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We show that local-global compatibility (at split primes) away from p holds at *all* points of the p-adic eigenvariety of a definite n-variable unitary group. We do this by interpolating the local Langlands correspondence for GL_n across the eigenvariety by considering the fibers of its defining coherent sheaf. We employ techniques of Chenevier and Scholze used in Scholze's proof of the local Langlands conjecture for GL_n.

1. Introduction

The goal of this paper is to study the interpolation the local Langlands correspondence across eigenvarieties of definite unitary groups, in the spirit of earlier works [Paulin 2011; Bellaïche and Chenevier 2009; Chenevier 2009]. Our approach is based on the construction of eigenvarieties in [Emerton 2006c] and utilizes techniques from Scholze's proof [2013b] of the local Langlands conjecture for GL_n. In the next few paragraphs we introduce notation in order to state our main result (Theorem 1.1 below).

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Let $p > 2$ be a prime, and fix an unramified CM extension F/F^+ which is split at all places v of F^+ above p. Suppose $U_{/F^+}$ is a unitary group in n variables which is quasisplit at all finite places and compact at infinity (see 2A for more details). Throughout Σ is a finite set of finite places of F^+ containing $\Sigma_p = \{v : v \mid p\}$, and we let $\Sigma_0 = \Sigma \backslash \Sigma_p$. We assume all places $v \in \Sigma$ split in F and we choose a divisor $\tilde{v}|v$ once and for all, which we use to make the identification $U(F_v^+) \xrightarrow{\sim} \mathrm{GL}_n(F_{\tilde{v}})$. We consider tame levels of the form $K^p = K_{\Sigma_0} K^{\Sigma}$, where $K^{\Sigma} = \prod_{v \notin \Sigma} K_v$ is a product of hyperspecial maximal compact subgroups, and $K_{\Sigma_0} = \prod_{v \in \Sigma_0} K_v$.

Our coefficient field is a sufficiently large finite extension $E/\mathbb{Q}_p$ with integers $\mathcal{O}$ and residue field $k = k_E$, and we start off with an absolutely irreducible[1] Galois representation $\bar{r} : \mathrm{Gal}_F \to \mathrm{GL}_n(k)$ which is automorphic of tame level K^p. We let $\mathfrak{m} = \mathfrak{m}_{\bar{r}}$ be the associated maximal ideal, viewed in various Hecke algebras (see Sections 2C and 2D for more details). In Sections 2E and 3B we introduce the universal deformation ring $R_{\bar{r}}$ and the deformation space $X_{\bar{r}} = \mathrm{Spf}(R_{\bar{r}})^{\mathrm{rig}}$. Each point $x \in X_{\bar{r}}$ carries a Galois representation r_x, which is a deformation of $\bar{r}$, and we let $\mathfrak{p}_x \subset R_{\bar{r}}$ be the associated prime ideal. The Banach representation of p-adic automorphic forms $\hat{S}(K^p, E)_{\mathfrak{m}}$ inherits a natural $R_{\bar{r}}$-module structure, and we consider its $\mathfrak{p}_x$-torsion $\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]$ and its dense subspace of locally analytic vectors $\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]^{\mathrm{an}}$, see Section 2B.

The eigenvariety $Y(K^p, \bar{r}) \subset X_{\bar{r}} \times \hat{T}$ equals the support of a certain coherent sheaf $\mathcal{M}$ on $X_{\bar{r}} \times \hat{T}$. Here $\hat{T}$ denotes the character space of the p-adic torus $T \subset U(F^+ \otimes \mathbb{Q}_p)$ isomorphic to $\prod_{v \mid p} T_{\mathrm{GL}(n)}(F_{\tilde{v}})$, see Section 3C below. We have $\hat{T} \simeq \mathcal{W} \times (\mathbb{G}_m^{\mathrm{rig}})^{n|\Sigma_p|}$, where $\mathcal{W}$ is weight space (parametrizing continuous characters of the maximal compact subgroup of T) which is a disjoint union of finitely many open unit balls of dimension $n[F^+ : \mathbb{Q}]$. By definition a point $y = (x, \delta) \in X_{\bar{r}} \times \hat{T}$ belongs to the eigenvariety $Y(K^p, \bar{r})$ if and only if the fiber $\mathcal{M}_y$ is nonzero. If y is E-rational the E-linear dual of $\mathcal{M}_y$ can be described as

$$\mathcal{M}'_y \simeq J_B^{\delta}(\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]^{\mathrm{an}}),$$

where J_B denotes Emerton's [2006a] locally analytic variant of the Jacquet functor and J_B^{δ} means the δ-eigenspace. Morally our main result states that $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_y$ interpolates the local Langlands correspondence for GL_n across the eigenvariety. In our formulation below we let $\pi_{x,v}$ be the irreducible smooth representation of $U(F_v^+) \xrightarrow{\sim} \mathrm{GL}_n(F_{\tilde{v}})$ associated with $r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}}$ via the local Langlands correspondence, i.e.,

$$\mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}})^{F-\mathrm{ss}} \simeq \mathrm{rec}\big(\mathrm{BC}_{\tilde{v}|v}(\pi_{x,v}) \otimes |\det|^{(1-n)/2}\big)$$

[1] This is mostly for convenience. The automorphic $\mathcal{O}$-lifts of $\bar{r}$ then arise from *cusp* forms on $\mathrm{GL}_n(\mathbb{A}_F)$, see Lemma 3.3.

with rec$(\cdot)$ normalized as in [Harris and Taylor 2001]. The notation $\mathrm{BC}_{\tilde{v}|v}(\pi_{x,v})$ signifies local base change, which simply amounts to viewing $\pi_{x,v}$ as a representation of $\mathrm{GL}_n(F_{\tilde{v}})$ via its identification with $U(F_v^+)$.

Here is the precise formulation of our main result.

Theorem 1.1. *Let $y = (x, \delta) \in Y(K^p, \bar{r})$ be an arbitrary point on the eigenvariety.*

(1) $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_y$ *has finite length as a $U(F^+_{\Sigma_0})$-representation, and every irreducible subquotient thereof has the same supercuspidal support as $\bigotimes_{v\in\Sigma_0} \pi_{x,v}$.*

(2) *If y is a point such that r_x is strongly generic at every $v \in \Sigma_0$* (see Definition 9.4 in the main text), *then there is an $m_y \in \mathbb{Z}_{>0}$ such that up to semisimplification*

$$\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_y \overset{\mathrm{ss}}{\simeq} \big(\bigotimes_{v\in\Sigma_0} \pi_{x,v}\big)^{\bigoplus m_y}.$$

When $\bigotimes_{v\in\Sigma_0} \pi_{x,v}$ is supercuspidal $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_y$ is semisimple.

(3) *If y is any point which appears at Iwahori level* (i.e., where the factors of K^p at places in Σ_0 are all Iwahori subgroups) *then $\bigotimes_{v\in\Sigma_0} \pi^{\mathrm{gen}}_{x,v}$ is the only generic irreducible subquotient of $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_y$ — and it* ***does*** *appear — where $\pi^{\mathrm{gen}}_{x,v}$ denotes the generic representation with the same supercuspidal support as $\pi_{x,v}$.*

Before proceeding we remark that part (1) is also know due to work of Bellaïche and Chenevier [2009] (finiteness) and Chenevier [2009] (compatibility with local Langlands).[2] A more detailed discussion of these works in relation to ours can be found in Section 10. Moving on, we note that part (1) of the theorem implies, in particular, that $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_y$ lies in the Bernstein component $\mathcal{R}^{\mathfrak{s}}(U(F^+_{\Sigma_0}))$ for the inertial class $\mathfrak{s}$ determined by y (see Section 9A). Our methods are based on p-adic interpolation of traces and do not give us any information about the monodromy operator.

The control of generic constituents in the case where K_{Σ_0} is a product of Iwahori subgroups (part (3) of the main theorem) is the most novel aspect of our paper; it employs a genericity criterion of Barbasch–Moy, recently generalized by Chan and Savin [2019]. In part (2) of Theorem 1.1 when $y = (x, \delta)$ is a point for which $\pi_{x,v}$ is supercuspidal for all $v \in \Sigma_0$ we can remove the "ss" since there are no self-extensions with central character that of $\pi_{x,v}$ (see Remark 9.6) by the projectivity and/or injectivity of $\pi_{x,v}$ in this category — this requires some attention to how the central character varies on the eigenvariety, see Section 8.

We expect that the length m_y of $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_y$ as a $U(F^+_{\Sigma_0})$-representation can be > 1 at certain singular points. If y is a classical point of noncritical slope (automatically étale by [Chenevier 2011, Theorem 4.10]) $m_y = 1$, see Proposition 4.2

[2] The latter part is [Chenevier 2009, Remarque 3.13], which the authors were unfortunately unaware of when making this paper public. We thank Chenevier for pointing it out to us.

below. Under certain mild nondegeneracy assumptions, m_y should be closely related to $\dim_E J_B^\delta(\Pi(\varrho_x)^{\mathrm{an}})$, which is finite by [Emerton 2007, Corollary 0.15]. Here $\varrho_x := \{r_x|_{\mathrm{Gal}_{F_{\tilde v}}}\}_{v\in\Sigma_p}$ and $\Pi(\varrho_x) := \widehat{\bigotimes}_{v\,|\,p}\Pi(r_x|_{\mathrm{Gal}_{F_{\tilde v}}})$, where $\Pi(\cdot)$ is the p-adic local Langlands correspondence for $\mathrm{GL}_n(F_{\tilde v})$ — as defined in [Caraiani et al. 2016] say, to fix ideas.[3] This expectation is based on the strong local-global compatibility results of [Emerton 2011; Chojecki and Sorensen 2017], which also seem to suggest that $\varinjlim_{K_{\Sigma_0}}\mathcal{M}'_y$ should in fact be semisimple — for generic points (otherwise the "generic" local Langlands correspondence gives a reducible indecomposable representation). We are not sure if this is an artifact of the $n=2$ case, or if it is supposed to be true more generally. It is certainly not true for trivial reasons since $\pi_{x,v}$ does admit nontrivial self-extensions. For example, by [Orlik 2005, Corollary 2] we have $\dim\mathrm{Ext}^i_{\mathrm{GL}_n}(\mathrm{St},\mathrm{St}) = \binom{n}{i}$. Even when $\pi_{x,v}$ is parabolically induced from a supercuspidal it does happen that $\mathrm{Ext}^1_{\mathrm{GL}_n(F_{\tilde v})}(\pi_{x,v},\pi_{x,v}) \neq 0$ (see Remark 9.6.).

We briefly outline the overall strategy behind the proof of Theorem 1.1: For classical points $y=(x,\delta)$ (i.e., those corresponding to automorphic representations) local-global compatibility away from p essentially gives an inclusion $\bigotimes_{v\in\Sigma_0}\pi_{x,v}\hookrightarrow\varinjlim_{K_{\Sigma_0}}\mathcal{M}'_y$ which is an isomorphism if δ moreover is of noncritical slope. We reinterpret this using ideas from Scholze's proof [2013b] of the local Langlands correspondence: he works with certain elements f_τ in the Bernstein center of $\mathrm{GL}_n(F_w)$, associated with $\tau\in W_{F_w}$, which act on an irreducible smooth representation Π via scaling by $\mathrm{tr}(\tau\,|\,\mathrm{rec}(\Pi))$; here and throughout this paragraph we ignore a twist by $|\det|^{(1-n)/2}$ for simplicity. For each tuple $\tau=(\tau_{\tilde v})\in\prod_{v\in\Sigma_0}W_{F_{\tilde v}}$ we thus have an element $f_\tau := \bigotimes_{v\in\Sigma_0} f_{\tau_{\tilde v}}$ of the Bernstein center of

$$U(F^+_{\Sigma_0}) \xrightarrow{\sim} \prod_{v\in\Sigma_0}\mathrm{GL}_n(F_{\tilde v}),$$

which we know how to evaluate on all irreducible smooth representations. In particular f_τ acts on $\varinjlim_{K_{\Sigma_0}}\mathcal{M}'_y$ via scaling by $\prod_{v\in\Sigma_0}\mathrm{tr}\big(\tau_{\tilde v}\,|\,\mathrm{rec}(\mathrm{BC}_{\tilde v|v}(\pi_{x,v}))\big)$ — still assuming y is classical and noncritical. Those points are Zariski dense in $Y(K^p,\bar r)$, and using this we interpolate this key scaling property to *all* points y as follows. By mimicking the standard proof of Grothendieck's monodromy theorem one can interpolate $\mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde v}}})$ in families. Namely, for each $\mathrm{Sp}(A)\subset X_{\bar r}$ we construct a Weil–Deligne representation $\mathrm{WD}_{\bar r,\tilde v}$ over A which specializes to $\mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde v}}})$ for all $x\in\mathrm{Sp}(A)$. Around the point y we find a neighborhood $\Omega\subset\mathrm{Sp}(A)\times\hat T$ and use the weight morphism $\omega: Y(K^p,\bar r)\to\mathcal{W}$, or rather its restriction $\omega|_\Omega$, to view $\Gamma(\Omega,\mathcal{M})$ as a finite type projective module over $\mathcal{O}_{\mathcal{W}}(\omega(\Omega))$, which allows us to show that f_τ acts on $\varinjlim_{K_{\Sigma_0}}\Gamma(\Omega,\mathcal{M})$ via scaling by $\prod_{v\in\Sigma_0}\mathrm{tr}(\tau_{\tilde v}\,|\,\mathrm{WD}_{\bar r,\tilde v})$. This is

[3] At least for the choice of $R_\infty\to\mathcal{O}$ in [Caraiani et al. 2016] compatible with $x: R_{\bar r}\to\mathcal{O}$ via the projection $R_\infty\twoheadrightarrow R_{\bar r}$.

the most technical part of our argument; in fact we glue and get the scaling property on the sheaf $\mathcal{M}$ itself. By specialization at y we deduce that f_τ acts on $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_y$ via scaling by $\prod_{v\in\Sigma_0} \mathrm{tr}\left(\tau_{\tilde{v}} | \mathrm{rec}(\mathrm{BC}_{\tilde{v}|v}(\pi_{x,v}))\right)$ as desired. This result tells us that every irreducible constituent $\bigotimes_{v\in\Sigma_0} \pi_v$ of $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_y$ has the same supercuspidal support as $\bigotimes_{v\in\Sigma_0} \pi_{x,v}$, and therefore is isomorphic to it if x is a strongly generic point. We also infer that $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_y$ has finite length since $\dim \mathcal{M}'_y < \infty$ and the constituents $\bigotimes_{v\in\Sigma_0} \pi_v$ have conductors bounded by the conductors of $\mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}})$.

Before finishing this introduction by discussing the structure of the paper, we wish to mention that Theorem 1.1 was motivated in part by the question of local-global compatibility for the Breuil–Herzig construction $\Pi(\rho)^{\mathrm{ord}}$, see [Breuil and Herzig 2015, Conjecture 4.2.5]. The latter is defined for upper triangular p-adic representations ρ of $\mathrm{Gal}_{\mathbb{Q}_p}$, and is supposed to model the largest subrepresentation of the "true" p-adic local Langlands correspondence built from unitary continuous principal series representations. We approach this problem starting from the inclusion (for unitary δ)

$$J_B^\delta(\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]^{\mathrm{an}}) \hookrightarrow \mathrm{ord}_B^\delta(\hat{S}(K^p, \mathcal{O})_{\mathfrak{m}}[\mathfrak{p}_x])[1/p]^{\mathrm{an}}, \tag{1-2}$$

as shown in [Sorensen 2017, Theorem 6.2]. Here ord_B is Emerton's functor of ordinary parts [Emerton 2010], which is right adjoint to parabolic induction $\mathrm{Ind}_{\overline{B}}$. If $y = (x, \delta)$ lies on $Y(K^p, \bar{r})$ the source of (1-2) is nonzero, and we deduce the existence of a nonzero map $\mathrm{Ind}_{\overline{B}}(\delta) \to \hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]$. If one could show that certain Weyl-conjugates $y_w = (x, w\delta)$ all lie on $Y(K^p, \bar{r})$ one would infer that there is a nontrivial map $\mathrm{soc}_{\mathrm{GL}_n(\mathbb{Q}_p)} \Pi(\rho)^{\mathrm{ord}} \to \hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]$ which one could hope to promote to a map $\Pi(\rho)^{\mathrm{ord}} \to \hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]$ using [Breuil and Herzig 2015, Corollary 4.3.11]. Here we take $\rho = r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}}$ (up to a twist which we ignore here) for some $v\,|\,p$ such that $F_{\tilde{v}} = \mathbb{Q}_p$, and x is a point where $r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}}$ is upper triangular with $\delta_{\tilde{v}}$ on the diagonal. In light of these speculations it is conceivable that Theorem 1.1 can be used to show *strong* local-global compatibility, in the sense that there is an embedding

$$\otimes_{v\in\Sigma_0}\pi_{x,v} \hookrightarrow \varinjlim_{K_{\Sigma_0}} \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Q}_p)}(\Pi(\rho)^{\mathrm{ord}}, \hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]).$$

Finally, we make a few remarks on the structure of the paper. In our first (rather lengthy) Section 2 we introduce in detail the notation and assumptions in force throughout; the unitary groups $U_{/F^+}$, automorphic forms $\hat{S}(K^p, E)$, Hecke algebras, Galois representations and their deformations. Section 3 then defines the eigenvarieties $Y(K^p, \bar{r})$ and the sheaves $\mathcal{M}_{K^p}$, essentially following [Breuil et al. 2017] and [Emerton 2006c]. In Section 4 we recall the notion of a noncritical classical point, and prove Theorem 1.1 for those. Section 5 interpolates the Weil–Deligne representations across reduced $\mathrm{Sp}(A) \subset X_{\bar{r}}$ by suitably adapting Grothendieck's argument. We recall Scholze's characterization of the local Langlands correspondence in Section 6, and introduce the functions f_τ in the Bernstein center. The goal of Section 7 is to show Proposition 7.9 on the action of f_τ on $\varinjlim_{K_{\Sigma_0}} \Gamma(\Omega, \mathcal{M}_{K^p})$, where Ω is a neighborhood of y as above. Finally in Section 9 we put the pieces together; we introduce the notion of a strongly generic point, and prove our main results. Section 9B focuses on the case where K_{Σ_0} is a product of Iwahori subgroups; we recall and use the genericity criterion of Chan–Savin to show the occurrence of $\bigotimes_{v\in\Sigma_0} \pi_{x,v}^{\mathrm{gen}}$.

2. Notation and terminology

We denote the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ of a field F by Gal_F.

2A. ***Unitary groups.*** Our setup will be identical to that of [Breuil et al. 2017] although we will adopt a slightly different notation, which we will introduce below.

We fix a CM field F with maximal totally real subfield F^+ and $\mathrm{Gal}(F/F^+) = \{1, c\}$. We assume the extension F/F^+ is unramified at all finite places, and split at all places $v \mid p$ of F^+ above a fixed prime p.

Let n be a positive integer. If n is even assume that $\frac{n}{2}[F^+ : \mathbb{Q}] \equiv 0 \bmod 2$. By [Clozel et al. 2008, §3.5] this guarantees the existence of a unitary group $U_{/F^+}$ in n variables such that

- $U \times_{F^+} F \xrightarrow{\sim} \mathrm{GL}_n$,
- U is quasisplit over F_v^+ (hence unramified) for all[4] finite places v,
- $U(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ is compact.

We let $G = \mathrm{Res}_{F^+/\mathbb{Q}}\, U$ be its restriction of scalars.

If v splits in F the choice of a divisor $w|v$ determines an isomorphism $i_w : U(F_v^+) \xrightarrow{\sim} \mathrm{GL}_n(F_w)$ well-defined up to conjugacy. Throughout we fix a finite set Σ of finite places of F^+ such that every $v \in \Sigma$ splits in F, and Σ contains $\Sigma_p = \{v : v \mid p\}$. We let $\Sigma_0 = \Sigma \backslash \Sigma_p$. We emphasize that unlike [Clozel et al. 2008] we do *not* assume the places in Σ_0 are banal.

For each $v \in \Sigma$ we choose a divisor $\tilde{v}|v$ once and for all and let $\tilde{\Sigma} = \{\tilde{v} : v \in \Sigma\}$. We also choose an embedding $\mathrm{Gal}_{F_{\tilde{v}}} \hookrightarrow \mathrm{Gal}_F$ for each such v. Moreover, we choose isomorphisms $i_{\tilde{v}}$ which we will tacitly use to identify $U(F_v^+)$ with $\mathrm{GL}_n(F_{\tilde{v}})$. For instance the collection $(i_{\tilde{v}})_{v\mid p}$ gives an isomorphism

$$G(\mathbb{Q}_p) = U(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p) \xrightarrow{\sim} \prod_{v\mid p} \mathrm{GL}_n(F_{\tilde{v}}). \tag{2-1}$$

[4]Convenient in Lemma 3.3 when considering local base change from $U(F_v^+)$ to $\mathrm{GL}_n(F_{\tilde{v}})$ — for unramified representations.

Similarly $U(F_{\Sigma}^{+}) \xrightarrow{\sim} \prod_{v\in\Sigma} \mathrm{GL}_n(F_{\tilde{v}})$ and analogously for $U(F_{\Sigma_0}^{+})$. When there is no risk of confusion we will just write G instead of $G(\mathbb{Q}_p)$. We let $B \subset G$ be the inverse image of the upper-triangular matrices under (2-1). In the same fashion T corresponds to the diagonal matrices, and N corresponds to the unipotent radical. Their opposites are denoted $\bar{B}$ and $\bar{N}$.

Below we will only consider tame levels $K^p \subset G(\mathbb{A}_f^p)$ of the form $K^p = \prod_{v\nmid p} K_v$, where $K_v \subset U(F_v^+)$ is a compact open subgroup which is assumed to be hyperspecial for $v \notin \Sigma$. Accordingly we factor it as $K^p = K_{\Sigma_0} K^{\Sigma}$, where $K^{\Sigma} = \prod_{v\notin\Sigma} K_v$ is a product of hyperspecials, and $K_{\Sigma_0} = \prod_{v\in\Sigma_0} K_v$.

2B. ***Automorphic forms.*** We work over a fixed finite extension $E/\mathbb{Q}_p$, which we assume is large enough in the sense that every embedding $F_v^+ \hookrightarrow \overline{\mathbb{Q}}_p$ factors through E for all $v \mid p$. We let $\mathcal{O}$ denote its valuation ring, ϖ is a choice of uniformizer, and $k = \mathcal{O}/(\varpi) \simeq \mathbb{F}_q$ is the residue field. We endow E with its normalized absolute value $|\cdot|$ for which $|\varpi| = q^{-1}$.

For a tame level $K^p \subset G(\mathbb{A}_f^p)$ we introduce the space of p-adic automorphic forms on $G(\mathbb{A})$ as follows (see Definition 3.2.3 in [Emerton 2006c]). First let

$$\hat{S}(K^p, \mathcal{O}) = \mathcal{C}\big(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K^p, \mathcal{O}\big) = \varprojlim_i \mathcal{C}^{\infty}\big(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K^p, \mathcal{O}/\varpi^i\mathcal{O}\big).$$

Here $\mathcal{C}$ is the space of continuous functions, $\mathcal{C}^{\infty}$ is the space of locally constant functions. Note that the space of locally constant functions in $\hat{S}(K^p, \mathcal{O})$ is ϖ-adically dense, so alternatively

$$\begin{aligned}\hat{S}(K^p, \mathcal{O}) &= \mathcal{C}^{\infty}\big(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K^p, \mathcal{O}\big)^{\wedge} \\ &= \varprojlim_i \mathcal{C}^{\infty}\big(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K^p, \mathcal{O}\big) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^i\mathcal{O}.\end{aligned}$$

These two viewpoints amount to thinking of $\hat{S}(K^p, \mathcal{O})$ as $\tilde{H}^0(K^p)$ or $\hat{H}^0(K^p)$ respectively in the notation of [Emerton 2006c], see (2.1.1) and Corollary 2.2.25 there. The reduction modulo ϖ is the space of mod p modular forms on $G(\mathbb{A})$,

$$S(K^p, k) = \mathcal{C}^{\infty}\big(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K^p, k\big) \simeq \hat{S}(K^p, \mathcal{O})/\varpi\hat{S}(K^p, \mathcal{O}),$$

which is an admissible (smooth) $k[G]$-module with $G = G(\mathbb{Q}_p)$ acting via right translations. Thus $\hat{S}(K^p, \mathcal{O})$ is a ϖ-adically admissible G-representation over $\mathcal{O}$, i.e., an object of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(\mathcal{O})$ (see Definition 2.4.7 in [Emerton 2010]). Since it is clearly flat over $\mathcal{O}$, it is the unit ball of a Banach representation

$$\hat{S}(K^p, E) = \hat{S}(K^p, \mathcal{O})[1/p] = \mathcal{C}\big(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K^p, E\big).$$

Here we equip the right-hand side with the supremum norm $\|f\| = \sup_{g\in G(\mathbb{A}_f)} |f(g)|$, and $\hat{S}(K^p, E)$ thus becomes an object of the category $\mathrm{Ban}_G(E)^{\leq 1}$ of Banach

E-spaces $(H, \|\cdot\|)$ for which $\|H\| \subset |E|$ endowed with an isometric G-action. $\hat{S}(K^p, E)$ is dubbed the space of p-adic automorphic forms on $G(\mathbb{A})$.

The connection to classical modular forms is through locally algebraic vectors as we now explain. Let V be an absolutely irreducible algebraic representation of $G \times_{\mathbb{Q}} E$. Thus V is a finite-dimensional E-vector space with an action of $G(E)$, which we restrict to $G(\mathbb{Q}_p)$. If $K_p \subset G(\mathbb{Q}_p)$ is a compact open subgroup we let it act on V and consider

$$S_V(K_pK^p, E) = \mathrm{Hom}_{K_p}(V, \hat{S}(K^p, E)).$$

If we assume E is large enough that $\mathrm{End}_G(V) = E$, the space of V-locally algebraic vectors in $\hat{S}(K^p, E)$ can be defined as the image of the natural map

$$\varinjlim_{K_p} V \otimes_E S_V(K_pK^p, E) \xrightarrow{\sim} \hat{S}(K^p, E)^{V-\mathrm{alg}} \hookrightarrow \hat{S}(K^p, E)$$

(see Proposition 4.2.4 in [Emerton 2017]). Then the space of all locally algebraic vectors decomposes as a direct sum $\hat{S}(K^p, E)^{\mathrm{alg}} = \bigoplus_V \hat{S}(K^p, E)^{V-\mathrm{alg}}$. Letting $\tilde{V}$ denote the contragredient representation, one easily identifies $S_V(K_pK^p, E)$ with the space of (necessarily continuous) functions

$$f : G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K^p \to \tilde{V}, \quad f(gk) = k^{-1} f(g), \quad \text{for all } k \in K_p.$$

In turn, considering the function $h(g) = gf(g)$ identifies it with the space of right K_pK^p-invariant functions $h : G(\mathbb{A}_f) \to \tilde{V}$ such that $h(\gamma g) = \gamma h(g)$ for all $\gamma \in G(\mathbb{Q})$. If we complexify this space along an embedding $\iota : E \hookrightarrow \mathbb{C}$ we obtain vector-valued automorphic forms. Thus we arrive at the decomposition

$$S_V(K_pK^p, E) \otimes_{E,\iota} \mathbb{C} \simeq \bigoplus_{\pi} m_G(\pi) \cdot \pi_p^{K_p} \otimes (\pi_f^p)^{K^p} \tag{2-2}$$

with π running over automorphic representations of $G(\mathbb{A})$ with $\pi_\infty \simeq V \otimes_{E,\iota} \mathbb{C}$. It is even known by now that all $m_G(\pi) = 1$, see [Mok 2015] and "the main global theorem" [Kaletha et al. 2014, Theorem 1.7.1, p. 89] (both based on the symplectic/orthogonal case [Arthur 2013]). Multiplicity one will be used below in Lemma 3.3.

Remark 2.3. For full disclosure we will only use multiplicity one for representations π whose base change $\Pi = \mathrm{BC}_{F/F^+}(\pi)$ to $\mathrm{GL}_n(\mathbb{A}_F)$ is cuspidal (see the proof of Lemma 3.3 below). Since Π_∞ is V-cohomological the Ramanujan conjecture holds in this case, i.e., Π is tempered. Therefore the packets in [Kaletha et al. 2014, Theorem 1.7.1] do not overlap and consist of irreducible representations; in particular $m_G(\pi) = 1$. Some of the authors of [Kaletha et al. 2014] have informed us that multiplicity one even holds for nontempered representations π, the point being that the groups $S^{\natural}_{\psi_v}$ in [loc. cit.] are abelian. As mentioned in the introduction to [loc. cit.], the nontempered case is the topic of a sequel.

2C. ***Hecke algebras.*** At each $v \nmid p$ we consider the Hecke algebra $\mathcal{H}(U(F_v^+), K_v)$ of K_v-biinvariant compactly supported functions $\phi : U(F_v^+) \to \mathcal{O}$ (with K_v-normalized convolution). The characteristic functions of double cosets $[K_v \gamma_v K_v]$ form an $\mathcal{O}$-basis.

Suppose v splits in F and K_v is hyperspecial. Choose a place $w|v$ and an isomorphism i_w which restricts to $i_w : K_v \xrightarrow{\sim} \mathrm{GL}_n(\mathcal{O}_{F_w})$. Then we identify $\mathcal{H}(U(F_v^+), K_v)$ with the spherical Hecke algebra for $\mathrm{GL}_n(F_w)$. We let $\gamma_{w,j} \in U(F_v^+)$ denote the element corresponding to

$$i_w(\gamma_{w,j}) = \mathrm{diag}(\underbrace{\varpi_{F_w}, \dots, \varpi_{F_w}}_{j}, 1, \dots, 1).$$

Then let $T_{w,j} = [K_v \gamma_{w,j} K_v]$ be the standard Hecke operators; $\mathcal{H}(U(F_v^+), K_v) = \mathcal{O}[T_{w,1}, \dots, T_{w,n}^{\pm 1}]$.

For a tame level K^p as above, the full Hecke algebra is the restricted tensor product relative to the characteristic functions char_{K_v} (below V runs over all finite sets of places $v \nmid p$):

$$\mathcal{H}(G(\mathbb{A}_f^p), K^p) = {\bigotimes_{v \nmid p}}' \mathcal{H}(U(F_v^+), K_v) = \varinjlim_V \Bigg(\bigotimes_{v \in V} \mathcal{H}(U(F_v^+), K_v) \Bigg).$$

It acts on $\hat{S}(K^p, E)$ by norm-decreasing morphisms, and hence preserves the unit ball $\hat{S}(K^p, \mathcal{O})$. This induces actions on $S(K^p, k)$ and $S_V(K_p K^p, E)$ as well given by the usual double coset operators. Let

$$\mathcal{H}(K_{\Sigma_0}) = \bigotimes_{v \in \Sigma_0} \mathcal{H}(U(F_v^+), K_v), \quad \mathcal{H}_s(K^{\Sigma}) = \bigotimes_{v \notin \Sigma \text{ split}} \mathcal{H}(U(F_v^+), K_v)$$

be the subalgebras of $\mathcal{H}(G(\mathbb{A}_f^p), K^p)$ generated by Hecke operators at $v \in \Sigma_0$, respectively $T_{w,1}, \dots, T_{w,n}^{\pm 1}$ for $v \notin \Sigma$ *split* in F and $w|v$ (the subscript s is for "split"). In what follows we ignore the Hecke action at the nonsplit places $v \notin \Sigma$. Note that $\mathcal{H}_s(K^{\Sigma})$ is commutative, but of course $\mathcal{H}(K_{\Sigma_0})$ need not be.

We define the Hecke polynomial $P_w(X) \in \mathcal{H}_s(K^{\Sigma})[X]$ to be

$$P_w(X) = X^n + \cdots + (-1)^j (Nw)^{j(j-1)/2} T_{w,j} X^{n-j} + \cdots + (-1)^n (Nw)^{n(n-1)/2} T_{w,n},$$

where Nw is the size of the residue field $\mathcal{O}_{F_w}/(\varpi_{F_w})$.

We denote by $\mathbb{T}_V(K_p K^p, \mathcal{O})$ the subalgebra of $\mathrm{End}\big(S_V(K_p K^p, E)\big)$ generated by the operators $\mathcal{H}_s(K^{\Sigma})$. This is reduced and finite over $\mathcal{O}$. In case V is the trivial representation we write $\mathbb{T}_0(K_p K^p, \mathcal{O})$. As K_p shrinks there are surjective transition maps between these (given by restriction) and we let

$$\hat{\mathbb{T}}(K^p, \mathcal{O}) = \varprojlim_{K_p} \mathbb{T}_0(K_p K^p, \mathcal{O}),$$

equipped with the projective limit topology (each term being endowed with the ϖ-adic topology). We refer to it as the "big" Hecke algebra. $\hat{\mathbb{T}}(K^p, \mathcal{O})$ clearly acts faithfully on $\hat{S}(K^p, E)$ and one can easily show that the natural map $\mathcal{H}_s(K^\Sigma) \to \hat{\mathbb{T}}(K^p, \mathcal{O})$ has dense image, see the discussion in [Emerton 2011, 5.2].

A maximal ideal $\mathfrak{m} \subset \mathcal{H}_s(K^\Sigma)$ is called automorphic (of tame level K^p) if it arises as the pullback of a maximal ideal in some $\mathbb{T}_V(K_p K^p, \mathcal{O})$. Shrinking K_p if necessary we may assume it is pro-p, in which case we may take V to be trivial ("Shimura's principle"). In particular there are only finitely many such $\mathfrak{m}$, and we interchangeably view them as maximal ideals of $\hat{\mathbb{T}}(K^p, \mathcal{O})$ (and use the same notation), which thus factors as a finite product of complete local $\mathcal{O}$-algebras

$$\hat{\mathbb{T}}(K^p, \mathcal{O}) = \prod_{\mathfrak{m}} \hat{\mathbb{T}}(K^p, \mathcal{O})_{\mathfrak{m}}.$$

Correspondingly we have a decomposition $\hat{S}(K^p, E) = \bigoplus_{\mathfrak{m}} \hat{S}(K^p, E)_{\mathfrak{m}}$, and similarly for $\hat{S}(K^p, \mathcal{O})$. This direct sum is clearly preserved by $\mathcal{H}(K_{\Sigma_0})$.

2D. *Galois representations.* If R is an $\mathcal{O}$-algebra, and $r : \mathrm{Gal}_F \to \mathrm{GL}_n(R)$ is an arbitrary representation which is unramified at all places w of F lying above a split $v \notin \Sigma$, we associate the eigensystem $\theta_r : \mathcal{H}_s(K^\Sigma) \to R$ determined by

$$\det(X - r(\mathrm{Frob}_w)) = \theta_r(P_w(X)) \in R[X]$$

for all such w. Here Frob_w denotes a geometric Frobenius. (Note that the coefficients of the polynomial determine $\theta_r(T_{w,j})$ since $Nw \in \mathcal{O}^\times$; and $\theta_r(T_{w,n}) \in R^\times$.) We say r is automorphic (for G) if θ_r factors through one of the quotients $\mathbb{T}_V(K_p K^p, \mathcal{O})$.

When $R = \mathcal{O}$ this means r is associated with one of the automorphic representations π contributing to (2-2) in the sense that $T_{w,j}$ acts on $\pi_v^{K_v}$ by scaling by $\iota(\theta_r(T_{w,j}))$ for all $w|v \notin \Sigma$ as above. Conversely, it is now known that to any such π (and a choice of isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$) one can attach a unique semisimple Galois representation $r_{\pi,\iota} : \mathrm{Gal}_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ with that property, see [Thorne 2012, Theorem 6.5] for a nice summary. It is polarized, meaning that $r_{\pi,\iota}^\vee \simeq r_{\pi,\iota}^c \otimes \epsilon^{n-1}$, where ϵ is the cyclotomic character, and one can explicitly write down its Hodge–Tate weights in terms of V.

When $R = k$ we let $\mathfrak{m}_r = \ker(\theta_r)$ be the corresponding maximal ideal of $\mathcal{H}_s(K^\Sigma)$. Then r is automorphic precisely when $\mathfrak{m}_r$ is automorphic, in which case we tacitly view it as a maximal ideal of $\mathbb{T}_V(K_p K^p, \mathcal{O})$ (with residue field k) for suitable V and K_p. In the other direction, starting from a maximal ideal $\mathfrak{m}$ in $\mathbb{T}_V(K_p K^p, \mathcal{O})$ (whose residue field is necessarily a finite extension of k) one can attach a unique semisimple representation

$$\bar{r}_{\mathfrak{m}} : \mathrm{Gal}_F \to \mathrm{GL}_n(\mathbb{T}_V(K_p K^p, \mathcal{O})/\mathfrak{m})$$

such that $\theta_{\bar{r}_{\mathfrak{m}}}(T_{w,j}) = T_{w,j} + \mathfrak{m}$ (and which is polarized), see [Thorne 2012, Proposition 6.6]. We say $\mathfrak{m}$ is non-Eisenstein if $\bar{r}_{\mathfrak{m}}$ is absolutely irreducible. Under this hypothesis $\bar{r}_{\mathfrak{m}}$ admits a (polarized) lift

$$r_{\mathfrak{m}} : \mathrm{Gal}_F \to \mathrm{GL}_n(\mathbb{T}_V(K_p K^p, \mathcal{O})_{\mathfrak{m}})$$

with the property that $\theta_{r_{\mathfrak{m}}}(T_{w,j}) = T_{w,j}$; it is unique up to conjugation, see [Thorne 2012, Proposition 6.7], and gives a well-defined deformation of $\bar{r}_{\mathfrak{m}}$. If we let K_p shrink to a pro-p subgroup we may take V to be trivial, i.e., $\mathfrak{m} \subset \mathbb{T}_{\mathbf{1}}(K_p K^p, \mathcal{O})$. Passing to the inverse limit yields a lift of $\bar{r}_{\mathfrak{m}}$ with coefficients in $\hat{\mathbb{T}}(K^p, \mathcal{O})_{\mathfrak{m}}$ which we will denote by $\hat{r}_{\mathfrak{m}}$. Throughout [Thorne 2012] it is assumed that $p > 2$; we adopt that hypothesis here.

All the representations discussed above ($r_{\pi,\iota}$, $\bar{r}_{\mathfrak{m}}$, $r_{\mathfrak{m}}$ etc.) extend[5] to continuous homomorphisms $\mathrm{Gal}_{F^+} \to \mathcal{G}_n(R)$ for various R, where $\mathcal{G}_n$ is the group scheme (over $\mathbb{Z}$) defined as a semidirect product $\{1, j\} \ltimes (\mathrm{GL}_n \times \mathrm{GL}_1)$, see [Thorne 2012, Definition 2.1]. We let $\nu : \mathcal{G}_n \to \mathrm{GL}_1$ be the natural projection. Thus $\nu \circ \bar{r}_{\mathfrak{m}} = \epsilon^{1-n}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}$ (and similarly for $r_{\mathfrak{m}}$), where δ_{F/F^+} is the nontrivial quadratic character of $\mathrm{Gal}(F/F^+)$ and $\mu_{\mathfrak{m}} \in \{0, 1\}$ is determined by the congruence $\mu_{\mathfrak{m}} \equiv n \bmod 2$ (see [Clozel et al. 2008, Theorem 3.5.1; Bellaïche and Chenevier 2011, Theorem 1.2]).

2E. ***Deformations.*** Now start with $\bar{r} : \mathrm{Gal}_{F^+} \to \mathcal{G}_n(k)$ such that its restriction $\bar{r} : \mathrm{Gal}_F \to \mathrm{GL}_n(k)$ is absolutely irreducible and automorphic, with corresponding maximal ideal $\mathfrak{m} = \mathfrak{m}_{\bar{r}}$, and $\nu \circ \bar{r} = \epsilon^{1-n}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}$. In particular $\bar{r}$ is unramified outside Σ.

We consider lifts and deformations of $\bar{r}$ to rings in $\mathcal{C}_{\mathcal{O}}$, the category of complete local Noetherian $\mathcal{O}$-algebras R with residue field $k \xrightarrow{\sim} R/\mathfrak{m}_R$, see [Thorne 2012, Definition 3.1]. Recall that a lift is a homomorphism $r : \mathrm{Gal}_{F^+} \to \mathcal{G}_n(R)$ such that r reduces to $\bar{r}$ mod $\mathfrak{m}_R$, and $\nu \circ r = \epsilon^{1-n}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}$ (thought of as taking values in $R^\times$). A deformation is a $(1+\mathfrak{m}_R M_n(R))$-conjugacy class of lifts.

For each $v \in \Sigma$ consider the restriction $\bar{r}_{\tilde{v}} = \bar{r}|_{\mathrm{Gal}_{F_{\tilde{v}}}}$ and its universal lifting ring $R^{\square}_{\bar{r}_{\tilde{v}}}$. Following [Thorne 2012] we let $\bar{R}^{\square}_{\bar{r}_{\tilde{v}}}$ denote its maximal reduced p-torsion free quotient, and consider the deformation problem

$$\mathcal{S} = \big(F/F^+, \Sigma, \tilde{\Sigma}, \mathcal{O}, \bar{r}, \epsilon^{1-n}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}, \{\bar{R}^{\square}_{\bar{r}_{\tilde{v}}}\}_{v\in\Sigma}\big).$$

The functor $\mathrm{Def}_{\mathcal{S}}$ of deformations of type $\mathcal{S}$ is then represented by an object $R^{\mathrm{univ}}_{\mathcal{S}}$ of $\mathcal{C}_{\mathcal{O}}$, see [Thorne 2012, Proposition 3.4] or [Clozel et al. 2008, Proposition 2.2.9].

[5]Once a choice of $\gamma_0 \in \mathrm{Gal}_{F^+} - \mathrm{Gal}_F$ is made, see [Clozel et al. 2008, Lemma 2.1.4]. See also Proposition 3.4.4 therein.

In what follows we will simply write $R_{\bar{r}}$ instead of $R_{\mathcal{S}}^{\mathrm{univ}}$, and keep in mind the underlying deformation problem $\mathcal{S}$. Similarly, $R_{\bar{r}}^{\square}$ is the universal lifting ring of type $\mathcal{S}$ (which is denoted by $R_{\mathcal{S}}^{\square}$ in [Thorne 2012, Proposition 3.4]). Note that $R_{\bar{r}}^{\square}$ is a power series $\mathcal{O}$-algebra in $|\Sigma|n^2$ variables over $R_{\bar{r}}$ ([Clozel et al. 2008, Proposition 2.2.9]); a fact we will not use in this paper.

The universal automorphic deformation $r_{\mathfrak{m}}$ is of type $\mathcal{S}$, so by universality it arises from a local homomorphism

$$\psi : R_{\bar{r}} \to \mathbb{T}_V(K_pK^p, \mathcal{O})_{\mathfrak{m}}.$$

These maps are compatible as we shrink K_p. Taking V to be trivial and passing to the inverse limit over K_p we obtain a map $\hat{\psi} : R_{\bar{r}} \to \hat{\mathbb{T}}(K^p, \mathcal{O})_{\mathfrak{m}}$ which we use to view $\hat{S}(K^p, E)_{\mathfrak{m}}$ as an $R_{\bar{r}}$-module.

3. Eigenvarieties

3A. *Formal schemes and rigid spaces.* In what follows $(\cdot)^{\mathrm{rig}}$ will denote Berthelot's functor (which generalizes Raynaud's construction for topologically finite type formal schemes $\mathfrak{X}$ over $\mathrm{Spf}(\mathcal{O})$, see [Raynaud 1974]). Its basic properties are nicely reviewed in [de Jong 1995, Chapter 7]. The source $\mathrm{FS}_{\mathcal{O}}$ is the category of locally Noetherian adic formal schemes $\mathfrak{X}$ which are formally of finite type over $\mathrm{Spf}(\mathcal{O})$ (i.e., their reduction modulo an ideal of definition is of finite type over $\mathrm{Spec}(k)$); the target Rig_E is the category of rigid analytic varieties over E, see Definition 9.3.1/4 in [Bosch et al. 1984]. For example, $\mathbb{B} = (\mathrm{Spf}\,\mathcal{O}\{y\})^{\mathrm{rig}}$ is the closed unit disc (at 0); $\mathbb{U} = (\mathrm{Spf}\,\mathcal{O}[[x]])^{\mathrm{rig}}$ is the open unit disc. For a general affine formal scheme $\mathfrak{X} = \mathrm{Spf}(A)$, where

$$A = \mathcal{O}\{y_1, \dots, y_r\}[\![x_1, \dots, x_s]\!]/(g_1, \dots, g_t),$$

$\mathfrak{X}^{\mathrm{rig}} \subset \mathbb{B}^r \times \mathbb{U}^s$ is the closed analytic subvariety cut out by the functions $g_1, \dots, g_t$, see [Bosch et al. 1984, 9.5.2]. In general $\mathfrak{X}^{\mathrm{rig}}$ is obtained by gluing affine pieces as in [de Jong 1995, §7.2]. The construction of $\mathfrak{X}^{\mathrm{rig}}$ in the affine case is actually completely canonical and free from coordinates: If $I \subset A$ is the largest ideal of definition, $A[I^n/\varpi]$ is the subring of $A \otimes_{\mathcal{O}} E$ generated by A and all i/ϖ with $i \in I^n$. Let $A[I^n/\varpi]^\wedge$ be its I-adic completion (equivalently, its ϖ-adic completion, see the proof of [de Jong 1995, Lemma 7.1.2]). Then $A[I^n/\varpi]^\wedge \otimes_{\mathcal{O}} E$ is an affinoid E-algebra and there is an admissible covering

$$\mathfrak{X}^{\mathrm{rig}} = \mathrm{Spf}(A)^{\mathrm{rig}} = \bigcup_{n=1}^{\infty} \mathrm{Sp}\big(A[I^n/\varpi]^\wedge \otimes_{\mathcal{O}} E\big).$$

In particular $A^{\mathrm{rig}} := \mathcal{O}(\mathrm{Spf}(A)^{\mathrm{rig}}) = \varprojlim_n A[I^n/\varpi]^\wedge \otimes_{\mathcal{O}} E$. The natural map $A \otimes_{\mathcal{O}} E \to A^{\mathrm{rig}}$ factors through the ring of bounded functions on $\mathrm{Spf}(A)^{\mathrm{rig}}$; the image of A lies in $\mathcal{O}^0(\mathrm{Spf}(A)^{\mathrm{rig}})$, the functions whose absolute value is bounded by 1, see [de Jong 1995, 7.1.8].

3B. ***Deformation space.*** We let $X_{\bar{r}} = \mathrm{Spf}(R_{\bar{r}})^{\mathrm{rig}}$ (a subvariety of $\mathbb{U}^s$ for some s). For a point $x \in X_{\bar{r}}$ we let $\kappa(x)$ denote its residue field, which is a finite extension of E, and let $\kappa(x)^0$ be its valuation ring; an $\mathcal{O}$-algebra with finite residue field $k(x)$. Note the different meanings of $\kappa(x)$ and $k(x)$. The evaluation map $R_{\bar{r}} \to \mathcal{O}^0(X_{\bar{r}}) \to \kappa(x)^0$ corresponds to a deformation

$$r_x : \mathrm{Gal}_{F^+} \to \mathcal{G}_n(\kappa(x)^0)$$

of $\bar{r} \otimes_k k(x)$. (We tacitly choose a representative r_x in the conjugacy class of lifts.) We let $\mathfrak{p}_x = \ker(R_{\bar{r}} \to \kappa(x)^0)$ be the prime ideal of $R_{\bar{r}}$ corresponding to x, see the bijection in [de Jong 1995, Lemma 7.1.9]. We will often assume for notational simplicity that x is E-rational, in which case $\kappa(x) = E$ and $k(x) = k$; so that r_x is a deformation of $\bar{r}$ over $\kappa(x)^0 = \mathcal{O}$.

3C. ***Character and weight space.*** Recall our choice of torus $T \subset G(\mathbb{Q}_p)$, and let T_0 be its maximal compact subgroup. Upon choosing uniformizers $\{\varpi_{F_{\tilde{v}}}\}_{v|p}$ we have an isomorphism $T \simeq T_0 \times \mathbb{Z}^{n|\Sigma_p|}$ of topological groups. Moreover,

$$T_0 \simeq \prod_{v|p} (\mathcal{O}_{F_{\tilde{v}}}^{\times})^n \simeq \underbrace{\Big(\prod_{v|p} \mu_\infty(F_{\tilde{v}})^n\Big)}_{\mu} \times \mathbb{Z}_p^{n[F^+:\mathbb{Q}]}.$$

Let $\hat{T} := \mathcal{W} \times (\mathbb{G}_m^{\mathrm{rig}})^{n|\Sigma_p|}$, where $\mathcal{W} := \big(\mathrm{Spf}(\mathcal{O}[[T_0]])\big)^{\mathrm{rig}}$. The weight space $\mathcal{W}$ is isomorphic to $|\mu|$ copies of the open unit ball $\mathbb{U}^{n[F^+:\mathbb{Q}]}$. From a more functorial point of view $\hat{T}$ represents the functor which takes an affinoid E-algebra to the set $\mathrm{Hom}_{\mathrm{cont}}(T, A^\times)$, and similarly for $\mathcal{W}$ and T_0. See [Emerton 2017, Proposition 6.4.5]. Thus $\hat{T}$ carries a universal continuous character $\delta^{\mathrm{univ}} : T \to \mathcal{O}(\hat{T})^\times$ which restricts to a character $T_0 \to \mathcal{O}^0(\mathcal{W})^\times$ via the canonical morphism $\hat{T} \to \mathcal{W}$. Henceforth we identify points of $\hat{T}$ with continuous characters $\delta : T \to \kappa(\delta)^\times$ for varying finite extensions $\kappa(\delta)$ of E (and analogously for $\mathcal{W}$).

3D. ***Definition of the eigenvariety.*** We follow [Breuil et al. 2017, §4.1] in defining the eigenvariety $Y(K^p, \bar{r})$ as the support of a certain coherent sheaf $\mathcal{M} = \mathcal{M}_{K^p}$ on $X_{\bar{r}} \times \hat{T}$. This is basically also the approach taken in Section (2.3) of [Emerton 2006c], except there $X_{\bar{r}}$ is replaced by Spec of a certain Hecke algebra. We define $\mathcal{M}$ as follows.

Let $(\cdot)^{\mathrm{an}}$ be the functor from [Schneider and Teitelbaum 2003, Theorem 7.1]. It takes an object H of $\mathrm{Ban}_G^{\mathrm{adm}}(E)$ to the dense subspace H^{an} of locally analytic vectors. H^{an} is a locally analytic G-representation (over E) of compact type whose strong dual $(H^{an})'$ is a coadmissible $D(G, E)$-module, see [Schneider and Teitelbaum 2003, p. 176].

We take $H = \hat{S}(K^p, E)_{\mathfrak{m}}$ and arrive at an admissible locally analytic G-representation $\hat{S}(K^p, E)_{\mathfrak{m}}^{an}$ which we feed into the Jacquet functor J_B defined in [Emerton 2006a, Definition 3.4.5]. By Theorem 0.5 of [loc. cit.] this yields an *essentially* admissible locally analytic T-representation $J_B(\hat{S}(K^p, E)_{\mathfrak{m}}^{an})$. See [Emerton 2017, Definition 6.4.9] for the notion of essentially admissible (the difference with admissibility lies in incorporating the action of the center Z, or rather viewing the strong dual as a module over $\mathcal{O}(\hat{Z}) \hat{\otimes} D(G, E)$).

We recall [Emerton 2006c, Proposition 2.3.2]: If $\mathcal{F}$ is a coherent sheaf on $\hat{T}$, see [Bosch et al. 1984, Definition 9.4.3/1], its global sections $\Gamma(\hat{T}, \mathcal{F})$ is a coadmissible $\mathcal{O}(\hat{T})$-module. Moreover, the functor $\mathcal{F} \rightsquigarrow \Gamma(\hat{T}, \mathcal{F})$ is an equivalence of categories (since $\hat{T}$ is quasi-Stein). Note that $\Gamma(\hat{T}, \mathcal{F})$ and its strong dual both acquire a T-action via δ^{univ}. Altogether the functor $\mathcal{F} \rightsquigarrow \Gamma(\hat{T}, \mathcal{F})'$ sets up an antiequivalence of categories between coherent sheaves on $\hat{T}$ and essentially admissible locally analytic T-representations (over E).

As pointed out at the end of Section 2E, $\hat{S}(K^p, E)_{\mathfrak{m}}$ is an $R_{\bar{r}}$-module via $\hat{\psi}$, and the G-action is clearly $R_{\bar{r}}$-linear. Thus $J_B(\hat{S}(K^p, E)_{\mathfrak{m}}^{an})$ inherits an $R_{\bar{r}}$-module structure. By suitably modifying the remarks of the preceding paragraph (as in Section 3.1 of [Breuil et al. 2017] where they define and study locally $R_{\bar{r}}$-analytic vectors, see Definition 3.2 in [loc. cit.]) one finds that there is a coherent sheaf $\mathcal{M} = \mathcal{M}_{K^p}$ on $X_{\bar{r}} \times \hat{T}$ for which

$$J_B(\hat{S}(K^p, E)_{\mathfrak{m}}^{an}) \simeq \Gamma(X_{\bar{r}} \times \hat{T}, \mathcal{M})'.$$

The *eigenvariety* is then defined as the (schematic) support of $\mathcal{M}$, see [Bosch et al. 1984, Proposition 9.5.2/4]. I.e.,

$$Y(K^p, \bar{r}) := \sup(\mathcal{M}) = \{y = (x, \delta) : \mathcal{M}_y \neq 0\} \subset X_{\bar{r}} \times \hat{T}.$$

Thus $Y(K^p, \bar{r})$ is an analytic subset of $X_{\bar{r}} \times \hat{T}$ with structure sheaf $\mathcal{O}_{X_{\bar{r}} \times \hat{T}}/\mathcal{I}$, where $\mathcal{I}$ is the ideal sheaf of annihilators of $\mathcal{M}$. That is $\mathcal{I}(U) = \mathrm{Ann}_{\mathcal{O}(U)}\Gamma(U, \mathcal{M})$ for admissible open U. One can show that $Y(K^p, \bar{r})$ is reduced, see part (3) of Lemma 7.7 below for precise references.

The fiber $\mathcal{M}_y = \big(\varinjlim_{U \ni y} \Gamma(U, \mathcal{M})\big) \otimes_{\mathcal{O}_{Y(K^p, \bar{r}), y}} \kappa(y)$ is finite-dimensional over $\kappa(y)$. Suppose $\kappa(y) \simeq E$ solely to simplify the notation. Then the full E-linear dual $\mathcal{M}_y' = \mathrm{Hom}_E(\mathcal{M}_y, E)$ has the following useful description.

Lemma 3.1. *Let $y = (x, \delta) \in (X_{\bar{r}} \times \hat{T})(E)$ be an E-rational point. Then there is an isomorphism*

$$\mathcal{M}_y' \simeq J_B^{\delta}(\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]^{an}). \tag{3-2}$$

(Here J_B^δ means the δ-eigenspace of J_B, and $[\mathfrak{p}_x]$ means taking $\mathfrak{p}_x$-torsion.)

Proof. First, since $X_{\bar{r}} \times \hat{T}$ is quasi-Stein, $\mathcal{M}_y$ is the largest quotient of $\Gamma(X_{\bar{r}} \times \hat{T}, \mathcal{M})$ which is annihilated by $\mathfrak{p}_x$ and on which T acts via δ, see [Breuil et al. 2017, §5.4]. Thus $\mathcal{M}'_y$ is the largest subspace of $J_B(\hat{S}(K^p, E)^{\mathrm{an}}_{\mathfrak{m}})$ with the same properties, i.e., $J_B^\delta(\hat{S}(K^p, E)^{\mathrm{an}}_{\mathfrak{m}})[\mathfrak{p}_x]$, as observed in Proposition 2.3.3(iii) of [Emerton 2006c]. Now,

$$J_B^\delta(\hat{S}(K^p, E)^{\mathrm{an}}_{\mathfrak{m}})[\mathfrak{p}_x] = J_B^\delta(\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]^{\mathrm{an}})$$

as follows easily from the exactness of $(\cdot)^{\mathrm{an}}$ and the left-exactness of J_B (using that $\mathfrak{p}_x$ is finitely generated to reduce to the principal case by induction on the number of generators), see the proof of [Breuil et al. 2017, Proposition 3.7]. □

The space in (3-2) can be made more explicit: Choose a compact open subgroup $N_0 \subset N$ and introduce the monoid $T^+ = \{t \in T : tN_0t^{-1} \subset N_0\}$. Then by [Emerton 2006a, Proposition 3.4.9],

$$J_B^\delta(\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]^{\mathrm{an}}) \simeq (\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]^{\mathrm{an}})^{N_0, T^+ = \delta},$$

where T^+ acts by double coset operators $[N_0 t N_0]$ on the space on the right. Observe that y lies on the eigenvariety $Y(K^p, \bar{r})$ precisely when the above space $\mathcal{M}'_y$ is nonzero.

Note that the Hecke algebra $\mathcal{H}(K_{\Sigma_0})$ acts on $J_B(\hat{S}(K^p, E)^{\mathrm{an}}_{\mathfrak{m}})$, and therefore on $\mathcal{M}$ and its fibers $\mathcal{M}_y$ (on the *right* since we are taking duals). The isomorphism (3-2) is $\mathcal{H}(K_{\Sigma_0})$-equivariant, and our first goal is to describe $\mathcal{M}'_y$ as a $\mathcal{H}(K_{\Sigma_0})$-module.

3E. *Classical points.* We say that a point $y = (x, \delta) \in Y(K^p, \bar{r})(E)$ is *classical* (of weight V) if the following conditions hold (see [Breuil et al. 2017, Definition 3.14] or the paragraph before [Emerton 2006c, Definition 0.6]):

(1) $\delta = \delta_{\mathrm{alg}}\delta_{\mathrm{sm}}$, where δ_{alg} is an algebraic character which is dominant relative to B (i.e., obtained from an element of $X^*(T \times_{\mathbb{Q}} E)^+$ by restriction to T), and δ_{sm} is a smooth character of T. In this case let V denote the irreducible algebraic representation of $G \times_{\mathbb{Q}} E$ of highest weight δ_{alg}.

(2) There exists an automorphic representation π of $G(\mathbb{A})$ such that
   (a) $(\pi_f^p)^{K^p} \neq 0$ and the $\mathcal{H}_s(K^\Sigma)$-action on this space is given by the eigensystem $\iota \circ \theta_{r_x}$,
   (b) $\pi_\infty \simeq V \otimes_{E,\iota} \mathbb{C}$,
   (c) π_p is a quotient of $\mathrm{Ind}_{\bar{B}}^G(\delta_{\mathrm{sm}}\delta_B^{-1})$.

These points comprise the subset $Y(K^p, \bar{r})_{\mathrm{cl}}$. Note that condition (a) is equivalent to the isomorphism $r_x \simeq r_{\pi,\iota}$ (both sides are irreducible since r_x is a lift of $\bar{r}$). In (c) δ_B denotes the modulus character of B; the reason we include it in condition (c) will become apparent in the proof of Proposition 4.2 below.

Lemma 3.3. *There is at most one automorphic π satisfying* (a)–(c) *above*; *and* $m_G(\pi) = 1$.

Proof. Let $\Pi = \mathrm{BC}_{F/F^+}(\pi)$ be a (strong) base change of π to $\mathrm{GL}_n(\mathbb{A}_F)$, where we view π as a representation of $U(\mathbb{A}_{F^+}) = G(\mathbb{A})$. For its existence see [Labesse 2011, §5.3]. Note that Π is cuspidal since $r_{\pi,\iota}$ is irreducible. In particular Π is globally generic, hence locally generic. By local-global compatibility, see [Barnet-Lamb et al. 2012; 2014; Caraiani 2014] for places $w|p$; [Taylor and Yoshida 2007; Shin 2011] for places $w\nmid p$,

$$\iota\,\mathrm{WD}(r_{\pi,\iota}|_{\mathrm{Gal}_{F_w}})^{F-\mathrm{ss}} \simeq \mathrm{rec}(\Pi_w \otimes |\det|^{(1-n)/2})$$

for *all* finite places w of F, with the local Langlands correspondence $\mathrm{rec}(\cdot)$ normalized as in [Harris and Taylor 2001]. This shows that Π_w is completely determined by r_x at all finite places w. Moreover, we have $\Pi_w = \mathrm{BC}_{w|v}(\pi_v)$ whenever the local base change on the right is defined, i.e., when either v splits or π_v is unramified. Our assumption that Σ consists of split places guarantees that $\mathrm{BC}_{w|v}(\pi_v)$ makes sense locally everywhere. Furthermore, unramified local base change is injective according to [Mínguez 2011, Corollary 4.2]. We conclude that π_f is determined by r_x, and $\pi_\infty \simeq V \otimes_{E,\iota} \mathbb{C}$. Thus π is unique. Multiplicity one was noted earlier at the end of Section 2B above, see Remark 2.3. □

4. The case of classical points of noncritical slope

Each point $x \in X_{\bar{r}}$ carries a Galois representation $r_x : \mathrm{Gal}_F \to \mathrm{GL}_n(\kappa(x))$ which we restrict to the various decomposition groups $\mathrm{Gal}_{F_{\tilde{v}}}$ for $v \in \Sigma$. When $v \in \Sigma_0$ there is a corresponding Weil–Deligne representation, see Section (4.2) in [Tate 1979], and we let $\pi_{x,v}$ be the representation of $U(F_v^+)$ (over $\kappa(x)$) such that

$$\mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}})^{F-\mathrm{ss}} \simeq \mathrm{rec}\big(\mathrm{BC}_{\tilde{v}|v}(\pi_{x,v}) \otimes |\det|^{(1-n)/2}\big) \tag{4-1}$$

Note that the local base change $\mathrm{BC}_{\tilde{v}|v}(\pi_{x,v})$ is just $\pi_{x,v}$ thought of as a representation of $\mathrm{GL}_n(F_{\tilde{v}})$ via the isomorphism $i_{\tilde{v}} : U(F_v^+) \xrightarrow{\sim} \mathrm{GL}_n(F_{\tilde{v}})$. We emphasize that $\pi_{x,v}$ is defined even for *non*classical points on the eigenvariety. If $y = (x, \delta)$ happens to be classical, $\pi_{x,v} \otimes_{E,\iota} \mathbb{C} \simeq \pi_v$, where π is the automorphic representation in Lemma 3.3. Below we relate $\bigotimes_{v\in\Sigma_0} \pi_{x,v}$ to the fiber $\mathcal{M}'_y$.

Proposition 4.2. *Let $y = (x, \delta) \in Y(K^p, \bar{r})(E)$ be a classical point. Then there exists an embedding of $\mathcal{H}(K_{\Sigma_0})$-modules $\bigotimes_{v\in\Sigma_0} \pi_{x,v}^{K_v} \hookrightarrow \mathcal{M}'_y$ which is an isomorphism if δ is of noncritical slope*, (see [Emerton 2006a, Definition 4.4.3], which is summarized below).

Proof. According to (0.14) in [Emerton 2006a] there is a closed embedding

$$J_B\big((\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]^{\mathrm{an}})^{V-\mathrm{alg}}\big) \hookrightarrow J_B\big(\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]^{\mathrm{an}}\big)^{V^N-\mathrm{alg}}.$$

Note that $V^N \simeq \delta_{\mathrm{alg}}$ so after passing to δ-eigenspaces we get a closed embedding

$$J_B^{\delta}\big((\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]^{\mathrm{an}})^{V-\mathrm{alg}}\big) \hookrightarrow J_B^{\delta}\big(\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]^{\mathrm{an}}\big). \tag{4-3}$$

The target is exactly $\mathcal{M}'_y$ by (3-2). On the other hand

$$(\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]^{\mathrm{an}})^{V-\mathrm{alg}} \simeq \bigoplus_{\pi} (V \otimes_E \pi_p) \otimes_E (\pi_f^p)^{K^p}$$

with π running over automorphic representations of $G(\mathbb{A})$ over E with $\pi_\infty \simeq V$ and such that θ_{r_x} gives the action of $\mathcal{H}_s(K^\Sigma)$ on $(\pi_f^p)^{K^p}$. As noted in Lemma 3.3 there is precisely one such π which we will denote by π_x throughout this proof (consistent with the notation $\pi_{x,v}$ introduced above). Note that $\bigotimes_{v \notin \Sigma} \pi_{x,v}^{K_v}$ is a line so

$$(\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]^{\mathrm{an}})^{V-\mathrm{alg}} \simeq (V \otimes_E \pi_{x,p}) \otimes_E \big(\textstyle\bigotimes_{v \in \Sigma_0} \pi_{x,v}^{K_v}\big).$$

Since J_B is compatible with the classical Jacquet functor, see [Emerton 2006a, Proposition 4.3.6], we identify the source of (4-3) with

$$(V^N \otimes_E (\pi_{x,p})_N)^{T=\delta} \otimes_E \big(\textstyle\bigotimes_{v \in \Sigma_0} \pi_{x,v}^{K_v}\big).$$

Now $V^N \simeq \delta_{\mathrm{alg}}$ is one-dimensional, and so is $(\pi_{x,p})_N^{T=\delta_{\mathrm{sm}}}$. Indeed, by Bernstein second adjointness,

$$(\pi_{x,p})_N^{T=\delta_{\mathrm{sm}}} \simeq \mathrm{Hom}_G\big(\mathrm{Ind}_{\bar{B}}^G(\delta_{\mathrm{sm}}\delta_B^{-1}), \pi_{x,p}\big).$$

The right-hand side is nonzero by condition (c) above, and in fact it is a line since $\mathrm{Ind}_{\bar{B}}^G(\delta_{\mathrm{sm}}\delta_B^{-1})$ has a unique generic constituent (namely $\pi_{x,p}$, see the proof of Lemma 3.3) which occurs with multiplicity one; this follows from the theory of derivatives [Bernstein and Zelevinsky 1977, Chapter 4]. From this observation we immediately infer that $\mathrm{Hom}_G\big(\tilde{\pi}_{x,p}, \mathrm{Ind}_{\bar{B}}^G(\delta_{\mathrm{sm}}^{-1}\delta_B)\big)$ is one-dimensional. To summarize, (4-3) is an embedding $\bigotimes_{v \in \Sigma_0} \pi_{x,v}^{K_v} \hookrightarrow \mathcal{M}'_y$. Finally, since $\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]^{\mathrm{an}}$ clearly admits a G-invariant norm (the sup norm), Theorem 4.4.5 in [Emerton 2006a] tells us that (4-3) is an isomorphism if δ is of noncritical slope. □

To aid the reader we briefly recall the notion of noncritical slope: To each $\delta \in \hat{T}(E)$ we assign the element $\mathrm{slp}(\delta) \in X^*(T \times_{\mathbb{Q}} E)$ defined as follows, see [Emerton 2006a, Definition 1.4.2]. First note that there is a natural surjection $T(E) \twoheadrightarrow X_*(T \times_{\mathbb{Q}} E)$; the cocharacter $\mu_t \in X_*(T \times_{\mathbb{Q}} E)$ associated with $t \in T(E)$ is given by $\langle \chi, \mu_t \rangle = \mathrm{ord}_E \chi(t)$ for all algebraic characters χ (here ord_E is the valuation on E normalized such that $\mathrm{ord}_E(\varpi_E) = 1$). Then the slope of δ is the algebraic character $\mathrm{slp}(\delta)$ satisfying $\langle \mathrm{slp}(\delta), \mu_t \rangle = \mathrm{ord}_E \delta(t)$ for all $t \in T$.

Definition 4.4. Let $\varrho = \frac{1}{2}\sum_{\alpha > 0} \alpha$. We say that $\delta = \delta_{\mathrm{alg}}\delta_{\mathrm{sm}}$ is of noncritical slope if there is *no* simple root α for which the element $s_\alpha(\delta_{\mathrm{alg}} + \varrho) + \mathrm{slp}(\delta_{\mathrm{sm}}) + \varrho$ lies in the $\mathbb{Q}_{\geq 0}$-cone generated by all simple roots.

5. Interpolation of the Weil–Deligne representations

Our goal in this section is to interpolate across deformation space $X_{\bar{r}}$, the Weil–Deligne representations $\mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}})$, for a fixed $v \in \Sigma_0$. More precisely, for any affinoid subvariety $\mathrm{Sp}(A) \subset X_{\bar{r}}$ we will define a rank n Weil–Deligne representation $\mathrm{WD}_{\bar{r},\tilde{v}}$ over A such that

$$\mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}}) \simeq \mathrm{WD}_{\bar{r},\tilde{v}} \otimes_{A,x} \kappa(x) \tag{5-1}$$

for all points $x \in \mathrm{Sp}(A)$. The usual proof of Grothendieck's monodromy theorem (see [Tate 1979, Corollary 4.2.2]) adapts easily to this setting, and this has already been observed by other authors. See for example [Bellaïche and Chenevier 2009, 7.8.3–7.8.14; Paulin 2011, 5.2; Emerton and Helm 2014, 4.1.6]. To make our article more self-contained (and to point out the "usual" assumption that A is reduced is unnecessary) we give the details for the convenience of the reader.

Proposition 5.2. *Let w be a place of F not dividing p, and let A be an affinoid E-algebra. For any continuous representation $\rho : \mathrm{Gal}_{F_w} \to \mathrm{GL}_n(A)$ there is a unique nilpotent $N \in M_n(A)$ such that the equality $\rho(\gamma) = \exp(t_p(\gamma)N)$ holds for all γ in an open subgroup $J \subset I_{F_w}$.* (Here $t_p : I_{F_w} \twoheadrightarrow \mathbb{Z}_p$ is a choice of homomorphism as in Section (4.2) of [Tate 1979].)

Proof. Choose a submultiplicative norm $\|\cdot\|$ on A relative to which A is complete (if A is reduced one can take the spectral norm, see [Bosch et al. 1984, 6.2.4]). Let A° be the (closed) unit ball. Then $I + p^i M_n(A^\circ)$ is an open (normal) subgroup of $\mathrm{GL}_n(A^\circ)$ for $i > 0$, so its inverse image $\rho^{-1}(I + p^i M_n(A^\circ)) = \mathrm{Gal}_{F_i}$ for some finite extension F_i of F_w. Note that F_{i+1}/F_i is a Galois extension whose Galois group is killed by p. Let us fix an $i > 0$ and work with the restriction $\rho|_{\mathrm{Gal}_{F_i}}$. Recall that wild inertia $P_{F_i} \subset I_{F_i}$ is the Sylow pro-ℓ subgroup where $w|\ell$. Since $\ell \neq p$ we deduce that $P_{F_i} \subset \mathrm{Gal}_{F_j}$ for all $j \geq i$. That is ρ factors through the tame quotient $I_{F_i}/P_{F_i} \simeq \prod_{q \neq \ell} \mathbb{Z}_q$. For the same reason ρ factors further through $t_p : I_{F_i} \twoheadrightarrow \mathbb{Z}_p$. Therefore we find an element $\alpha \in I + p^i M_n(A^\circ)$ (the image of $1 \in \mathbb{Z}_p$ under ρ) such that $\rho(\gamma) = \alpha^{t_p(\gamma)}$ for all $\gamma \in I_{F_i}$. We let $N := \log(\alpha)$. If we choose i large enough ($i > 1$ suffices, see the discussion in [Schneider 2011, p. 220]) all power series converge and we arrive at $\rho(\gamma) = \exp(t_p(\gamma)N)$ for $\gamma \in I_{F_i}$. We conclude that we may take $J := I_{F_2}$. (The uniqueness of N follows by taking log on both sides.)

To see that N is nilpotent note the standard relation $\rho(w)N\rho(w)^{-1} = \|w\|N$ for $w \in W_{F_i}$. If we take w to be a (geometric) Frobenius this shows that all specializations of N^n at points $x \in \mathrm{Sp}(A)$ are 0 (by considering the eigenvalues in $\overline{\kappa(x)}$ as usual). Thus all matrix entries of N^n are nilpotent (by the maximum modulus principle [Bosch et al. 1984, 6.2.1]). Therefore N itself is nilpotent since A is Noetherian. □

If we choose a geometric Frobenius Φ from W_{F_w} (keeping the notation of the previous Proposition) we can thus define a Weil–Deligne representation $(\tilde{\rho}, N)$ on A^n by the usual formula [Tate 1979, 4.2.1]:

$$\rho(\Phi^s\gamma) = \tilde{\rho}(\Phi^s\gamma)\exp(t_p(\gamma)N),$$

where $s \in \mathbb{Z}$ and $\gamma \in I_{F_w}$. With this definition $\tilde{\rho} : W_{F_w} \to \mathrm{GL}_n(A)$ is a representation which is trivial on the open subgroup $J \subset W_{F_w}$ (so continuous for the discrete topology on A).

As already hinted at above we apply this construction to $r^{\mathrm{univ}}|_{\mathrm{Gal}_{F_{\tilde{v}}}}$ for a fixed place $v \in \Sigma_0$, and an affinoid $\mathrm{Sp}(A) \subset X_{\bar{r}}$. We view the universal deformation $r^{\mathrm{univ}} : \mathrm{Gal}_F \to \mathrm{GL}_n(R_{\bar{r}})$ as a representation on A^n by composing with $R_{\bar{r}} \to \mathcal{O}(X_{\bar{r}}) \to A$. This gives a Weil–Deligne representation $\mathrm{WD}_{\bar{r},\tilde{v}}$ over A with the interpolative property (5-1).

6. The local Langlands correspondence for GL_n after Scholze

Scholze [2013b] gave a new purely local characterization of the local Langlands correspondence. His trace identity (see Theorem 1.2 in [loc. cit.]) takes the following form. Let Π be an irreducible smooth representation of $\mathrm{GL}_n(F_w)$, where w is an arbitrary finite place of F. Suppose we are given $\tau = \Phi^s\gamma$ with $\gamma \in I_{F_w}$ and $s \in \mathbb{Z}_{>0}$, together with a $\mathbb{Q}$-valued "cut-off" function $h \in \mathcal{C}_c^\infty(\mathrm{GL}_n(\mathcal{O}_{F_w}))$. First Scholze associates a $\mathbb{Q}$-valued function $\phi_{\tau,h} \in \mathcal{C}_c^\infty(\mathrm{GL}_n(F_{w,s}))$, where $F_{w,s}$ denotes the unramified degree s extension of F_w. The function $\phi_{\tau,h}$ is defined by taking the trace of $\tau \times h^\vee$ on (alternating sums of) certain formal nearby cycle sheaves à la Berkovich on deformation spaces of ϖ-divisible $\mathcal{O}_{F_w}$-modules; and $h^\vee(g) = h({}^tg^{-1})$. See the discussion leading up to [Scholze 2013b, Theorem 2.6] for more details. Next one selects a function $f_{\tau,h} \in \mathcal{C}_c^\infty(\mathrm{GL}_n(F_w))$ which is *associated* with $\phi_{\tau,h}$ in the sense that their (twisted) orbital integrals match. More precisely, with suitable normalizations one has the identity $TO_\delta(\phi_{\tau,h}) = O_\gamma(f_{\tau,h})$ for regular $\gamma = \mathcal{N}\delta$, see [Clozel 1987, Theorem 2.1]. With our normalization of $\mathrm{rec}(\cdot)$, Scholze's trace identity reads

$$\mathrm{tr}(f_{\tau,h}|\Pi) = \mathrm{tr}\big(\tau\,|\,\mathrm{rec}(\Pi \otimes |\det|^{(1-n)/2})\big)\cdot \mathrm{tr}(h|\Pi).$$

We will make use of a variant of $f_{\tau,h}$ which lives in the Bernstein center of $\mathrm{GL}_n(F_w)$. We refer to Section 3 of [Haines 2014] for a succinct review of the basic properties and different characterizations of the Bernstein center. This variant f_τ has the property that $\mathrm{tr}(f_{\tau,h}|\Pi) = \mathrm{tr}(f_\tau * h|\Pi)$ and is defined for all $\tau \in W_{F_w}$ by decreeing that f_τ acts on any irreducible smooth representation Π via scaling by

$$f_\tau(\Pi) = \mathrm{tr}\big(\tau\,|\,\mathrm{rec}(\Pi \otimes |\det|^{(1-n)/2})\big).$$

For the existence of f_τ see the proofs of [Scholze 2013b, Lemma 3.2; 2013a, Lemma 6.1; 2011, Lemma 9.1]. These f_τ also appear in [Chenevier 2009, Proposition 3.11], see Section 10 below for a more thorough discussion.

We apply this construction to each of the places $\tilde{v}$ with $v \in \Sigma_0$. Now $\tau = (\tau_{\tilde{v}})$ denotes a tuple of Weil elements $\tau_{\tilde{v}} \in W_{F_{\tilde{v}}}$. Via our isomorphisms $i_{\tilde{v}}$ we view $f_{\tau_{\tilde{v}}}$ as an element of the Bernstein center of $U(F_v^+)$, say $\mathfrak{Z}(U(F_v^+))$, and consider the element $f_\tau := \bigotimes_{v\in\Sigma_0} f_{\tau_{\tilde{v}}} \in \bigotimes_{v\in\Sigma_0} \mathfrak{Z}(U(F_v^+))$.

Lemma 6.1. *Let $x \in X_{\bar{r}}$ be arbitrary. Then f_τ acts on $\bigotimes_{v\in\Sigma_0} \pi_{x,v}$ via scaling by*

$$f_\tau(\otimes_{v\in\Sigma_0}\pi_{x,v}) = \prod_{v\in\Sigma_0} \mathrm{tr}(\tau_{\tilde{v}} \mid \mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}})).$$

Proof. If $\{\pi_v\}_{v\in\Sigma_0}$ is a family of irreducible smooth representations, f_τ acts on $\bigotimes_{v\in\Sigma_0} \pi_v$ via scaling by

$$f_\tau(\otimes_{v\in\Sigma_0}\pi_v) = \prod_{v\in\Sigma_0} \mathrm{tr}\big(\tau_{\tilde{v}} \mid \mathrm{rec}(\mathrm{BC}_{\tilde{v}|v}(\pi_v) \otimes |\det|^{(1-n)/2})\big).$$

Now use the defining property (4-1) of the representations $\pi_{x,v}$ attached to the point x. □

7. Interpolation of traces

As above let $\mathfrak{Z}(U(F_v^+))$ denote the Bernstein center of $U(F_v^+)$, and $\mathcal{Z}(U(F_v^+), K_v)$ the center of the Hecke algebra $\mathcal{H}(U(F_v^+), K_v)$. There is a canonical homomorphism $\mathfrak{Z}(U(F_v^+)) \to \mathcal{Z}(U(F_v^+), K_v)$ obtained by letting the Bernstein center act on $\mathcal{C}_c^\infty(K_v\backslash U(F_v^+))$, see [Haines 2014, 3.2]. We let $f_{\tau_{\tilde{v}}}^{K_v}$ be the image of $f_{\tau_{\tilde{v}}}$ under this map, and consider $f_\tau^{K_{\Sigma_0}} := \bigotimes_{v\in\Sigma_0} f_{\tau_{\tilde{v}}}^{K_v}$ belonging to $\mathcal{Z}(K_{\Sigma_0}) := \bigotimes_{v\in\Sigma_0} \mathcal{Z}(U(F_v^+), K_v)$ which is the center of $\mathcal{H}(K_{\Sigma_0})$. In particular this operator $f_\tau^{K_{\Sigma_0}}$ acts on the sheaf $\mathcal{M}$ and its fibers $\mathcal{M}_y$.

If $y = (x, \delta) \in Y(K^p, \bar{r})(E)$ is a classical point of noncritical slope, and we combine Proposition 4.2 and Lemma 6.1, we deduce that $f_\tau^{K_{\Sigma_0}}$ acts on $\mathcal{M}'_y \simeq \bigotimes_{v\in\Sigma_0} \pi_{x,v}^{K_v}$ via scaling by

$$\prod_{v\in\Sigma_0} \mathrm{tr}(\tau_{\tilde{v}} \mid \mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}})).$$

The goal of this section is to extrapolate this property to *all* points y. As a first observation we note that the above factor can be interpolated across deformation space $X_{\bar{r}}$. Indeed, let $\mathrm{Sp}(A) \subset X_{\bar{r}}$ be an affinoid subvariety and let $\mathrm{WD}_{\bar{r},\tilde{v}}$ be the Weil–Deligne representation on A^n constructed after Proposition 5.2.

Lemma 7.1. *For each tuple* $\tau = (\tau_{\tilde{v}}) \in \prod_{v\in\Sigma_0} W_{F_{\tilde{v}}}$, *the element*

$$a_\tau := \prod_{v\in\Sigma_0} \operatorname{tr}(\tau_{\tilde{v}} \,|\, \mathrm{WD}_{\bar{r},\tilde{v}}) \in A$$

satisfies the following interpolative property: *For every point* $x \in \mathrm{Sp}(A)$ *the function* a_τ *specializes to*

$$a_\tau(x) = \prod_{v\in\Sigma_0} \operatorname{tr}(\tau_{\tilde{v}} \,|\, \mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}})) \in \kappa(x).$$

Proof. This is clear from the interpolative property of $\mathrm{WD}_{\bar{r},\tilde{v}}$ by taking traces in (5-1). □

Our main result in this section (Proposition 7.9 below) shows that a_τ extends naturally to a function defined on the whole eigenvariety $Y(K^p, \bar{r})$ in such a way that $f_\tau^{K_{\Sigma_0}} : \mathcal{M} \to \mathcal{M}$ is multiplication by a_τ.

First we need to recall a couple of well-known facts from rigid analytic geometry.

Lemma 7.2. *Let X be an irreducible rigid analytic space* (over some unspecified nonarchimedean field) *and let $Y \subset X$ be a nonempty Zariski open subset* (see [Bosch et al. 1984, Definition 9.5.2/1]). *Then Y is irreducible.*

Proof. Let $\widetilde{X} \to X$ be the (irreducible) normalization of X. The pullback of Y to $\widetilde{X}$ is a normalization $\widetilde{Y} \to Y$ and it suffices to show that the Zariski open subset $\widetilde{Y} \subset \widetilde{X}$ is connected (see [Conrad 1999, Definition 2.2.2]). Suppose $\widetilde{Y} = U \coprod V$ is an admissible covering with U, V proper admissible open subsets of $\widetilde{Y}$. By Bartenwerfer's Hebbarkeitssatz [1976, p. 159] the idempotent function on $\widetilde{Y}$ which is 1 on U and 0 on V extends to an analytic function on $\widetilde{X}$, which is necessarily a nontrivial idempotent by the uniqueness in Bartenwerfer's theorem "Riemann I." This contradicts the irreducibility of $\widetilde{X}$ (by [Conrad 1999, Lemma 2.2.3]), so $\widetilde{Y}$ must be connected. □

Definition 7.3. A Zariski dense subset Z of a rigid space X is called *very Zariski dense* (or *Zariski dense and accumulation*, see [Chenevier 2011, Proposition 2.6]) if for $z \in Z$ and an affinoid open neighborhood $z \in U \subset X$, there is an affinoid open neighborhood $z \in V \subset U$ such that $Z \cap V$ is Zariski dense in V.

Lemma 7.4. *Let X be a rigid space and let $Z \subset X$ be a very Zariski dense subset. Let $Y \subset X$ be a Zariski open subset which is Zariski dense. Then $Y \cap Z$ is very Zariski dense in Y.*

Proof. We first note that it suffices to prove that $Y \cap Z$ is Zariski dense in Y. Very Zariski density then follows immediately from very Zariski density of Z in X. We show that Z is Zariski dense in every irreducible component of Y. By [Conrad 1999, Corollary 2.2.9] these irreducible components are given by the subsets $Y \cap C$, where C is an irreducible component of X. Denote by C° the Zariski open subset of X given by removing the intersections with all other irreducible components from C. Then $Y \cap C^\circ$ is irreducible by Lemma 7.2 and meets Z since it is Zariski open in X. It follows from very Zariski density of Z in X that Z is Zariski dense in $Y \cap C^\circ$. We deduce that Z is Zariski dense in $Y \cap C$, as desired. □

In order to deal with the *non étale* points below, the following generic freeness lemma will be crucial.

Lemma 7.5. *Let X be a reduced rigid space and let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. Then there is a Zariski open and dense subset $X_{\mathcal{M}} \subset X$ over which $\mathcal{M}$ is locally free.*

Proof. We follow an argument from the proof of [Hansen 2017, Theorem 5.1.2]: The regular locus X^{reg} of X is Zariski open and dense, by the excellence of affinoid algebras. If $U \subset X$ is an affinoid open $\mathcal{M}$ is locally free at a regular point $x \in U$ if and only if x is not in the support of $\bigoplus_{i=1}^{\dim U} \operatorname{Ext}^i_{\mathcal{O}(U)}(\mathcal{M}(U), \mathcal{O}(U))$. This shows that $\mathcal{M}$ is locally free over a Zariski open subset $X_{\mathcal{M}}$ which is the intersection of X^{reg} and another Zariski open subset of X — the complement of the support. Namely, if $U \subset X^{\text{reg}}$ is a connected affinoid open (so $\mathcal{O}(U)$ is a regular domain) then the support of $\bigoplus_{i=1}^{\dim U} \operatorname{Ext}^i_{\mathcal{O}(U)}(\mathcal{M}(U), \mathcal{O}(U))$ in $\operatorname{Spec}(\mathcal{O}(U))$ has dimension $< \dim(U)$, by [Bruns and Herzog 1993, Corollary 3.5.11(c)] and therefore its complement is dense. We deduce that $X_{\mathcal{M}}$ is dense in X. □

The following observation lies at the heart of our interpolation argument.

Lemma 7.6. *Let $w : X \to W$ be a map of reduced equidimensional rigid spaces and let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. We assume that X admits a covering by affinoid opens V such that*

1. *$w(V) \subset W$ is affinoid open,*
2. *The restriction $w|_V : V \to w(V)$ is finite,*
3. *$\mathcal{M}(V)$ is a finite projective $\mathcal{O}(w(V))$-module.*

Let $Z \subset X$ be a very Zariski dense subset, and suppose $\phi \in \operatorname{End}_{\mathcal{O}_X}(\mathcal{M})$ induces the zero map $\phi_z = 0$ on the fibers $\mathcal{M}_z = \mathcal{M} \otimes_{\mathcal{O}_X} \kappa(z)$ for all $z \in Z$. Then $\phi = 0$.

Proof. First we restrict to the Zariski open and dense set $X_{\mathcal{M}}$ from Lemma 7.5. Since $\mathcal{M}$ is locally free over $X_{\mathcal{M}}$, the locus in $X_{\mathcal{M}}$ where ϕ vanishes is a Zariski closed subset. By Lemma 7.4, this locus also contains a Zariski dense set of points (namely $Z \cap X_{\mathcal{M}}$) so we infer that $\phi|_{X_{\mathcal{M}}} = 0$.

Now we let $V \subset X$ be an affinoid open forming part of the cover described in the statement. Let $w(V)_0 \subset w(V)$ be the (Zariski open and dense — since W is reduced) locus where the map $V \to w(V)$ is finite étale.

Since $X\backslash X_{\mathcal{M}} \subset X$ is a Zariski closed subset of dimension $< \dim X$, the set $W_1 := w(V \cap (X\backslash X_{\mathcal{M}}))$ is a Zariski closed subset of $w(V)$ with dimension $< \dim X = \dim W$. So $w(V)\backslash W_1$ is Zariski open and dense in $w(V)$.

We deduce that $w(V)_0 \cap (w(V)\backslash W_1)$ is a Zariski dense subset of $w(V)$. Moreover, ϕ induces the zero map on the fibers $\mathcal{M}(V) \otimes_{\mathcal{O}(w(V))} \kappa(y)$ for all y in this dense intersection: Use that $w|_V$ is étale at y, so if $x_1, \dots, x_r$ are the preimages of y in V, then

$$\mathcal{M}(V) \otimes_{\mathcal{O}(w(V))} \kappa(y) \simeq \bigoplus_{i=1}^{r} \mathcal{M}(V) \otimes_{\mathcal{O}(V)} \kappa(x_i)$$

and we know that ϕ acts as zero on each $\mathcal{M}(V) \otimes_{\mathcal{O}(V)} \kappa(x_i)$ since $x_i \in X_{\mathcal{M}}$ (otherwise $y = w(x_i) \in W_1$), as observed in the first paragraph of the proof. We conclude that $\phi = 0$ on $\mathcal{M}(V)$: Indeed $\mathcal{M}(V)$ is a finite projective $\mathcal{O}(w(V))$-module so the points $y \in w(V)$ where ϕ vanishes on the fiber form a Zariski closed subset which contains $w(V)_0 \cap (w(V)\backslash W_1)$. Since W is reduced $\phi_{\mathcal{M}(V)} = 0$. Since V was arbitrary, we must have $\phi = 0$ on $\mathcal{M}$ as desired. □

We now return to the notation of Section 3. We have defined the eigenvariety $Y(K^p, \bar{r})$ to be the (scheme-theoretic) support of the coherent sheaf $\mathcal{M}$ over $X_{\bar{r}} \times \hat{T}$. It comes equipped with a natural weight morphism $\omega : Y(K^p, \bar{r}) \to \mathcal{W}$ defined as the composition of maps

$$Y(K^p, \bar{r}) \hookrightarrow X_{\bar{r}} \times \hat{T} \xrightarrow{\text{pr}} \hat{T} \xrightarrow{\text{can}} \mathcal{W}.$$

The following lemma summarizes some important facts about $Y(K^p, \bar{r})$ and ω.

Lemma 7.7. *The eigenvariety $Y(K^p, \bar{r})$ satisfies the following properties.*

(1) *$Y(K^p, \bar{r})$ has an admissible cover by open affinoids $(U_i)_{i\in I}$ such that for all i there exists an open affinoid $W_i \subset \mathcal{W}$ which fulfills* (a) *and* (b) *below*:
   (a) *The weight morphism $\omega : Y(K^p, \bar{r}) \to \mathcal{W}$ induces, upon restriction to each irreducible component $C \subset U_i$, a finite surjective map $C \to W_i$.*
   (b) *Each $\mathcal{O}(U_i)$ is isomorphic to an $\mathcal{O}(W_i)$-subalgebra of* $\mathrm{End}_{\mathcal{O}(W_i)}(P_i)$ *for some finite projective $\mathcal{O}(W_i)$-module P_i.*

(2) *The classical points of noncritical slope are very Zariski dense in $Y(K^p, \bar{r})$.*

(3) *$Y(K^p, \bar{r})$ is reduced.*

Proof. These can be proved in a similar way to the analogous statements in [Breuil et al. 2017]. More precisely, we refer to Proposition 3.11, Theorem 3.19 and Corollary 3.20 of that paper. (Note that in the proof of Corollary 3.20 we can, in our setting, replace the reference to [Caraiani et al. 2016] with the well-known assertion that the Hecke operators at good places act semisimply on spaces of cuspidal automorphic forms.) □

Remark 7.8. In [Breuil et al. 2017, p. 1610] there is a "neatness" assumption on the tame level K^p. Namely that (in our notation) $G(\mathbb{Q}) \cap hK_pK^ph^{-1} = \{1\}$ for all $h \in G(\mathbb{A}_f)$, which can always be ensured by shrinking K^p. This assumption is necessary for the patching argument of [Caraiani et al. 2016]. However, to avoid future potential confusion, we stress that neatness is not essential in the context of eigenvarieties — such as [Breuil et al. 2017, Proposition 3.11] which we cited in the proof of Lemma 7.7 above. This observation is crucial in Section 9B below, where the level is hyperspecial/Iwahori at all places and therefore not neat.

Since $Y(K^p, \bar{r})$ projects to $X_{\bar{r}}$, its ring of functions $\mathcal{O}(Y(K^p, \bar{r}))$ becomes an $R_{\bar{r}}$-algebra via the natural map $R_{\bar{r}} \to \mathcal{O}^0(X_{\bar{r}})$. Pushing forward the universal deformation of $\bar{r}$ (with a fixed choice of basis) then yields a continuous representation

$$r : \mathrm{Gal}_F \to \mathrm{GL}_n\big(\mathcal{O}(Y(K^p, \bar{r}))\big).$$

In particular, for every open affinoid $U \subset Y(K^p, \bar{r})$ we may specialize r further and arrive at a continuous representation $r : \mathrm{Gal}_F \to \mathrm{GL}_n(\mathcal{O}(U))$. We may in fact take $\mathcal{O}^0(U)$ here (the functions bounded by one), but we will not need that.

It follows from Proposition 5.2 that for $v \in \Sigma_0$, an open affinoid $U \subset Y(K^p, \bar{r})$, and a fixed choice of lift of geometric Frobenius $\Phi = \Phi_{\tilde{v}}$ in $W_{F_{\tilde{v}}}$, we obtain a Weil–Deligne representation $\mathrm{WD}_{\bar{r},\tilde{v}}(U)$ over $\mathcal{O}(U)$. Moreover, this construction is obviously compatible as we vary U in the sense that if $U' \subset U$, then $\mathrm{WD}_{\bar{r},\tilde{v}}(U)$ pulls back to $\mathrm{WD}_{\bar{r},\tilde{v}}(U')$ over U' (by the uniqueness in Proposition 5.2). To be precise, there is a natural isomorphism of Weil–Deligne representations over $\mathcal{O}(U')$,

$$\mathrm{WD}_{\bar{r},\tilde{v}}(U') \simeq \mathrm{WD}_{\bar{r},\tilde{v}}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U').$$

Now, for a tuple of Weil elements $\tau = (\tau_{\tilde{v}}) \in \prod_{v \in \Sigma_0} W_{F_{\tilde{v}}}$ we obtain functions

$$a_{\tau,U} := \prod_{v \in \Sigma_0} \mathrm{tr}(\tau_{\tilde{v}} | \mathrm{WD}_{\bar{r},\tilde{v}}(U)) \in \mathcal{O}(U),$$

as defined above in Lemma 7.1. By the compatibility just mentioned, $a_{\tau,U'} = \mathrm{res}_{U,U'}(a_{\tau,U})$ when $U' \subset U$. It follows that we may glue the $a_{\tau,U}$ and get a function $a_\tau = a_{\tau,Y(K^p,\bar{r})}$ on the whole eigenvariety $Y(K^p, \bar{r})$ with the interpolation property in Lemma 7.1.

Proposition 7.9. *The operator $f_\tau^{K_{\Sigma_0}}$ acts on $\mathcal{M}$ via scaling by a_τ, for every $\tau \in \prod_{v \in \Sigma_0} W_{F_{\tilde{v}}}$.*

Proof. We must show the endomorphism $\phi := f_\tau^{K_{\Sigma_0}} - a_\tau$ of $\mathcal{M}$ equals zero. By the discussion at the beginning of this section (just prior to 7.1) we know ϕ induces the zero map on the fibers of $\mathcal{M}$ at classical points of noncritical slope. We are now done by Lemma 7.6 (together with Lemma 7.7). □

By specialization at any point $y=(x,\delta)\in Y(K^p,\bar r)$ we immediately find that $f_\tau^{K_{\Sigma_0}}$ acts on the fiber $\mathcal{M}_y$ (and hence its dual $\mathcal{M}'_y$) via scaling by $a_\tau(x)$. We summarize this below.

Corollary 7.10. *Let $y\in Y(K^p,\bar r)$ be an arbitrary point. Then $f_\tau^{K_{\Sigma_0}}$ acts on $\mathcal{M}'_y$ via scaling by*

$$\prod_{v\in\Sigma_0}\operatorname{tr}(\tau_{\tilde v}|\operatorname{WD}(r_x|_{\operatorname{Gal}_{F_{\tilde v}}})).$$

Proof. This is an immediate consequence of Proposition 7.9. □

8. Interpolation of central characters

In this section we will reuse parts of the argument from the previous Section 7 to interpolate the central characters $\omega_{\pi_{x,v}}$ across the eigenvariety. We include it here mostly for future reference. It will only be used in this paper in the very last paragraph of Remark 9.6 below.

For $v\in\Sigma_0$ we let $Z(U(F_v^+))$ be the center of $U(F_v^+)$ (recall that its *Bernstein* center is denoted by $\mathfrak{Z}$). There is a natural homomorphism

$$Z(U(F_v^+))\to\mathcal{Z}(U(F_v^+),K_v)^\times$$

which takes ξ_v to the double coset operator $[K_v\xi_vK_v]$. Taking the product over $v\in\Sigma_0$ we get an analogous map $Z(U(F_{\Sigma_0}^+))\to\mathcal{Z}(K_{\Sigma_0})^\times$ which we will denote $\xi=(\xi_v)_{v\in\Sigma_0}\mapsto h_\xi^{K_{\Sigma_0}}=\bigotimes_{v\in\Sigma_0}[K_v\xi_vK_v]$. Thus $h_\xi^{K_{\Sigma_0}}$ operates on $\mathcal{M}$ and its fibers.

If $y=(x,\delta)\in Y(K^p,\bar r)(E)$ is a classical point of noncritical slope the action of $h_\xi^{K_{\Sigma_0}}$ on $\mathcal{M}'_y\simeq\bigotimes_{v\in\Sigma_0}\pi_{x,v}^{K_v}$ is clearly just multiplication by $\prod_{v\in\Sigma_0}\omega_{\pi_{x,v}}(\xi_v)$. This property extrapolates to *all* points y by mimicking the proof in Section 7, as we will now explain.

For $\operatorname{Sp}(A)\subset X_{\bar r}$ we have the Weil–Deligne representation $\operatorname{WD}_{\bar r,\tilde v}$ on A^n. Consider its determinant $\det(\operatorname{WD}_{\bar r,\tilde v})$ as a character $F_{\tilde v}^\times\to A^\times$ via local class field theory. Note that $Z(U(F_v^+))\simeq Z(\operatorname{GL}_n(F_{\tilde v}))\simeq F_{\tilde v}^\times$ which allows us to view the product $\prod_{v\in\Sigma_0}\det(\operatorname{WD}_{\bar r,\tilde v})$ as a character $\omega:Z(U(F_{\Sigma_0}^+))\to A^\times$. Clearly the specialization of ω at any $x\in\operatorname{Sp}(A)$ is $\omega_x=\bigotimes_{v\in\Sigma_0}\omega_{\pi_{x,v}}:Z(U(F_{\Sigma_0}^+))\to\kappa(x)^\times$ by the interpolative property of $\operatorname{WD}_{\bar r,\tilde v}$.

By copying the proof of Proposition 7.9 almost verbatim, one easily deduces the following.

Proposition 8.1. *There is a homomorphism $\omega:Z(U(F_{\Sigma_0}^+))\to\mathcal{O}(Y(K^p,\bar r))^\times$ such that $h_\xi^{K_{\Sigma_0}}:\mathcal{M}\to\mathcal{M}$ is multiplication by $\omega(\xi)$ for all ξ. In particular, for any point $y=(x,\delta)\in Y(K^p,\bar r)$, the action of $h_\xi^{K_{\Sigma_0}}$ on $\mathcal{M}'_y$ is scaling by $\prod_{v\in\Sigma_0}\omega_{\pi_{x,v}}(\xi_v)$.*

9. Proof of the main result

We now vary K_{Σ_0} and reinstate the notation $\mathcal{M}_{K^p}$ (instead of just writing $\mathcal{M}$) to stress the dependence on $K^p = K_{\Sigma_0}K^{\Sigma}$. Suppose $K'_{\Sigma_0} \subset K_{\Sigma_0}$ is a compact open subgroup, and let $K'^p = K'_{\Sigma_0}K^{\Sigma}$. Recall that the global sections of $\mathcal{M}_{K^p}$ is the *dual* of $J_B(\hat{S}(K^p, E)^{\mathrm{an}}_{\mathfrak{m}})$. Thus we find a natural transition map $\mathcal{M}_{K'^p} \twoheadrightarrow \mathcal{M}_{K^p}$ of sheaves on $X_{\bar{r}} \times \hat{T}$. Taking their support we find that $Y(K^p, \bar{r}) \hookrightarrow Y(K'^p, \bar{r})$. Passing to the dual fibers at a point $y \in Y(K^p, \bar{r})$ yields an embedding $\mathcal{M}'_{K^p,y} \hookrightarrow \mathcal{M}'_{K'^p,y}$ which is equivariant for the Hecke action (i.e., compatible with the map $\mathcal{H}(K'_{\Sigma_0}) \twoheadrightarrow \mathcal{H}(K_{\Sigma_0})$ given by $e_{K_{\Sigma_0}} \star (\cdot) \star e_{K_{\Sigma_0}}$). The limit $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_{K^p,y}$ thus becomes an admissible representation of $U(F^+_{\Sigma_0}) \xrightarrow{\sim} \prod_{v\in\Sigma_0} \mathrm{GL}_n(F_{\tilde{v}})$ with coefficients in $\kappa(y)$. Subsequently we will use the next lemma to show it is of finite length.

Lemma 9.1. *Let $y \in Y(K^p, \bar{r})$ be any point. Let $\bigotimes_{v\in\Sigma_0} \pi_v$ be an arbitrary irreducible subquotient*[6] *of $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_{K^p,y}$. Then for all places $v \in \Sigma_0$ we have an isomorphism*

$$\mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}})^{\mathrm{ss}} \simeq \mathrm{rec}\big(\mathrm{BC}_{\tilde{v}|v}(\pi_v) \otimes |\det|^{(1-n)/2}\big)^{\mathrm{ss}}.$$

(Here ss means semisimplification of the underlying representation $\tilde{\rho}$ of $W_{F_{\tilde{v}}}$, and setting $N = 0$.)

Proof. By Corollary 7.10 we know that f_τ acts on $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_{K^p,y}$ via scaling by $a_\tau(x)$. On the other hand, by the proof of Lemma 6.1 we know what $f_\tau\big(\bigotimes_{v\in\Sigma_0} \pi_v\big)$ is. By comparing the two expressions we find that

$$\prod_{v\in\Sigma_0} \mathrm{tr}(\tau_{\tilde{v}} \,|\, \mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}})) = \prod_{v\in\Sigma_0} \mathrm{tr}\big(\tau_{\tilde{v}} \,|\, \mathrm{rec}(\mathrm{BC}_{\tilde{v}|v}(\pi_v) \otimes |\det|^{(1-n)/2})\big)$$

for all tuples τ. This shows that $\mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}})$ and $\mathrm{rec}\big(\mathrm{BC}_{\tilde{v}|v}(\pi_v) \otimes |\det|^{(1-n)/2}\big)$ have the same semisimplification for all $v \in \Sigma_0$ by "linear independence of characters." □

We employ Lemma 9.1 to show $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_{K^p,y}$ has finite length (which for an admissible representation is equivalent to being finitely generated by Howe's Theorem, see [Bernšteĭn and Zelevinskiĭ 1976, 4.1]).

Lemma 9.2. *The length of $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_{K^p,y}$ as a $U(F^+_{\Sigma_0})$-representation is finite, and uniformly bounded in y on quasicompact subvarieties of $Y(K^p, \bar{r})$.*

Proof. We first show finiteness. Any admissible smooth representation contains a simple subrepresentation. Therefore, if $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_{K^p,y}$ is of infinite length we can write down an infinite proper ascending chain of $U(F^+_{\Sigma_0})$-invariant subspaces

$$0 = V_0 \subset V_1 \subset V_2 \subset V_3 \subset \cdots \subset \varinjlim_{K_{\Sigma_0}} \mathcal{M}'_{K^p,y}, \quad V_{i+1}/V_i \neq 0 \text{ simple}.$$

[6] Such exist by Zorn's lemma; any finitely generated subrepresentation admits an irreducible quotient.

Taking K_{Σ_0}-invariants (which is exact as $\mathrm{char}_E = 0$) we find an increasing chain of $\mathcal{H}(K_{\Sigma_0})$-submodules $V_i^{K_{\Sigma_0}} \subset \mathcal{M}'_{K^p,y}$. The fiber is finite-dimensional so this chain must become stationary. I.e., V_{i+1}/V_i has *no* nonzero K_{Σ_0}-invariants for i large enough. If we can show that every irreducible subquotient $\bigotimes_{v\in\Sigma_0}\pi_v$ of $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_{K^p,y}$ *has* nonzero K_{Σ_0}-invariants, we are done. We will show that we can find a small enough K_{Σ_0} with this last property.

The local Langlands correspondence preserves ϵ-factors, and hence conductors. (See [Jacquet et al. 1981] for the definition of conductors in the GL_n-case, and [Tate 1979, p. 21] for the Artin conductor of a Weil–Deligne representation.) Therefore, for every place $v \in \Sigma_0$ we get a bound on the conductor of $\mathrm{BC}_{\tilde{v}|v}(\pi_v)$:

$$\begin{aligned} c(\pi_v) &:= c(\mathrm{BC}_{\tilde{v}|v}(\pi_v)) \\ &= c\big(\mathrm{rec}(\mathrm{BC}_{\tilde{v}|v}(\pi_v)\otimes|\det|^{(1-n)/2})\big) \\ &\le c\big(\mathrm{rec}(\mathrm{BC}_{\tilde{v}|v}(\pi_v)\otimes|\det|^{(1-n)/2})^{\mathrm{ss}}\big)+n \\ &\overset{9.1}{=} c\big(\mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}})^{\mathrm{ss}}\big)+n. \end{aligned} \tag{9-3}$$

In the inequality we used the following general observation: If $(\tilde{\rho},\mathcal{N})$ is a Weil–Deligne representation on a vector space S, its conductor is

$$c(\tilde{\rho}) + \dim S^I - \dim(\ker\mathcal{N})^I,$$

where I is shorthand for inertia; $c(\tilde{\rho})$ is the usual Artin conductor, which is clearly invariant under semisimplification: $c(\tilde{\rho})$ only depends on $\tilde{\rho}|_I$ which is semisimple because it has finite image. This shows $c(\pi_v)$ is bounded in terms of x. If we take K_{Σ_0} small enough, say $K_{\Sigma_0} = \prod_{v\in\Sigma_0} K_v$, where

$$K_v = i_{\tilde{v}}^{-1}\big\{g \in \mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}}) : (g_{n1},\ldots,g_{nn}) \equiv (0,\ldots,1) \bmod \varpi_{F_{\tilde{v}}}^N\big\}$$

with N greater than the right-hand side of the inequality (9-3), then every constituent $\bigotimes_{v\in\Sigma_0}\pi_v$ as above satisfies $\pi_v^{K_v} \neq 0$ as desired. This shows the length is finite.

To get a uniform bound in K^p and $\bar{r}$ we improve on the bound (9-3) using [Livné 1989, Proposition 1.1]: Since $r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}}$ is a lift of $\bar{r}|_{\mathrm{Gal}_{F_{\tilde{v}}}}$, that proposition implies that

$$c(\mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}})) \le c(\bar{r}|_{\mathrm{Gal}_{F_{\tilde{v}}}}) + n.$$

(One can improve this bound but the point here is to get uniformity.) Taking K_{Σ_0} as above with N greater than $c(\bar{r}|_{\mathrm{Gal}_{F_{\tilde{v}}}}) + 2n$ the above argument guarantees that the $U(F_{\Sigma_0}^+)$-length of $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_{K^p,y}$ is the same as the $\mathcal{H}(K_{\Sigma_0})$-length of $\mathcal{M}'_{K_{\Sigma_0}K^\Sigma,y}$, which is certainly at most $\dim_E \mathcal{M}'_{K_{\Sigma_0}K^\Sigma,y}$. This dimension is uniformly bounded when y is constrained to a quasicompact subspace of $Y(K^p,\bar{r})$. □

9A. ***Strongly generic representations.*** Fix a place $v \in \Sigma_0$ and recall the definition of $\pi_{x,v}$ in (4-1). We call x a *generic* point if $\pi_{x,v}$ is a generic representation (i.e., when it has a Whittaker model) for all $v \in \Sigma_0$. For instance, all classical points are generic (see the proof of Lemma 3.3). We will impose a stronger condition on $r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}}$ which ensures that $\pi_{x,v}$ is fully induced from a supercuspidal representation of a Levi subgroup (thus in particular is generic, see [Bernstein and Zelevinsky 1977]). This rules out that $\pi_{x,v}$ is Steinberg for instance, and bypasses difficulties arising from having nonzero monodromy.

Definition 9.4. Decompose $\mathrm{WD}(r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}})^{\mathrm{ss}} \simeq \tilde{\rho}_1 \oplus \cdots \oplus \tilde{\rho}_t$ into a sum of irreducible representations $\tilde{\rho}_i : W_{F_{\tilde{v}}} \to \mathrm{GL}_{n_i}(\overline{\mathbb{Q}}_p)$. We say $r_x|_{\mathrm{Gal}_{F_{\tilde{v}}}}$ is *strongly generic* if $\tilde{\rho}_i \not\sim \tilde{\rho}_j \otimes \epsilon$ for all $i \neq j$, where $\epsilon : \mathrm{Gal}_{F_{\tilde{v}}} \to \mathbb{Z}_p^\times$ is the cyclotomic character.

For the rest of this section we will assume r_x is strongly generic at each $v \in \Sigma_0$. In the notation of Definition 9.4, each $\tilde{\rho}_i$ corresponds to a supercuspidal representation $\tilde{\pi}_i$ of $\mathrm{GL}_{n_i}(F_{\tilde{v}})$. More precisely $\mathrm{WD}(\tilde{\rho}_i) = \mathrm{rec}(\tilde{\pi}_i \otimes |\det|^{(1-n_i)/2})$. Letting $\mathrm{Ind}_{P_{n_1,\dots,n_t}}^{\mathrm{GL}_n}$ denote normalized parabolic induction from the upper block-triangular parabolic subgroup with Levi $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_t}$, we have

$$\pi_{x,v} \otimes |\det|^{(1-n)/2} \simeq \mathrm{Ind}_{P_{n_1,\dots,n_t}}^{\mathrm{GL}_n}\big((\tilde{\pi}_1 \otimes |\det|^{(1-n_1)/2}) \otimes \cdots \otimes (\tilde{\pi}_t \otimes |\det|^{(1-n_t)/2})\big)$$

since the induced representation is irreducible, see [Bernstein and Zelevinsky 1977]. Indeed $\tilde{\pi}_i \not\sim \tilde{\pi}_j(1)$ for all $i \neq j$. (The twiddles above ρ_i and π_i should not be confused with taking the contragredient.)

By Lemma 9.1, for any irreducible subquotient $\bigotimes_{v\in\Sigma_0} \pi_v$ of $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_{K^p,y}$, the factor π_v has the same supercuspidal support as $\pi_{x,v}$. Since the latter is fully induced from $P_{n_1,\dots,n_t}$ they must be isomorphic. In summary we have arrived at this result:

Corollary 9.5. *Let $y = (x, \delta) \in Y(K^p, \bar{r})$ be a point at which r_x is strongly generic at every $v \in \Sigma_0$. Then $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_{K^p,y}$ has finite length, and every irreducible subquotient is isomorphic to $\bigotimes_{v\in\Sigma_0} \pi_{x,v}$.*

Altogether this proves Theorem 1.1 in the Introduction.

Remark 9.6. Naively one might hope to remove the “ss” in Theorem 1.1 by showing that $\pi_{x,v}$ has no nonsplit self-extensions; $\mathrm{Ext}^1_{\mathrm{GL}_n(F_{\tilde{v}})}(\pi_{x,v}, \pi_{x,v}) = 0$. However, this is false even if we assume $\pi_{x,v} \simeq \mathrm{Ind}_P^{\mathrm{GL}_n}(\sigma)$ with $\sigma = \bigotimes_{j=1}^t \tilde{\pi}_j$ supercuspidal (as above). Let us explain why. For simplicity we assume σ is regular, which means $w\sigma \simeq \sigma \Rightarrow w = 1$ for all block-permutations $w \in S_n$. In other words $\tilde{\pi}_i \not\simeq \tilde{\pi}_j$ for $i \neq j$ with $n_i = n_j$. Under this assumption the “geometric lemma” (see [Casselman 1995, Proposition 6.4.1]) gives an actual direct sum decomposition of the N-coinvariants:

$$(\pi_{x,v})_N \simeq \bigoplus_w w\sigma$$

with w running over block-permutations as above. The usual adjointness property of $(\cdot)_N$ is easily checked to hold for Ext^i (see [Prasad 2013, Proposition 2.9]). Therefore

$$\mathrm{Ext}^1_{\mathrm{GL}_n}(\pi_{x,v},\pi_{x,v}) \simeq \mathrm{Ext}^1_M((\pi_{x,v})_N,\sigma) \simeq \prod_w \mathrm{Ext}^1_M(w\sigma,\sigma) \simeq \mathrm{Ext}^1_M(\sigma,\sigma).$$

In the last step we used [Casselman 1995, Corollary 5.4.4] to conclude that $\mathrm{Ext}^1_M(w\sigma,\sigma)=0$ for $w\neq 1$. However, $\mathrm{Ext}^1_M(\sigma,\sigma)$ is always nontrivial. For example, consider the principal series case where $P=B$ and σ is a smooth character of T. Here $\mathrm{Ext}^1_T(\sigma,\sigma)\simeq \mathrm{Ext}^1_T(\mathbf{1},\mathbf{1})\simeq \mathrm{Hom}(T,E)\simeq E^n$. In general, if σ is an irreducible representation of M with central character ω, there is a short exact sequence

$$0\to \mathrm{Ext}^1_{M,\omega}(\sigma,\sigma)\to \mathrm{Ext}^1_M(\sigma,\sigma)\to \mathrm{Hom}(Z_M,E)\to 0$$

(see [Paškūnas 2010, Proposition 8.1] whose proof works verbatim with coefficients E instead of $\bar{\mathbb{F}}_p$). If σ is supercuspidal it is projective and/or injective in the category of smooth M-representations with central character ω, and vice versa (see [Casselman 1995, Theorem 5.4.1; Adler and Roche 2004]). In particular $\dim_E \mathrm{Ext}^1_M(\sigma,\sigma)=\dim(Z_M)$.

By Proposition 8.1 all the self-extensions of $\pi_{x,v}$ arising from $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_{K^p,y}$ actually live in the full subcategory of smooth representations with central character $\omega_{\pi_{x,v}}$. As we just pointed out, supercuspidal is equivalent to being projective and/or injective in this category. Thus at least in the case where $\bigotimes_{v\in\Sigma_0}\pi_{x,v}$ is supercuspidal we can remove the "ss" in Theorem 1.1.

Remark 9.7. We comment on the multiplicity m_y in the analogous case of $\mathrm{GL}(2)_{/\mathbb{Q}}$. Replacing our unitary group U with $\mathrm{GL}(2)_{/\mathbb{Q}}$, and replacing $\hat{S}(K^p,E)$ with the completed cohomology of modular curves $\hat{H}^1(K^p)_E$ with tame level $K^p\subset \mathrm{GL}_2(\mathbb{A}_f^p)$, a statement analogous to Theorem 1.1 is a consequence of Emerton's local–global compatibility theorem [Emerton 2011, Theorem 1.2.1], under the assumption that $\bar{r}|_{\mathrm{Gal}_{\mathbb{Q}_p}}$ is not isomorphic to a twist of $\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1&*\\0&\epsilon\end{smallmatrix}\right)$. With this assumption, the multiplicities m_y are (at least predicted to be) equal to 2 (coming from the two-dimensional Galois representation r_x), and the representations of $\mathrm{GL}_2(\mathbb{Q}_{\Sigma_0})$ which appear are semisimple.

Indeed, it follows from [loc. cit.] that we have $m_y=2\dim_E J_B^\delta(\Pi(\varrho_x)^{\mathrm{an}})$, where $\varrho_x:=r_x|_{\mathrm{Gal}_{\mathbb{Q}_p}}$. When ϱ_x is absolutely irreducible, it follows from [Dospinescu 2014, Theorems 1.1 and 1.2] (see also [Colmez 2014, Theorem 0.6]) that $J_B^\delta(\Pi(\varrho_x)^{\mathrm{an}})$ has dimension at most 1. If ϱ_x is reducible, then [Emerton 2006b, Conjecture 3.3.1(8), Lemma 4.1.4] predicts that $J_B^\delta(\Pi(\varrho_x)^{\mathrm{an}})$ again has dimension at most 1, unless ϱ_x is of the form $\eta\oplus\eta$ for some continuous character $\eta:\mathrm{Gal}_{\mathbb{Q}_p}\to E^\times$.

In the exceptional case with $\varrho_x \simeq \eta \oplus \eta$ scalar, where [Emerton 2011, Theorem 1.2.1(2)] does not apply, we have

$$\dim_E J_B^\delta(\Pi(\varrho_x)^{\mathrm{an}}) = 2, \quad \text{when } \delta = \eta|\cdot| \otimes \eta\epsilon|\cdot|^{-1},$$

and therefore [Emerton 2011, Conjecture 1.1.1] predicts that we have $m_y = 4$ for $y = (x, \eta|\cdot| \otimes \eta\epsilon|\cdot|^{-1})$. Again the representation of $\mathrm{GL}_2(\mathbb{Q}_{\Sigma_0})$ which appears is predicted to be semisimple.

9B. ***The general case at Iwahori level.*** In this section we assume $\bar{r}$ is automorphic of tame level $K^p = K_{\Sigma_0} K^\Sigma$, where $K_{\Sigma_0} = \prod_{v \in \Sigma_0} K_v$ is a product of *Iwahori* subgroups. This can usually be achieved by a solvable base change; i.e., by replacing $\bar{r}$ with its restriction $\bar{r}|_{\mathrm{Gal}_{F'}}$ for some solvable Galois extension F'/F (see the "Skinner–Wiles trick" [Skinner and Wiles 2001]). We make this assumption to employ a genericity criterion of Barbasch and Moy [1994], which was recently strengthened by Chan and Savin [2018; 2019].

9B1. *Genericity and Iwahori-invariants.* The setup of [Chan and Savin 2018] is the following. Let G be a split group over a p-adic field F, with a choice of Borel subgroup $B = TU$. We assume these are defined over $\mathcal{O} = \mathcal{O}_F$, and let $I \subset G(\mathcal{O})$ be the Iwahori subgroup (the inverse image of B over the residue field $\mathbb{F}_q$). The Iwahori–Hecke algebra $\mathcal{H}$ has basis $T_w = [IwI]$, where $w \in W_{\mathrm{ex}}$ runs over the extended affine Weyl group $W_{\mathrm{ex}} = N_G(T)/T(\mathcal{O})$. The basis vectors satisfy the usual relations

$$\begin{aligned} T_{w_1} T_{w_2} &= T_{w_1 w_2}, && \text{when } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2), \\ (T_s - q)(T_s + 1) &= 0, && \text{when } \ell(s) = 1. \end{aligned}$$

Here $\ell : W_{\mathrm{ex}} \to \mathbb{Z}$ denotes the length function defined by $q^{\ell(w)} = |IwI/I|$. Inside of $\mathcal{H}$ we have the subalgebra $\mathcal{H}_W$ of functions supported on $G(\mathcal{O})$, which has basis $\{T_w\}_{w \in W}$ where W is the (actual) Weyl group. The algebra $\mathcal{H}_W$ carries a natural one-dimensional representation $\mathrm{sgn} : \mathcal{H}_W \to \mathbb{C}$ which sends T_w to $(-1)^{\ell(w)}$, and we are interested in the sgn-isotypic subspaces of $\mathcal{H}$-modules.

Definition 9.8. For a smooth G-representation π (over $\mathbb{C}$) we introduce the following subspace of the Iwahori-invariants

$$\mathbb{S}(\pi) = \bigcap_{w \in W} (\pi^I)^{T_w = (-1)^{\ell(w)}}.$$

In other words the (possibly trivial) subspace of π^I where $\mathcal{H}_W$ acts via the sgn-character.

Fix a nontrivial continuous unitary character $\psi : F \to \mathbb{C}^\times$ and extend it to a character of U as in [Chan and Savin 2018, Section 4]. For a smooth G-representation

π we let $\pi_{U,\psi}$ be the "top derivative" of ψ-coinvariants (whose dual is exactly the space of ψ-Whittaker functionals on π).

Theorem 9.9 (Barbasch–Moy, Chan–Savin). *Let π be a smooth G-representation which is generated by π^I. Then the natural map $\mathbb{S}(\pi) \hookrightarrow \pi \twoheadrightarrow \pi_{U,\psi}$ is an isomorphism.*

Proof. This is [Chan and Savin 2018, Corollary 4.5] which is a special case of [Chan and Savin 2019, Theorem 3.5]. □

In particular, an irreducible representation π with $\pi^I \neq 0$ is *generic* if and only if $\mathbb{S}(\pi) \neq 0$, in which case $\dim \mathbb{S}(\pi) = 1$. This is the genericity criterion we will use below.

9B2. *The $\mathbb{S}$-part of the eigenvariety.* We continue with the usual setup and notation. We run the eigenvariety construction with $\hat{S}(K^p, E)_{\mathfrak{m}}$ replaced by its $\mathbb{S}$-subspace. More precisely, for each $v \in \Sigma_0$ we have the functor $\mathbb{S}_v$ (Definition 9.8) taking smooth $\mathrm{GL}_n(F_{\tilde{v}})$-representations to vector spaces over E. We apply their composition $\mathbb{S} = \circ_{v \in \Sigma_0} \mathbb{S}_v$ to $\varinjlim_{K_{\Sigma_0}} \hat{S}(K^p, E)_{\mathfrak{m}}$. I.e., we take

$$\Pi := \bigcap_{v \in \Sigma_0} \bigcap_{w \in W_v} (\hat{S}(K^p, E)_{\mathfrak{m}})^{T_w = (-1)^{\ell(w)}}.$$

Clearly Π is a closed subspace of $\hat{S}(K^p, E)_{\mathfrak{m}}$, and therefore an admissible Banach representation of $G = G(\mathbb{Q}_p)$. As a result $J_B(\Pi^{\mathrm{an}})'$ is coadmissible (see [Breuil et al. 2017, Proposition 3.4]) and hence the global sections $\Gamma(X_{\bar{r}} \times \hat{T}, \mathcal{M}_\Pi)$ of a coherent sheaf $\mathcal{M}_\Pi$ on $X_{\bar{r}} \times \hat{T}$. We let

$$Y_\Pi(K^p, \bar{r}) = \sup(\mathcal{M}_\Pi)$$

be its schematic support with the usual annihilator ideal sheaf. Mimicking the proof of Lemma 3.1 we obtain the following description of the dual fiber of $\mathcal{M}_\Pi$ at a point $y = (x, \delta) \in Y_\Pi(K^p, \bar{r})$;

$$\mathcal{M}'_{\Pi,y} \simeq J_B^\delta(\Pi[\mathfrak{p}_x]^{\mathrm{an}}) \simeq \bigcap_{v \in \Sigma_0} \bigcap_{w \in W_v} J_B^\delta(\hat{S}(K^p, E)_{\mathfrak{m}}[\mathfrak{p}_x]^{\mathrm{an}})^{T_w = (-1)^{\ell(w)}}.$$

This clearly shows $Y_\Pi(K^p, \bar{r})$ is a closed subvariety of $Y(K^p, \bar{r})$. Our immediate goal is to show equality.

Lemma 9.10. $$Y_\Pi(K^p, \bar{r}) = Y(K^p, \bar{r}).$$

Proof. Since the classical points are Zariski dense in $Y(K^p, \bar{r})$ we just have to show each classical $y = (x, \delta)$ in fact lies in $Y_\Pi(K^p, \bar{r})$. Let π be an automorphic representation such that $r_x \simeq r_{\pi,\iota}$. This is an irreducible Galois representation (since $\bar{r}$ is) and thus $\mathrm{BC}_{F/F^+}(\pi)$ is a cuspidal and therefore generic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. In particular the factors of $\bigotimes_{v \in \Sigma_0} \pi_v$ are generic.

Taking T_w-eigenspaces of the embedding $\bigotimes_{v\in\Sigma_0}\pi_v^{K_v}\hookrightarrow\mathcal{M}'_y$ from Proposition 4.2 yields a map $\bigotimes_{v\in\Sigma_0}\mathbb{S}_v(\pi_v)\hookrightarrow\mathcal{M}'_{\Pi,y}$. Finally, by Theorem 9.9 we conclude that $\bigotimes_{v\in\Sigma_0}\mathbb{S}_v(\pi_v)\neq 0$ so that $\mathcal{M}'_{\Pi,y}\neq 0$. □

9B3. *Conclusion.* Now let $y\in Y(K^p,\bar{r})$ be an arbitrary point. By Lemma 9.10 we now know $\mathcal{M}'_{\Pi,y}\neq 0$. Note that $\mathcal{M}'_{\Pi,y}=\mathbb{S}(\varinjlim_{K_{\Sigma_0}}\mathcal{M}'_y)$ and we immediately infer that $\varinjlim_{K_{\Sigma_0}}\mathcal{M}'_y$ does have *some* generic constituent (by Theorem 9.9).

Suppose $\bigotimes_{v\in\Sigma_0}\pi_v$ is *any* generic constituent of $\varinjlim_{K_{\Sigma_0}}\mathcal{M}'_y$. Lemma 9.1 tells us π_v and $\pi_{x,v}$ have the same supercuspidal support. By the theory of Bernstein-Zelevinsky derivatives $\mathrm{Ind}_{P_{n_1,\dots,n_t}}^{\mathrm{GL}_n}(\tilde{\pi}_1\otimes\cdots\otimes\tilde{\pi}_t)$ has a *unique* generic constituent (where the $\tilde{\pi}_i$ are supercuspidals, or rather twists $\tilde{\pi}_i\otimes|\det|^{(1-n_i)/2}$ as in Section 9A). Consequently, there is a unique generic representation $\pi_{x,v}^{\mathrm{gen}}$ with the same supercuspidal support as $\pi_{x,v}$, and $\pi_v\simeq\pi_{x,v}^{\mathrm{gen}}$. Of course, under the Iwahori assumption the $\tilde{\pi}_i$ are unramified characters, so here $\pi_{x,v}^{\mathrm{gen}}$ is the generic constituent of an unramified principal series. Note however that this does *not* mean $\pi_{x,v}^{\mathrm{gen}}$ is necessarily a twisted Steinberg representation (when the principal series is reducible). For instance, for GL(3) one could have an induced-from-Steinberg representation $\chi_1\,\mathrm{St}_{\mathrm{GL}(2)}\times\chi_2$ and so on, see [Sorensen 2006, Table A, p. 1757].

We summarize our findings:

Theorem 9.11. *Let $y=(x,\delta)\in Y(K^p,\bar{r})$ be an arbitrary point, where K_{Σ_0} is a product of Iwahori subgroups. Then the following holds*:

(1) $\bigotimes_{v\in\Sigma_0}\pi_{x,v}^{\mathrm{gen}}$ ***occurs*** *as a constituent of* $\varinjlim_{K_{\Sigma_0}}\mathcal{M}'_y$ (possibly with multiplicity).

(2) *Every generic constituent of $\varinjlim_{K_{\Sigma_0}}\mathcal{M}'_y$ is isomorphic to $\bigotimes_{v\in\Sigma_0}\pi_{x,v}^{\mathrm{gen}}$.*

Here $\pi_{x,v}^{\mathrm{gen}}$ is the generic representation of $\mathrm{GL}_n(F_{\tilde{v}})$ with the same supercuspidal support as $\pi_{x,v}$.

It would be interesting to relax the assumption that K_v is Iwahori for $v\in\Sigma_0$. In [Chan and Savin 2019] they consider more general $\mathfrak{s}$ in the Bernstein spectrum of GL_{mr} (where the Levi is $\mathrm{GL}_r\times\cdots\times\mathrm{GL}_r$ and the supercuspidal representation is $\tau\otimes\cdots\otimes\tau$). For such an $\mathfrak{s}$-type (J,ρ) one can identify the Hecke algebra $\mathcal{H}(J,\rho)$ with the Iwahori–Hecke algebra of GL_m — but over a possibly larger p-adic field. This is used to define the subalgebra $\mathcal{H}_{S_m}\subset\mathcal{H}(J,\rho)$ which carries the sgn-character. If $\pi\in\mathcal{R}^{\mathfrak{s}}(\mathrm{GL}_{mr})$ is an admissible representation, [Chan and Savin 2019, Theorem 3.5] shows that a certain adjunction map $\mathbb{S}_\rho(\pi)\to\pi_{U,\psi}$ is an isomorphism, where $\mathbb{S}_\rho(\pi)$ denotes the sgn-isotypic subspace of $\mathrm{Hom}_J(\rho,\pi)$. (In the case $r=1$ and $\tau=\mathbf{1}$ this recovers Theorem 9.9 above; the type is $(I,\mathbf{1})$.) Instead of considering $\hat{S}(K_{\Sigma_0}K^\Sigma,E)_{\mathfrak{m}}$ in the eigenvariety construction one could take $K_{\Sigma_0}=\prod_{v\in\Sigma_0}J_v$ and $\rho=\bigotimes_{v\in\Sigma_0}\rho_v$ for certain types (J_v,ρ_v) and consider the space $\mathrm{Hom}_{K_{\Sigma_0}}(\rho,\hat{S}(K^\Sigma,E)_{\mathfrak{m}})$ which would result in an eigenvariety $Y_\rho(K_{\Sigma_0}K^\Sigma,\bar{r})$ which of course sits as a closed subvariety of $Y(K'_{\Sigma_0}K^\Sigma,\bar{r})$ for $K'_{\Sigma_0}\subset\ker(\rho)$.

If we take an arbitrary point $y \in Y_\rho(K_{\Sigma_0}K^{\Sigma}, \bar{r})$ we know $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_y$ lies in the $\mathfrak{s}_v$-component (for each $v \in \Sigma_0$) and it is at least plausible the above arguments with $\mathbb{S}$ replaced by $\mathbb{S}_\rho$ would allow us to draw the same conclusion: $\varinjlim_{K_{\Sigma_0}} \mathcal{M}'_y$ admits $\bigotimes_{v\in\Sigma_0} \pi^{\mathrm{gen}}_{x,v}$ as its unique generic irreducible subquotient (up to multiplicity). The inertial classes $\mathfrak{s}$ considered in [Chan and Savin 2019] are somewhat limited. However, Savin has communicated to us a more general (unpublished) genericity criterion — without restrictions on $\mathfrak{s}$.

10. A brief comparison with work of Bellaïche and Chenevier

As noted in the introduction, the papers [Bellaïche and Chenevier 2009; Chenevier 2009] contain results of the nature of those of this paper. In particular, Theorem 1.1(1) appears as [Chenevier 2009, remarque 3.13]. This section is an attempt to give a slightly more detailed comparison. The theory of eigenvarieties used by Bellaïche and Chenevier are those constructed in [Chenevier 2004]. In [Bellaïche and Chenevier 2009, §7.4], they construct, on an eigenvariety X, a sheaf Π_S of admissible $G(\mathbb{A}_S)$-representations, where S is a finite set of places away from p. As in our paper, this sheaf is constructed using the natural coherent sheaf coming from their construction.[7] Bellaïche and Chenevier then study how the fibers $\Pi_{S,x}$ vary with $x \in X$, and in particular show the finiteness property stated in Theorem 1.1(1). Each point x has an associated Hecke eigensystem $\psi_x : \mathcal{H} \to \kappa(x)$ and one considers a certain generalized eigenspace $\mathcal{S}^{\psi_x}$ of p-adic automorphic forms; $\Pi_S^{\psi_x}$ is then the $G(\mathbb{A}_S)$-representation over $\mathcal{O}_{X,x}/\mathfrak{m}_{\omega(x)}\mathcal{O}_{X,x}$ generated by $\mathcal{S}^{\psi_x}$. A rough "dictionary" between this paper and [Bellaïche and Chenevier 2009] is

$$G(\mathbb{A}_S) \leftrightsquigarrow U(F^+_{\Sigma_0}), \quad \Pi_{S,x} \leftrightsquigarrow \varinjlim_{K_{\Sigma_0}} \mathcal{M}'_y, \quad \Pi_S^{\psi_x} \leftrightsquigarrow \otimes_{v\in\Sigma_0}\pi_{x,v}.$$

We remark that the eigenvarieties used in [Bellaïche and Chenevier 2009] are isomorphic to those used here (when one uses the same input data in terms of groups, Hecke operators and so forth) by work of Loeffler [2011]. In fact even more is true, the coherent sheaves produced by the two different constructions agree.[8]

Let us now discuss the local-global compatibility of [Chenevier 2009]. Both his and our approach rely on the use of Bernstein center elements. Chenevier's very elegant approach is to build the elements he needs into his eigenvariety; this new eigenvariety is then an open and closed subset of the original eigenvariety. By contrast, we use the action on the coherent sheaf on an eigenvariety without any Hecke operators at ramified places.

[7]Recall that all known eigenvariety constructions equip the eigenvariety with a coherent sheaf that remembers the finite slope part of the spaces used to construct it.

[8]This is presumably well known to experts, and can be deduced from an extension of the method of [Loeffler 2011], though as far as we know this result does not currently appear in the literature.

We now go into slightly more detail. In this paragraph we work locally and let GL_n denote $\mathrm{GL}_n(F_{\tilde{v}})$ for some $v \in \Sigma_0$. For a fixed Bernstein component $\mathcal{R}^{\mathfrak{s}}(\mathrm{GL}_n)$ with center $\mathfrak{Z}^{\mathfrak{s}}$ Chenevier defines a continuous n-dimensional pseudocharacter

$$T^{\mathfrak{s}} : W_{F_{\tilde{v}}} \to \mathfrak{Z}^{\mathfrak{s}}$$

uniquely characterized by the following property (see [Chenevier 2009, Proposition 3.11]): For every irreducible π in $\mathcal{R}^{\mathfrak{s}}(\mathrm{GL}_n)$, on which $\mathfrak{Z}^{\mathfrak{s}}$ acts via the character $z_\pi : \mathfrak{Z}^{\mathfrak{s}} \to E$, one has the identity

$$(z_\pi \circ T^{\mathfrak{s}})(\tau) = \mathrm{tr}\big(\tau \,|\, \mathrm{rec}(\pi \otimes |\det|^{(1-n)/2})\big)$$

for all $\tau \in W_{F_{\tilde{v}}}$. (Note the different normalizations of the local Langlands correspondence; Chenevier takes the trace of τ on $\mathcal{L}_W(\pi) = \mathrm{rec}(\pi \otimes |\det|^{(1-n)/2})^{\mathrm{ss}}$.) In particular our Bernstein center element f_τ coincides with $T^{\mathfrak{s}}(\tau)$ on representations in $\mathcal{R}^{\mathfrak{s}}(\mathrm{GL}_n)$.

As mentioned earlier, in [Chenevier 2009] the eigenvariety Y comes with a choice of Bernstein components $(\mathfrak{s}_v)_{v \in \Sigma_0}$ and a homomorphism

$$\mathcal{H} = \mathcal{H}^{\Sigma} \otimes \big(\textstyle\bigotimes_{v \in \Sigma_0} \mathfrak{Z}^{\mathfrak{s}_v}\big) \to \mathcal{O}(Y)$$

(where $\mathcal{H}^{\Sigma}$ is the product of the spherical Hecke algebras away from Σ). For each $v \in \Sigma_0$ one composes $T^{\mathfrak{s}_v}$ with $\mathfrak{Z}^{\mathfrak{s}_v} \to \mathcal{O}(Y)$ and gets a pseudocharacter $T'_v : W_{F_{\tilde{v}}} \to \mathcal{O}(Y)$. On the other hand, one can restrict the Galois pseudocharacter $T : \mathrm{Gal}_F \to \mathcal{O}(Y)$ to the Weil group. By [Chenevier 2009, Lemma 3.12] they coincide:

$$T|_{W_{F_{\tilde{v}}}} = T'_v.$$

Consequently, to any $\tau \in W_{F_{\tilde{v}}}$ one can attach a function $a_\tau \in \mathcal{O}(Y)$ which specializes to $\mathrm{tr}\,(r_x(\tau))$ for any $y = (x, \delta) \in Y$. (Simply take a_τ to be the image of $T^{\mathfrak{s}_v}(\tau)$ under the map $\mathfrak{Z}^{\mathfrak{s}_v} \to \mathcal{O}(Y)$.)

The goal of [Chenevier 2009] is to use the p-adic deformation arguments above to remove a regularity assumption on the weight, and attach Galois representations $r_{\pi,\iota}$ to any automorphic representation π of $G(\mathbb{A})$. Théorème 3.3 in [loc. cit.] achieves this goal and proves local-global compatibility (up to semisimplification):

$$\mathrm{WD}(r_{\pi,\iota}|_{\mathrm{Gal}_{F_{\tilde{v}}}})^{\mathrm{ss}} \simeq \mathrm{rec}\big(\mathrm{BC}_{\tilde{v}|v}(\pi_v) \otimes |\det|^{(1-n)/2}\big)^{\mathrm{ss}}. \tag{10-1}$$

In fact Bellaïche and Chenevier can prove a stronger result and even compare the monodromy operators with respect to the usual partial order on partitions, see [Chenevier 2009, Theorem 3.5]. With our definition of $\pi_{x,v}$, (10-1) amounts to $\pi_{x,v}$ and π_v having the same supercuspidal support, for classical points $y = (x, \delta)$.

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