these Γ_E is dense in Γ_K by the Cebotarev Density Theorem: Every place w of K, outside $S(K)$, splits in some $E \in \mathcal{I}_K$, so the union contains Γ_{K_w}. To check property (b), fix prime degree extensions K and K' as above. Note that

$$(\rho_K|_{\Gamma_{KK'}})|_{\Gamma_{EE'}} \simeq (\rho_{K'}|_{\Gamma_{KK'}})|_{\Gamma_{EE'}}, \quad \forall E \in \mathcal{I}_K, \quad \forall E' \in \mathcal{I}_{K'}.$$

We finish the proof by showing that the union of these $\Gamma_{EE'}$ is dense in $\Gamma_{KK'}$. Let w be an arbitrary place of KK' such that $w|_F$ does not lie in S. Choose an extension $E \in \mathcal{I}_K$ linearly disjoint from KK' over K, in which $w|_K$ splits. Then pick an extension $E' \in \mathcal{I}_{K'}$ linearly disjoint from EK' over K', in which $w|_{K'}$ splits. By elementary splitting theory, w splits in EK', and any place of EK' above w splits in EE'. Consequently, w splits in EE', see the diagram:



The union then contains the Galois group of $(KK')_w$. Done by Cebotarev. □

The previous result should be compared to the patching lemma in [Har, v.1].

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