LOCAL LANGLANDS IN RIGID FAMILIES

CLAUS SORENSEN

Abstract. We show that local-global compatibility away from \( p \) holds at points on the \((p\text{-adic})\) eigenvariety of a definite unitary group which are étale over weight space. The novelty is we allow non-classical points. More precisely we interpolate the local Langlands correspondence across the eigenvariety by considering the stalks of its defining coherent sheaf. We employ techniques of Scholze from his recent new approach to the local Langlands conjecture.

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1. Introduction

The proof of (strong) local-global compatibility in the \( p \)-adic Langlands program for \( GL(2)/\mathbb{Q} \), cf. [Em11b], led Emerton and Helm to speculate on whether the local Langlands correspondence for \( GL(n) \) can be interpolated in Zariski families: Say \( L/\mathbb{Q}_\ell \) is a finite extension, with absolute Galois group \( \text{Gal}_L \), and \( R \) is a reduced complete local Noetherian ring with residual characteristic \( p \neq \ell \). If \( \rho : \text{Gal}_L \to GL_n(R) \) is a continuous representation it is shown in [EH14] (cf. their Theorem 1.2.1) that there is at most one admissible \( GL_n(L) \)-representation \( \tilde{\pi}(\rho) \) over \( R \) satisfying a short list of natural desiderata:

(a) \( \tilde{\pi}(\rho) \) is \( R \)-torsion free;
(b) If \( p \subset R \) is a minimal prime, with residue field \( \kappa(p) \) – a finite extension of \( \mathbb{Q}_p \), then
\[
\tilde{\pi}(\rho \otimes_R \kappa(p)) \simto \tilde{\pi}(\rho) \otimes_R \kappa(p),
\]
where the source is the representation of \( \text{GL}_n(L) \) associated with \( \rho \otimes_R \kappa(p) \) under the local Langlands correspondence (suitably modified and normalized);

(c) The reduction \( \tilde{\pi}(\rho) \otimes_R R/\mathfrak{m}_R \) has an absolutely irreducible and generic cosocle (i.e., the largest semisimple quotient), and no other generic constituents.

They conjecture ([EH14, Conj. 1.4.1]) that such a \( \tilde{\pi}(\rho) \) always exists, and this is now known in many cases; in particular when \( p \) is banal (meaning that \( 1, q, \ldots, q^n \) are distinct modulo \( p \), where \( q \) is the size of the residue field of \( L \)), see section 8 of [Hel16].

In this paper we take a different point of view, and interpolate the local Langlands correspondence in rigid families – across eigenvarieties of definite unitary groups; in much the same spirit as [Pau11] for the eigencurve, except that our approach is more representation-theoretic than geometric and based on the construction of eigenvarieties in [Em06a] and [BHS16]. In the next few paragraphs we introduce notation in order to state our main result (Theorem 1.1 below).

Let \( p > 2 \) be a prime, and fix an unramified CM extension \( F/F^+ \) which is split at all places \( v \) of \( F^+ \) above \( p \). Suppose \( U/F^+ \) is a unitary group in \( n \) variables which is quasi-split at all finite places and compact at infinity (see 2.1 for more details). Throughout \( \Sigma \) is a finite set of finite places of \( F^+ \) containing \( \Sigma_p = \{ v : v|p \} \), and we let \( \Sigma_0 = \Sigma \setminus \Sigma_p \). We assume all places \( v \in \Sigma \) split in \( F \) and we choose a divisor \( \tilde{v}|v \) once and for all, which we use to make the identification \( U(F^+_v) \simto \text{GL}_n(F_v) \). We consider tame levels of the form \( K^p = K_{\Sigma_0} K_\Sigma \) where \( K_\Sigma = \prod_{v \notin \Sigma} K_v \) is a product of hyperspecial maximal compact subgroups, and \( K_{\Sigma_0} = \prod_{v \in \Sigma_0} K_v \).

Our coefficient field is sufficiently large finite extension \( E/\mathbb{Q}_p \) with residue field \( k = k_E \), and we start off with an absolutely irreducible Galois representation \( \tilde{\rho} : \text{Gal}_F \rightarrow \text{GL}_n(k) \) which is automorphic of tame level \( K^p \). We let \( m = m_\rho \) be the associated maximal ideal, viewed in various Hecke algebras (see sections 2.3 and 2.4 for more details). In 2.5 and 3.2 we introduce the universal deformation ring \( R_{\tilde{\rho}} \) and the deformation space \( X_{\tilde{\rho}} \). Each point \( x \in X_{\tilde{\rho}} \) carries a Galois representation \( r_x \), which is a deformation of \( \tilde{\rho} \), and we let \( p_x \subset R_{\tilde{\rho}} \) be the associated prime ideal. The Banach representation of \( p \)-adic automorphic forms \( \tilde{S}(K^p, E)_m \) inherits a natural \( K_{\tilde{\rho}} \)-module structure, and we consider its \( p_x \)-torsion \( \tilde{S}(K^p, E)_m[p_x] \) and its dense subspace of locally analytic vectors \( \tilde{S}(K^p, E)_m[p_x]^{an} \), cf. section 2.2.

The eigenvariety \( Y(K^p, \tilde{\rho}) \subset X_{\tilde{\rho}} \times \tilde{T} \) is defined as the support of a certain coherent sheaf \( M \) on \( X_{\tilde{\rho}} \times \tilde{T} \). Here \( \tilde{T} \) denotes the character space of the \( p \)-adic torus \( T \subset U(F^+ \otimes \mathbb{Q}_p) \) isomorphic to \( \prod_{v|p} \text{GL}(n)(F_v) \), see 3.3 below. We have \( \tilde{T} \simeq \mathcal{W} \times (G^{\text{uni}}_{\text{rig}})^{n|\Sigma_0} \) where \( \mathcal{W} \) is weight space (parametrizing continuous characters of the maximal compact subgroup of \( T \)) which is a disjoint union of finitely many open unit balls of dimension \( n[F^+ : \mathbb{Q}] \). By definition a point \( y = (x, \delta) \in X_{\tilde{\rho}} \times \tilde{T} \) belongs to the eigenvariety \( Y(K^p, \tilde{\rho}) \) if and only if the stalk \( M_y \) is nonzero. If \( y \) is \( E \)-rational the \( E \)-linear dual of \( M_y \) can be described as \( M_y^* \simeq J_B^*(\tilde{S}(K^p, E)_m[p_x]^{an}) \) where \( J_B \) denotes Emerton’s locally analytic variant of the Jacquet functor ([Em06b]) and \( J_B^* \) means the \( \delta \)-eigenspace. Morally our main result states that \( \lim_{\rho \rightarrow K_{\Sigma_0}} M_y^* \) interpolates the local Langlands correspondence across the eigenvariety (at the \( \mathcal{\acute{e}tale} \) points; but possibly non-classical). Here is the precise formulation.
Theorem 1.1. Let \( y = (x, \delta) \in Y(K^p, \hat{r})(E) \) be a point which is étale over weight space \( W \), and such that \( r_x \) is generic at every \( v \in \Sigma_0 \) (cf. Def. 8.4 below). Then

\[
\lim_{K_{\Sigma_0}} J_B^y \left( \hat{S}(K^p, E)_m[p_x]^{an} \right)
\]

is an admissible representation of \( U(F_{\Sigma_0}^+) \) isomorphic to \( \prod_{v \in \Sigma_0} GL_n(F_v) \) of finite length, and all its irreducible subquotients are isomorphic to \( \otimes_{v \in \Sigma_0} \pi_{x,v} \), where \( \pi_{x,v} \) is the irreducible smooth representation of \( U(F_v^+) \) associated with \( r_x |_{Gal_{F_v}} \) via the local Langlands correspondence. More precisely,

\[
WD(r_x |_{Gal_{F_v}})^{F-ss} \simeq \text{rec}(BC_{\hat{r}})(\pi_{x,v}) \otimes | \det(1-n)/2 |
\]

with the local Langlands correspondence \( \text{rec}() \) normalized as in [HT01].

The notation \( BC_{\hat{r}}(\pi_{x,v}) \) used in the theorem signifies local base change, which simply amounts to viewing \( \pi_{x,v} \) as a representation of \( GL_n(F_v) \) via its identification with \( U(F_v^+) \).

By \( y \) being étale we mean that there is a neighborhood \( \Omega \) such that the weight map \( \omega : Y(K^p, \hat{r}) \to W \) restricts to an isomorphism \( \Omega \to \omega(\Omega) \). This may be a bit ad hoc.

Our result is actually slightly more general (cf. Lemma 8.1) in that it deals with non-generic points as well. In that generality one can only say that any irreducible subquotient of \( \lim_{v \in \Sigma_0} M'_y \) has the same supercuspidal support as \( \otimes_{v \in \Sigma_0} \pi_{x,v} \) (we have no information about the monodromy operator).

The length of \( \lim_{v \in \Sigma_0} M'_y \) as a \( U(F_{\Sigma_0}^+) \)-representation is believed to measure how critical \( \delta \) is. For example, if \( y \) is a classical point of non-critical slope (automatically étale by [Che11, Thm. 4.10]) the length is one, cf. Proposition 4.2 below, which also shows that the length can be \( \geq 1 \) if \( y \) is critical.

Theorem 1.1 was motivated in part by the question of local-global compatibility for the Breuil-Herzig construction \( \Pi(\rho)_{\text{ord}} \), cf. [BH15, Conj. 4.2.5]. The latter is defined for upper triangular \( p \)-adic representations \( \rho \) of \( Gal_{Q_p} \), and is supposed to model the largest subrepresentation of the “true” \( p \)-adic local Langlands correspondence built from unitary continuous principal series representations. We approach this problem starting from the inclusion (for unitary \( y \))

\[
(1.2) \quad J_B^y \left( \hat{S}(K^p, E)_m[p_x]^{an} \right) \hookrightarrow \text{Ord}_B^y \left( \hat{S}(K^p, O)_m[p_x]\right)[1/p]^{an},
\]

as shown in [Sor15, Thm. 6.2]. Here \( \text{Ord}_B^y \) is Emerton’s functor of ordinary parts ([Em10]), which is right adjoint to parabolic induction \( \text{Ind}_B^y \). If \( y = (x, \delta) \) lies on \( Y(K^p, \hat{r}) \) the source of \( (1.2) \) is nonzero, and we deduce the existence of a nonzero (norm-decreasing) equivariant map \( \text{Ind}_B^y(\delta) \to \hat{S}(K^p, E)_m[p_x] \). If one could show that certain Weyl-conjugates \( y_w = (x, \omega \delta) \) all lie on \( Y(K^p, \hat{r}) \) one would infer that there is a non-trivial map \( \text{soc}_{GL_n(O_p)}(\Pi(\rho)_{\text{ord}}) \to \hat{S}(K^p, E)_m[p_x] \) which one could hope to promote to a map \( \Pi(\rho)_{\text{ord}} \to \hat{S}(K^p, E)_m[p_x] \) using [BH15, Cor. 4.3.11]. Here we take \( \rho = r_x |_{Gal_{F_p}} \) (up to a twist which we ignore here) for some \( v|p \) such that \( F_v = Q_p \), and \( x \) is a point where \( r_x |_{Gal_{F_v}} \) is upper triangular with \( \delta_v \) on the diagonal. With a little more work, if \( y \) is étale over \( W \), Theorem 1.1 would give strong local-global compatibility – in the sense that there is an embedding

\[
\otimes_{v \in \Sigma_0} \pi_{x,v} \hookrightarrow \lim_{K_{\Sigma_0}} \text{Hom}_{GL_n(O_p)}(\Pi(\rho)_{\text{ord}}, \hat{S}(K^p, E)_m[p_x]).
\]

We hope to return to these questions on another occasion.

We briefly outline the overall strategy behind the proof of Theorem 1.1: For classical points \( y = (x, \delta) \) (i.e., those corresponding to automorphic representations) local-global compatibility away from \( p \)
essentially gives an inclusion $\otimes_{v \in \Sigma_0} \pi_{x,v} \hookrightarrow \lim_{\rightarrow K_{\Sigma_0}} M'_y$ which is an isomorphism if $\delta$ moreover is of non-critical slope. We reinterpret this using ideas from Scholze’s proof of the local Langlands correspondence ([Sc13b]). He works with certain elements $f_\tau$ in the Bernstein center of $GL_n(F_\omega)$, associated with $\tau \in W_{F_\omega}$, which act on an irreducible smooth representation $\Pi$ via scaling by $tr(\tau|rec(\Pi))$; here and throughout this paragraph we ignore a twist by $| \det |^{(1-n)/2}$ for simplicity. For each tuple $\tau = (\tau_v) \in \prod_{v \in \Sigma_0} W_{F_v}$ we thus have an element $f_\tau := \otimes_{v \in \Sigma_0} f_{\tau_v}$ of the Bernstein center of $U(F_{\Sigma_0}^+) \xrightarrow{\sim} \prod_{v \in \Sigma_0} GL_n(F_v)$ which we know how to evaluate on all irreducible smooth representations. In particular $f_\tau$ acts on $\lim_{\rightarrow K_{\Sigma_0}} M'_y$ via scaling by $\prod_{v \in \Sigma_0} tr(\tau_v|rec(B_{\Sigma|_v}(\pi_{x,v})))$ – still assuming $y$ is classical and non-critical. Those points are dense in $Y(K^p, \tilde{r})$, and using this we interpolate this last scaling property to all étale points $y$ as follows. By mimicking the standard proof of Grothendieck’s monodromy theorem one can interpolate $WD(r_x|Gal_{F_\omega})$ in families. Namely, for each reduced $Sp(A) \subset X_r$ we construct a Weil-Deligne representation $WD_{r,\tilde{r}}$ over $A$ which specializes to $WD(r_x|Gal_{F_\omega})$ for all $x \in Sp(A)$. For étale points $y$ we find a neighborhood $\Omega \subset Sp(A) \times \tilde{T}$ such that the weight morphism $\omega : Y(K^p, \tilde{r}) \to \mathcal{W}$ restricts to an isomorphism $\Omega \xrightarrow{\sim} \omega(\Omega)$. This ensures that $\Gamma(\Omega, M)$ is a finite type projective module over $O_{Y(K^p, \tilde{r})}(\Omega)$, which in turn allows us to show that $f_\tau$ acts on $\lim_{\rightarrow K_{\Sigma_0}} \Gamma(\Omega, M)$ via scaling by $\prod_{v \in \Sigma_0} tr(\tau_v|WD_{r,\tilde{r}})$, which uses the density of the classical non-critical points in $\Omega$. By specialization at $y$ we deduce that $f_\tau$ acts on $\lim_{\rightarrow K_{\Sigma_0}} M'_y$ via scaling by $\prod_{v \in \Sigma_0} tr(\tau_v|rec(B_{\Sigma|_v}(\pi_{x,v})))$ as desired. This result tells us that every irreducible constituent $\otimes_{v \in \Sigma_0} \pi_v$ of $\lim_{\rightarrow K_{\Sigma_0}} M'_y$ has the same supercuspidal support as $\otimes_{v \in \Sigma_0} \pi_{x,v}$, and therefore is isomorphic to it if $x$ is a generic point. We also infer that $\lim_{\rightarrow K_{\Sigma_0}} M'_y$ has finite length since $\dim M'_y < \infty$ and the constituents $\otimes_{v \in \Sigma_0} \pi_v$ have conductors bounded by the conductors of $WD(r_x|Gal_{F_\omega})$.

We finish with a few remarks on the structure of the paper. In our first (rather lengthy) section 2 we introduce in detail the notation and assumptions in force throughout; the unitary groups $U_{/F^+}$, automorphic forms $S(K^p, E)$, Hecke algebras, Galois representations and their deformations. Section 3 then defines the eigenvarieties $Y(K^p, \tilde{r})$ and the sheaves $M_{K^+}$, essentially following [BHS16] and [Em06a]. In section 4 we recall the notion of a non-critical classical point, and prove Theorem 1.1 for those. Section 5 interpolates the Weil-Deligne representations across reduced $Sp(A) \subset X_r$ by suitably adapting Grothendieck’s argument. We recall Scholze’s characterization of the local Langlands correspondence in section 6, and introduce the functions $f_\tau$ in the Bernstein center. The goal of section 7 is to show Proposition 7.2 on the action of $f_{\tau}$ on $\lim_{\rightarrow K_{\Sigma_0}} \Gamma(\Omega, M_{K^r})$ where $\Omega$ is a neighborhood of an étale point. Finally in section 8 we put the pieces together; we introduce the notion of a generic point, and prove our main result.

2. Notation and terminology

We denote the absolute Galois group $Gal(F^{sep}/F)$ of a field $F$ by $Gal_F$.

2.1. Unitary groups. Our setup will be identical to that of [BHS16] although we will adopt a slightly different notation, which we will introduce below.

We fix a CM field $F$ with maximal totally real subfield $F^+$ and $Gal(F/F^+) = \{1, c\}$. We assume the extension $F/F^+$ is unramified at all finite places, and split at all places $v|p$ of $F^+$ above a fixed prime $p$.

Let $n$ be a positive integer. If $|F^+ : \mathbb{Q}|$ is odd assume $n$ is odd as well. This guarantees the existence of a unitary group $U_{/F^+}$ in $n$ variables such that

- $U \times_{F^+} F \xrightarrow{\sim} GL_n,$
• $U$ is quasi-split over $F_v^+$ for all finite places $v$,
• $U(F^+ \otimes \mathbb{R})$ is compact.

We let $G = \text{Res}_{F^+/\mathbb{Q}} U$ be its restriction of scalars.

If $v$ splits in $F$ the choice of a divisor $v|v$ determines an isomorphism $i_v : U(F_v^+) \xrightarrow{\sim} GL_n(F_v)$ well-defined up to conjugacy. Throughout we fix a finite set $\Sigma$ of finite places of $F$, and $\Sigma$ contains $\Sigma_p = \{ v : v|p \}$. We let $\Sigma_0 = \Sigma \setminus \Sigma_p$. For each $v \in \Sigma$ we choose a divisor $\tilde{v}|v$ once and for all and let $\tilde{\Sigma} = \{ \tilde{v} : v \in \Sigma \}$. We also choose an embedding $\text{Gal}_{F_0} \hookrightarrow \text{Gal}_F$ for each such $v$. Moreover, we choose isomorphisms $i_{\tilde{v}}$ which we will tacitly use to identify $U(F_{\tilde{v}}^+)$ with $GL_n(F_{\tilde{v}})$. For instance the collection $(i_{\tilde{v}})_{\tilde{v}p}$ gives an isomorphism

$$(2.1) \quad G(\mathbb{Q}_p) = U(F^+ \otimes \mathbb{Q}_p) \xrightarrow{\sim} \prod_{v|p} GL_n(F_v).$$

Similarly $U(F_{\tilde{v}}^+) \xrightarrow{\sim} \prod_{v \in \Sigma} GL_n(F_v)$ and analogously for $U(F_{\Sigma_0}^+)$. When there is no risk of confusion we will just write $G$ instead of $G(\mathbb{Q}_p)$. We let $B \subset G$ be the inverse image of the upper-triangular matrices under (2.1). In the same fashion $T$ corresponds to the diagonal matrices, and $N$ corresponds to the unipotent radical. Their opposites are denoted $\tilde{B}$ and $\tilde{N}$.

Below we will only consider tame levels $K^p \subset G(\mathbb{A})^p$ of the form $K^p = \prod_{v|p} K_v$ where $K_v \subset U(F_v^+)$ is a compact open subgroup which is assumed to be hyperspecial for $v \notin \Sigma$ (in particular $\Sigma$ has to be large enough to contain the places where $G$ is ramified). Accordingly we factor it as $K^p = K_{\Sigma_0} K^\Sigma$ where $K^\Sigma = \prod_{v \notin \Sigma} K_v$ is a product of hyperspecials, and $K_{\Sigma_0} = \prod_{v \in \Sigma_0} K_v$.

2.2. Automorphic forms. We work over a fixed finite extension $E/\mathbb{Q}_p$, which we assume is large enough in the sense that every embedding $F_v^+ \hookrightarrow \mathbb{Q}_p$ factors through $E$ for all $v|p$. We let $\mathcal{O}$ denote its valuation ring, $\varpi$ is a choice of uniformizer, and $k = \mathcal{O}/(\varpi) \simeq \mathbb{F}_q$ is the residue field. We endow $E$ with its normalized absolute value $| \cdot |$ for which $|\varpi| = q^{-1}$.

For a tame level $K^p \subset G(\mathbb{A})^p$ we introduce the space of $p$-adic automorphic forms on $G(\mathbb{A})$ as follows (cf. Definition 3.2.3 in [Em06a]). First let

$$\tilde{S}(K^p, \mathcal{O}) = C(G(\mathbb{Q})\backslash G(A_f)/K^p, \mathcal{O}) = \lim_{\leftarrow i} C^\infty(G(\mathbb{Q})\backslash G(A_f)/K^p, \mathcal{O}/\varpi^i \mathcal{O}).$$

Here $C$ is the space of continuous functions, $C^\infty$ is the space of locally constant functions. Note that the space of locally constant functions in $\tilde{S}(K^p, \mathcal{O})$ is $\varpi$-dially dense, so alternatively

$$\tilde{S}(K^p, \mathcal{O}) = C^\infty(G(\mathbb{Q})\backslash G(A_f)/K^p, \mathcal{O})^\wedge = \lim_{\leftarrow i} C^\infty(G(\mathbb{Q})\backslash G(A_f)/K^p, \mathcal{O}) \otimes \mathcal{O}/\varpi^i \mathcal{O}.$$  

These two viewpoints amount to thinking of $\tilde{S}(K^p, \mathcal{O})$ as $\tilde{H}^0(K^p)$ or $\tilde{H}^0(K^p)$ respectively in the notation of [Em06a], cf. (2.1.1) and Corollary 2.2.25 there. The reduction modulo $\varpi$ is the space of mod $p$ modular forms on $G(\mathbb{A})$,

$$S(K^p, k) = C^\infty(G(\mathbb{Q})\backslash G(A_f)/K^p, k) \simeq \tilde{S}(K^p, \mathcal{O})/\varpi \tilde{S}(K^p, \mathcal{O}),$$

which is an admissible (smooth) $k[G]$-module with $G = G(\mathbb{Q}_p)$ acting via right translations. Thus $\tilde{S}(K^p, \mathcal{O})$ is a $\varpi$-adically admissible $G$-representation over $\mathcal{O}$, i.e. an object of $\text{Mod}_{G, \varpi\text{-adm}}(\mathcal{O})$ (cf. Definition 2.4.7 in [Em10]). Since it is clearly flat over $\mathcal{O}$, it is the unit ball of a Banach representation

$$\tilde{S}(K^p, E) = \tilde{S}(K^p, \mathcal{O})[1/p] = C(G(\mathbb{Q})\backslash G(A_f)/K^p, E).$$
Here we equip the right-hand side with the supremum norm \( \|f\| = \sup_{g \in G(\mathbb{A})} |f(g)| \), and \( \hat{S}(K^p, E) \) thus becomes an object of the category \( \text{Ban}_{\mathbb{R}}(E)^{\leq 1} \) of Banach \( E \)-spaces \( (H, \|\cdot\|) \) for which \( \|H\| \subset |E| \) endowed with an isometric \( G \)-action. \( \hat{S}(K^p, E) \) is dubbed the space of \( p \)-adic automorphic forms on \( G(\mathbb{A}) \).

The connection to classical modular forms is through locally algebraic vectors as we now explain. Let \( V \) be an absolutely irreducible algebraic representation of \( G \times E \). Thus \( V \) is a finite-dimensional \( E \)-vector space with an action of \( G(E) \), which we restrict to \( G(\mathbb{Q}_p) \). If \( K_p \subset G(\mathbb{Q}_p) \) is a compact open subgroup we let it act on \( V \) and consider

\[
S_V(K^p, K^p, E) = \text{Hom}_{K_p}(V, \hat{S}(K^p, E)).
\]

If we assume \( E \) is large enough that \( \text{End}_G(V) = E \), the space of \( V \)-locally algebraic vectors in \( \hat{S}(K^p, E) \) can be defined as the image of the natural map

\[
\lim_{K_p} V \otimes_E S_V(K^p, K^p, E) \xrightarrow{\sim} \hat{S}(K^p, E)^{V - \text{alg}} \hookrightarrow \hat{S}(K^p, E)
\]

(cf. Proposition 4.2.4 in [Em11a]). Then the space of all locally algebraic vectors decomposes as a direct sum \( \hat{S}(K^p, E)^{\text{alg}} = \bigoplus V \hat{S}(K^p, E)^{V - \text{alg}} \). Letting \( \hat{V} \) denote the contragredient representation, one easily identifies \( S_V(K^p, K^p, E) \) with the space of (necessarily continuous) functions

\[
f : G(\mathbb{Q}) \backslash G(\mathbb{A}) / K^p \longrightarrow \hat{V}, \quad f(gk) = k^{-1}f(g) \quad \forall k \in K_p.
\]

In turn, considering the function \( h(g) = gf(g) \) identifies it with the space of right \( K_p \)-invariant functions \( h : G(\mathbb{A}) \rightarrow \hat{V} \) such that \( h(\gamma g) = \gamma h(g) \) for all \( \gamma \in G(\mathbb{Q}) \). If we complexify this space along an embedding \( \iota : E \hookrightarrow \mathbb{C} \) we obtain vector-valued automorphic forms. Thus we arrive at the decomposition

\[(2.2)\]

\[
S_V(K^p, K^p, E) \otimes_{E, \iota} \mathbb{C} \simeq \bigoplus_{\pi} m_G(\pi) \cdot \pi_p^K \otimes (\pi_p^\iota)^{K^p}
\]

with \( \pi \) running over automorphic representations of \( G(\mathbb{A}) \) with \( \pi_{\infty} \simeq V \otimes_{E, \iota} \mathbb{C} \). It is even known by now that all \( m_G(\pi) = 1 \), cf. [Mok15] and [KMSW] (both based on the symplectic/orthogonal case [Art13]).

2.3. Hecke algebras. At each \( \nu \|

\mathbb{p} \) we consider the Hecke algebra \( \mathcal{H}(U(F_\nu^+), K_\nu) \) of \( K_\nu \)-biinvariant compactly supported functions \( \phi : U(F_\nu^+) \rightarrow \mathcal{O} \) (with \( K_\nu \)-normalized convolution). The characteristic functions of double cosets \( [K_\nu\gamma, K_\nu] \) form an \( \mathcal{O} \)-basis.

Suppose \( \nu \) splits in \( F \) and \( K_\nu \) is hyperspecial. Choose a place \( w|\nu \) and an isomorphism \( i_w \) which restricts to \( i_w : K_\nu \xrightarrow{\sim} \text{GL}_n(\mathcal{O}_F) \). Then we identify \( \mathcal{H}(U(F_\nu^+), K_\nu) \) with the spherical Hecke algebra for \( \text{GL}_n(F_w) \). We let \( \gamma_{w,j} \in U(F_\nu^+) \) denote the element corresponding to

\[
i_w(\gamma_{w,j}) = \text{diag}(\varpi_{F_w}, \ldots, \varpi_{F_w}, 1, \ldots, 1).
\]

Then let \( T_{w,j} = [K_\nu \gamma_{w,j}, K_\nu] \) be the standard Hecke operators; \( \mathcal{H}(U(F_\nu^+), K_\nu) = \mathcal{O}[T_{w,1}, \ldots, T_{w,n}^{\pm 1}] \).

For a tame level \( K^p \) as above, the full Hecke algebra

\[
\mathcal{H}(G(\mathbb{A}_F^p), K^p) = \bigotimes_{\nu | p} \mathcal{H}(U(F_\nu^+), K_\nu)
\]
acts on $\hat{S}(K_p, E)$ by norm-decreasing morphisms, and hence preserves the unit ball $\hat{S}(K_p, O)$. This induces actions on $S(K_p, k)$ and $S_V(K_p, K^p, E)$ as well given by the usual double coset operators. Let

$$\mathcal{H}(K_{\Sigma_0}) = \bigotimes_{v \in \Sigma_0} \mathcal{H}(U(F_v^+), K_v), \quad \mathcal{H}_s(K^\Sigma) = \bigotimes_{v \notin \Sigma \text{ split}} \mathcal{H}(U(F_v^+), K_v)$$

be the subalgebras of $\mathcal{H}(G(F_p), K^p)$ generated by Hecke operators at $v \in \Sigma_0$, respectively $T_w, 1, \ldots, T_{w/n}^\pm$ for $v \notin \Sigma$ split in $F$ and $w|v$ (the subscript $s$ is for "split"). In what follows we ignore the Hecke action at the non-split places $v \notin \Sigma$. Note that $\mathcal{H}_s(K^\Sigma)$ is commutative, but of course $\mathcal{H}(K_{\Sigma_0})$ need not be.

We define the Hecke polynomial $P_w(X) \in \mathcal{H}_s(K^\Sigma)[X]$ to be

$$P_w(X) = X^{n} + \cdots + (-1)^j (Nw)^{j-1}/j! T_{w,j} X^{n-j} + \cdots + (-1)^n (Nw)^{n-1}/n! T_{w,n}$$

where $Nw$ is the size of the residue field $O_{F_w}/(\mathfrak{a}_{F_w})$.

We denote by $T_V(K_p, K^p, O)$ the subalgebra of $End S_V(K_p, K^p, E)$ generated by the operators $\mathcal{H}_s(K^\Sigma)$. This is reduced and finite over $O$. In case $V$ is the trivial representation we write $T_0(K_p, K^p, O)$. As $K_p$ shrinks there are surjective transition maps between these (given by restriction) and we let

$$\hat{T}(K_p, O) = \varprojlim_{K_p} T_0(K_p, K^p, O),$$

equipped with the projective limit topology (each term being endowed with the $\mathfrak{a}$-adic topology). We refer to it as the "big" Hecke algebra. $\hat{T}(K_p, O)$ clearly acts faithfully on $\hat{S}(K_p, E)$ and one can easily show that the natural map $\mathcal{H}_s(K^\Sigma) \to \hat{T}(K_p, O)$ has dense image, cf. the discussion in [Em11b, 5.2].

A maximal ideal $m \subset \mathcal{H}_s(K^\Sigma)$ is called automorphic (of tame level $K^p$) if it arises as the pullback of a maximal ideal in some $T_V(K_p, K^p, O)$. Shrinking $K_p$ if necessary we may assume it is pro-$p$, in which case we may take $V$ to be trivial ("Shimura’s principle"). In particular there are only finitely many such $m$, and we interchangeably view them as maximal ideals of $\hat{T}(K_p, O)$ (and use the same notation), which thus factors as a finite product of complete local $O$-algebras

$$\hat{T}(K_p, O) = \prod_m \hat{T}(K_p, O)_m.$$ Correspondingly we have a decomposition $\hat{S}(K_p, E) = \bigoplus_m \hat{S}(K_p, E)_m$, and similarly for $\hat{S}(K_p, O)$. This direct sum is clearly preserved by $\mathcal{H}(K_{\Sigma_0})$.

2.4. Galois representations. If $R$ is an $O$-algebra, and $r : \text{Gal}_F \to \text{GL}_n(R)$ is an arbitrary representation which is unramified at all places $w$ of $F$ lying above a split $v \notin \Sigma$, we associate the eigensystem $\theta_r : \mathcal{H}_s(K^\Sigma) \to R$ determined by

$$\det(X - r(\text{Frob}_w)) = \theta_r(P_w(X)) \in R[X]$$

for all such $w$. Here $\text{Frob}_w$ denotes a geometric Frobenius. (Note that the coefficients of the polynomial determine $\theta_r(T_{w,j})$ since $Nw \in O^\times$; and $\theta_r(T_{w,n}) \in R^\times$.) We say $r$ is automorphic (for $G$) if $\theta_r$ factors through one of the quotients $T_V(K_p, K^p, O)$.

When $R = O$ this means $r$ is associated with one of the automorphic representations $\pi$ contributing to (2.2) in the sense that $T_{w,j}$ acts on $\pi^K_v$ by scaling by $\iota(\theta_r(T_{w,j}))$ for all $w|v \notin \Sigma$ as above. Conversely, it is now known that to any such $\pi$ (and a choice of isomorphism $\iota : \hat{Q}_p \sim \hat{C}$) one can attach a unique semisimple Galois representation $r_{\pi,\iota} : \text{Gal}_F \to \text{GL}_n(\hat{Q}_p)$ with that property, cf. [Tho12, Theorem 6.5].
for a nice summary. It is polarized, meaning that $r_{\pi,i}^\vee \simeq r_{\pi,i}^c \otimes \epsilon^{n-1}$ where $\epsilon$ is the cyclotomic character, and one can explicitly write down its Hodge-Tate weights in terms of $V$.

When $R = k$ we let $m_r = \ker(\theta_r)$ be the corresponding maximal ideal of $\mathcal{H}_s(K_p)$. Then $r$ is automorphic precisely when $m_r$ is automorphic, in which case we tacitly view it as a maximal ideal of $\mathbb{T}_V(K_p K^p, \mathcal{O})$ (with residue field $k$) for suitable $V$ and $K_p$. In the other direction, starting from a maximal ideal $m$ in $\mathbb{T}_V(K_p K^p, \mathcal{O})$ (whose residue field is necessarily a finite extension of $k$) one can attach a unique semisimple representation

$$\tilde{r}_m : \text{Gal}_F \longrightarrow \text{GL}_n(\mathbb{T}_V(K_p K^p, \mathcal{O})/m)$$

such that $\theta_{\tilde{r}_m}(T_{w,j}) = T_{w,j} + m$ (and which is polarized), cf. [Tho12, Prop. 6.6]. We say $m$ is non-Eisenstein if $\tilde{r}_m$ is absolutely irreducible. Under this hypothesis $\tilde{r}_m$ admits a (polarized) lift

$$r_m : \text{Gal}_F \longrightarrow \text{GL}_n(\mathbb{T}_V(K_p K^p, \mathcal{O})_m)$$

with the property that $\theta_{r_m}(T_{w,j}) = T_{w,j};$ it is unique up to conjugation, cf. [Tho12, Prop. 6.7], and gives a well-defined deformation of $\tilde{r}_m$. If we let $K_p$ shrink we may take $V$ to be trivial. Passing to the inverse limit yields a lift of $\tilde{r}_m$ with coefficients in $\mathbb{T}(K_p, \mathcal{O})$ which we will denote by $\tilde{r}_m$. Throughout [Tho12] it is assumed that $p > 2$; we adopt that hypothesis here.

All the representations discussed above ($r_{\pi,i}, \tilde{r}_m, r_m$ etc.) extend canonically to continuous homomorphisms $\text{Gal}_{F^+} \to G_n(R)$ for various $R$, where $G_n$ is the group scheme (over $\mathbb{Z}$) defined as the semi-direct product $\{1, j\} \ltimes (\text{GL}_n \times \text{GL}_1)$, cf. [Tho12, Def. 2.1]. We let $\nu : G_n \to \text{GL}_1$ be the natural projection. Thus $\nu \circ \tilde{r}_m = \epsilon^{1-n} \delta_{F/F^+}^{\mu_m}$ (and similarly for $r_m$) where $\delta_{F/F^+}$ is the non-trivial quadratic character of $\text{Gal}(F/F^+)$ and $\mu_m \in \{0, 1\}$.

### 2.5. Deformations.

Now start with $\tilde{r} : \text{Gal}_{F^+} \to G_n(k)$ such that its restriction $\tilde{r} : \text{Gal}_F \to G_n(k)$ is absolutely irreducible, with corresponding maximal ideal $m = m_r$, and $\nu \circ \tilde{r} = \epsilon^{1-n} \delta_{F/F^+}^{\mu_m}$. In particular $\tilde{r}$ is unramified outside $\Sigma$.

We consider lifts and deformations of $\tilde{r}$ to rings in $C_O$, the category of complete local Noetherian $O$-algebras $R$ with residue field $k \iso R/m_R$, cf. [Tho12, Def. 3.1]. Recall that a lift is a homomorphism $r : \text{Gal}_{F^+} \to G_n(R)$ such that $r$ reduces to $\tilde{r}$ mod $m_R$, and $\nu \circ r = \epsilon^{1-n} \delta_{F/F^+}^{\mu_m}$ (thought of as taking values in $R^\times$). A deformation is a $(1 + M_n(R))$-conjugacy class of lifts.

For each $v \in \Sigma$ consider the restriction $\tilde{r}_v = \tilde{r}|_{\text{Gal}_{F_v}}$ and its universal lifting ring $R^{\Sigma}_{\tilde{r}_{v}}$. Following [Tho12] we let $R^{\Sigma}_{\tilde{r}}$ denote its maximal reduced $p$-torsion free quotient, and consider the deformation problem

$$S = \left( F/F^+, \Sigma, \tilde{r}, \mathcal{O}, \epsilon^{1-n} \delta_{F/F^+}^{\mu_m}, \{R^{\Sigma}_{\tilde{r}_v}\}_{v \in \Sigma} \right).$$

The functor $\text{Def}_S$ of deformations of type $S$ is then represented by an object $R^\Sigma_{\text{univ}}$ of $C_O$, cf. [Tho12, Prop. 3.4] or [CHT08, Prop. 2.2.9]. In what follows we will simply write $R_r$ instead of $R^\Sigma_{\text{univ}}$, and keep in mind the underlying deformation problem $S$. Similarly, $R^{\Sigma}_{\tilde{r}}$ is the universal lifting ring of type $S$ (which is denoted by $R^\Sigma_{\tilde{r}}$ in [Tho12, Prop. 3.4]). Note that $R^{\Sigma}_{\tilde{r}}$ is a power series $O$-algebra in $[\Sigma][n^2]$ variables over $R_r$ ([CHT08, Prop. 2.2.9]); a fact we will not use this paper.

The universal automorphic deformation $r_m$ is of type $S$, so by universality it arises from a local homomorphism

$$\psi : R_r \longrightarrow T_V(K_p K^p, \mathcal{O})_m.$$
These maps are compatible as we shrink $K_p$. Taking $V$ to be trivial and passing to the inverse limit over $K_p$ we obtain a map $\hat{\psi} : R^e \to \hat{\mathcal{T}}(K^p, \mathcal{O})_m$ which we use to view $\hat{S}(K^p, E)_m$ as an $R^e$-module.

3. Eigenvarieties

3.1. Formal schemes and rigid spaces. In what follows $(-)^\text{rig}$ will denote Berthelot’s functor (which generalizes Raynaud’s construction for finite type formal schemes $X$ over $\text{Spf}(\mathcal{O})$, cf. [Ray74]). Its basic properties are nicely reviewed in [dJ95, Ch. 7]. The source $\text{FS}_{\mathcal{O}}$ is the category of locally Noetherian adic formal schemes $X$ over $\text{Spf}(\mathcal{O})$ whose mod $\varpi$ reduction is of finite type over $\text{Spec}(k)$; the target $\text{Rig}_E$ is the category of rigid analytic varieties over $E$, cf. Definition 9.3.1/4 in [BGR84]. For example, $\mathcal{B} = (\text{Spf}\mathcal{O}[y])^\text{rig}$ is the closed unit disc (at $0$); $U = (\text{Spf}\mathcal{O}[x])^\text{rig}$ is the open unit disc. For a general affine formal scheme $\bar{X} = \text{Spf}(A)$ where

$$A = \mathcal{O}\{y_1, \ldots, y_r\}[[x_1, \ldots, x_k]]/(g_1, \ldots, g_t),$$

$\mathfrak{X}^\text{rig} \subset \mathfrak{B} \times \set{t}$ is the closed analytic subvariety cut out by the functions $g_1, \ldots, g_t$, cf. [BGR84, 9.5.2]. In general $\mathfrak{X}^\text{rig}$ is obtained by gluing affine pieces as in [dJ95, 7.2]. The construction of $\mathfrak{X}^\text{rig}$ in the affine case is actually completely canonical and free from coordinates: If $I \subset A$ is the largest ideal of definition, $A[I^n/\varpi]$ is the subring of $A \otimes E$ generated by $A$ and all $i/\varpi$ with $i \in I^n$. Let $A[I^n/\varpi]^\wedge$ be its $I$-adic completion (equivalently, its $\varpi$-adic completion, see the proof of [dJ95, Lem. 7.1.2]). Then $A[I^n/\varpi]^\wedge \otimes E$ is an affinoid $E$-algebra and there is an admissible covering

$$\mathfrak{X}^\text{rig} = \text{Spf}(A)^\text{rig} = \bigcup_{n=1}^{\infty} \text{Sp}(A[I^n/\varpi]^\wedge \otimes E).$$

In particular $A^\text{rig} := \mathcal{O}\text{(Spf}(A)^\text{rig}) = \varprojlim_n A[I^n/\varpi]^\wedge \otimes E$. The natural map $A \otimes E \to A^\text{rig}$ factors through the ring of bounded functions on $\text{Spf}(A)^\text{rig}$; the image of $A$ lies in $\mathcal{O}^0(\text{Spf}(A)^\text{rig})$, the functions whose absolute value is bounded by $1$, cf. [dJ95, 7.1.8].

3.2. Deformation space. We let $X_r := \text{Spf}(R_r)^\text{rig}$ (a subvariety of $\mathfrak{B}^s$ for some $s$). For a point $x \in X_r$ we let $\kappa(x)$ denote its residue field (a finite extension of $E$) and $\kappa(x)^0$ its valuation ring; an $\mathcal{O}$-algebra with (finite) residue field $k(x)$. The evaluation map $R_r \to \mathcal{O}^0(X_r) \to \kappa(x)^0$ corresponds to a deformation

$$r_x : \text{Gal}_{\mathbb{Q}^s} \to \mathcal{G}_n(\kappa(x)^0)$$

of $\bar{\mathfrak{r}} \otimes k(x)$. (We tacitly choose a representative $r_x$ in the conjugacy class of lifts.) We let $p_x := \ker(R_r \to \kappa(x)^0)$ be the prime ideal of $R_r$ corresponding to $x$, cf. the bijection in [dJ95, Lem. 7.1.9]. We will often assume (for simplicity) that $x$ is $E$-rational, in which case $\kappa(x) = E$ and $k(x) = k$; and $r_x$ is a deformation of $\bar{\mathfrak{r}}$ over $\kappa(x)^0 = \mathcal{O}$.

3.3. Character and weight space. Recall our choice of torus $T \subset G(\mathbb{Q}_p)$, and let $T_0$ be its maximal compact subgroup; $T \simeq T_0 \times \mathbb{Z}^{n[\Sigma_p]}$ as topological groups, and

$$T_0 \simeq \prod_{v|p} (\mathcal{O}_p^\times)^n \simeq \left( \prod_{v|p} \mu_\infty(F_v)^n \right) \times \mathbb{Z}_p^{n[F^+ : E]}.$$

Let $\hat{T} := \mathcal{W} \times (\mathbb{G}^m)^{n[\Sigma_p]}$ where $\mathcal{W} := (\text{Spf}\mathcal{O}[[T_0]])^\text{rig}$. The weight space $\mathcal{W}$ is isomorphic to $|\mu|$ copies of the open unit ball $\set{t}^{n[F^+ : E]}$. From a more functorial point of view $\hat{T}$ represents the functor which
takes an affinoid $E$-algebra to the set $\text{Hom}_{\text{cont}}(T, A^\times)$, and similarly for $W$ and $T_0$. See [Em11a, Prop. 6.4.5]. Thus $\hat{T}$ carries a universal continuous character $\delta^{\text{uni}} : T \to \mathcal{O}(\hat{T})^\times$ which restricts to a character $T_0 \to \mathcal{O}^0(W)^\times$ via the canonical morphism $\hat{T} \to W$. Henceforth we identify points of $\hat{T}$ with continuous characters $\delta : T \to \kappa(\delta)^\times$ for varying finite extensions $\kappa(\delta)$ of $E$ (and analogously for $W$).

3.4. Definition of the eigenvariety. We follow [BHS16, 4.1] in defining the eigenvariety $Y(K^p, \hat{r})$ as the support of a certain coherent sheaf $\mathcal{M} = \mathcal{M}_{K^p}$ on $X_r \times \hat{T}$. This is basically also the approach taken in section (2.3) of [Em06a], except there $X_r$ is replaced by $\hat{S}$pec of a certain Hecke algebra. We define $\mathcal{M}$ as follows.

Let $(-)_{\text{an}}$ be the functor from [ST03, Thm. 7.1]. It takes an object $H$ of Ban$_G^\text{adm}(E)$ to the dense subspace $H_{\text{an}}$ of locally analytic vectors. $H_{\text{an}}$ is a locally analytic $G$-representation (over $E$) of compact type whose strong dual $(H_{\text{an}})'$ is a coadmissible $D(G, E)$-module, cf. [ST03, p. 176].

We take $H = \hat{S}(K^p, E)_{\text{m}}$ and arrive at an admissible locally analytic $G$-representation $\hat{S}(K^p, E)_{\text{m}}^{\text{an}}$ which we feed into the Jacquet functor $J_B$ defined in [Em06b, Def. 3.4.5]. By Theorem 0.5 of loc. cit. this yields an essentially admissible locally analytic $T$-representation $J_B(\hat{S}(K^p, E)_{\text{m}}^{\text{an}})$. See [Em11a, Def. 6.4.9] for the notion of essentially admissible (the difference with admissibility lies in incorporating the action of the center $Z$, or rather viewing the strong dual as a module over $\mathcal{O}(Z) \otimes D(G, E)$).

We recall [Em06a, Prop. 2.3.2]: If $\mathcal{F}$ is a coherent sheaf on $\hat{T}$, cf. [BGR84, Def. 9.4.3/1], its global sections $\Gamma(\hat{T}, \mathcal{F})$ is a coadmissible $\mathcal{O}(\hat{T})$-module. Moreover, the functor $\mathcal{F} \mapsto \Gamma(\hat{T}, \mathcal{F})$ is an equivalence of categories (since $\hat{T}$ is quasi-Stein). Note that $\Gamma(\hat{T}, \mathcal{F})$ and its strong dual both acquire a $T$-action via $\delta^{\text{uni}}$. Altogether the functor $\mathcal{F} \mapsto \Gamma(\hat{T}, \mathcal{F})'$ sets up an anti-equivalence of categories between coherent sheaves on $\hat{T}$ and essentially admissible locally analytic $T$-representations (over $E$).

As pointed out at the end of section 2.5, $\hat{S}(K^p, E)_{\text{m}}$ is an $R_r$-module via $\hat{\psi}$, and the $G$-action is clearly $R_r$-linear. Thus $J_B(\hat{S}(K^p, E)_{\text{m}}^{\text{an}})$ inherits an $R_r$-module structure. By suitably modifying the remarks of the preceding paragraph (as in section 3.1 of [BHS16] where they define and study locally $R_r$-analytic vectors, cf. Def. 3.2 in loc. cit.) one finds that there is a coherent sheaf $\mathcal{M} = \mathcal{M}_{K^p}$ on $X_r \times \hat{T}$ for which

$$J_B(\hat{S}(K^p, E)_{\text{m}}^{\text{an}}) \simeq \Gamma(X_r \times \hat{T}, \mathcal{M})'.$$

The eigenvariety is then defined as the (schematic) support of $\mathcal{M}$, i.e.

$$Y(K^p, \hat{r}) := \text{supp}(\mathcal{M}) = \{ y = (x, \delta) : \mathcal{M}_y \neq 0 \} \subset X_r \times \hat{T}.$$

Thus $Y(K^p, \hat{r})$ is an analytic subset of $X_r \times \hat{T}$, cf. [BGR84, 9.5.2], and we always endow $Y(K^p, \hat{r})$ with the analytic structure for which it becomes a reduced closed analytic subvariety, cf. [BGR84, Prop. 9.5.3/4].

The stalk $\mathcal{M}_y$ is a finitely generated module over $\mathcal{O}_{X_r \times \hat{T}, y}$, hence is finite-dimensional over $E$. The (full) $E$-linear dual $\mathcal{M}_y' = \text{Hom}_E(\mathcal{M}_y, E)$ has the following useful description.

**Lemma 3.1.** Let $y = (x, \delta) \in (X_r \times \hat{T})(E)$ be an $E$-rational point. Then there is an isomorphism

$$\mathcal{M}_y' \simeq J_B^\delta(\hat{S}(K^p, E)_{\text{m}}[p_x]^{\text{an}}).$$

(Here $J_B^\delta$ means the $\delta$-eigenspace of $J_B$, and $[p_x]$ means taking $p_x$-torsion.)

**Proof.** First, since $X_r \times \hat{T}$ is quasi-Stein, $\mathcal{M}_y$ is the largest quotient of $\Gamma(X_r \times \hat{T}, \mathcal{M})$ which is annihilated by $p_x$ and on which $T$ acts via $\delta$, cf. [BHS16, 5.4]. Thus $\mathcal{M}_y'$ is the largest subspace of $J_B(\hat{S}(K^p, E)_{\text{m}}^{\text{an}})$
with the same properties, i.e. \( J_B^\delta(S(K^p, E_m^\an)[p_x]) \), as observed in Proposition 2.3.3 (iii) of [Em06a]. Now,

\[
J_B^\delta(S(K^p, E_m^\an)[p_x]) = J_B^\delta(S(K^p, E_m)[p_x]^{\an})
\]

as follows easily from the exactness of \((-)^{\an}\) and the left-exactness of \(J_B\) (using that \(p_x\) is finitely generated to reduce to the principal case by induction on the number of generators), cf. the proof of [BHS16, Prop. 3.7].

The space in (3.2) can be made more explicit: Choose a compact open subgroup \( N_0 \subset N \) and introduce the monoid \( T^+ = \{ t \in T : tN_0t^{-1} \subset N_0 \} \). Then by [Em06b, Prop. 3.4.9],

\[
J_B^\delta(S(K^p, E_m)[p_x]^{\an}) \simeq (S(K^p, E_m)[p_x]^{\an})_{N_0,T^+=\delta}
\]

where \( T^+ \) acts by double coset operators \([N_0tN_0]\) on the space on the right. Observe that \( y \) lies on the eigenvariety \( Y(K^p, \bar{r}) \) precisely when the above space \( \mathcal{M}'_y \) is nonzero.

Note that the Hecke algebra \( \mathcal{H}(K_{\Sigma}) \) acts on \( J_B(S(K^p, E_m)^{\an}) \), and therefore on \( \mathcal{M} \) and its stalks \( \mathcal{M}_y \) (on the right since we are taking duals). The isomorphism (3.2) is \( \mathcal{H}(K_{\Sigma}) \)-equivariant, and our first goal is to describe \( \mathcal{M}_y \) as a \( \mathcal{H}(K_{\Sigma}) \)-module.

### 3.5. Classical points.

We say that a point \( y = (x, \delta) \in Y(K^p, \bar{r})(E) \) is classical (of weight \( V \)) if the following conditions hold (cf. [BHS16, Def. 3.14] or the paragraph before [Em06a, Def. 0.6]):

1. \( \delta = \delta_{alg} \delta_{sm} \), where \( \delta_{alg} \) is an algebraic character which is dominant relative to \( B \) (i.e., obtained from an element of \( X^*(T \times Q E)^+ \) by restriction to \( T \)), and \( \delta_{sm} \) is a smooth character of \( T \). In this case let \( V \) denote the irreducible algebraic representation of \( G \times Q E \) of highest weight \( \delta_{alg} \).

2. There exists an automorphic representation \( \pi \) of \( G(\hat{A}) \) such that
   - (a) \( (\pi^p)^{K^p} \neq 0 \) and the \( H_{s}(K^\Sigma) \)-action on this space is given by the eigensystem \( t \circ \theta_{r_x} \),
   - (b) \( \pi_{\infty} \simeq V \otimes_{E,\lambda} \mathbb{C} \),
   - (c) \( \pi_p \) is a quotient of \( \text{Ind}_B^G(S_{\delta_{sm},T^+=\delta}^B) \).

These points comprise the subset \( Y(K^p, \bar{r})_{cl} \). Note that condition (a) is equivalent to the isomorphism \( r_x \simeq r_{\pi, \lambda} \) (both sides are irreducible since \( r_x \) is a lift of \( \bar{r} \)). In (c) \( \delta_B \) denotes the modulus character of \( B \); the reason we include it in condition (c) will become apparent in the proof of Prop. 4.2 below.

**Lemma 3.3.** There is at most one automorphic \( \pi \) satisfying (a)–(c) above; and \( m_c(\pi) = 1 \).

**Proof.** Let \( \Pi = BC_{F/F^+}(\pi) \) be a (strong) base change of \( \pi \) to \( GL_n(\mathbb{A}_F) \), where we view \( \pi \) as a representation of \( U(\mathbb{A}_F, \mathbb{A}_F) = G(\hat{A}) \). For its existence see [Lab11, 5.3]. Note that \( \Pi \) is cuspidal since \( r_{x, \lambda} \) is irreducible. In particular \( \Pi \) is globally generic, hence locally generic. By local-global compatibility, cf. [BGGT1], [BGGT2], and [Car14] for places \( w | p \); [TY07] and [Shi11] for places \( w \nmid p \),

\[
\iota WD(r_{x, \lambda} |_{Gal_{w}})^{F - ss} \simeq \text{rec}(\Pi_w \otimes | \det |^{(1-n)/2})
\]

for all finite places \( w \) of \( F \), with the local Langlands correspondence \( \text{rec}(\cdot) \) normalized as in [HT01]. This shows that \( \Pi_w \) is completely determined by \( r_x \) at all finite places \( w \). Moreover, we have \( \Pi_w = BC_{w|\mathbb{A}(\pi_v)} \) whenever the local base change on the right is defined, i.e. when either \( v \) splits or \( \pi_v \) is unramified. Our assumption that \( \Sigma \) consists of split places guarantees that \( BC_{w|\mathbb{A}(\pi_v)} \) makes sense locally everywhere. Furthermore, unramified local base change is injective according to [Min11, Cor. 4.2]. We conclude that
\(\pi_f\) is determined by \(r_x\), and \(\pi_\infty \simeq V \otimes_{E,\ell} \mathbb{C}\). Thus \(\pi\) is unique. Multiplicity one was noted earlier at the end of section 2.2 above. \(\square\)

4. The case of classical points of non-critical slope

Each point \(x \in X_f\) carries a Galois representation \(r_x : \text{Gal}_F \to \text{GL}_n(\kappa(x))\) which we restrict to the various decomposition groups \(\text{Gal}_{F_v}\) for \(v \in \Sigma\). When \(v \in \Sigma_0\) there is a corresponding Weil-Deligne representation, cf. section (4.2) in [Tat79], and we let \(\pi_{x,v}\) be the representation of \(U(F_v^+)\) (over \(\kappa(x)\)) such that

\[
\text{WD}(r_x|_{\text{Gal}_{F_v}})^F\simeq \text{rec}(BC_{\varphi}|(\pi_{x,v}) \otimes |\det(1-\eta/2)|)
\]

Note that the local base change \(BC_{\varphi}(\pi_{x,v})\) is just \(\pi_{x,v}\) thought of as a representation of \(\text{GL}_n(\mathbb{C})\) via the isomorphism \(\varphi : U(F_v^+) \simeq \text{GL}_n(\mathbb{C})\). We emphasize that \(\pi_{x,v}\) is defined even for non-classical points on the eigenvariety. If \(y = (x, \delta)\) happens to be classical, \(\pi_{x,v} \otimes_{E,\ell} \mathbb{C} \simeq \pi_v\) where \(\pi\) is the automorphic representation in Lemma 3.3. Below we relate \(\otimes_{v \in \Sigma_0} \pi_{x,v}\) to the fiber \(\mathcal{M}'_y\).

**Proposition 4.2.** Let \(y = (x, \delta) \in Y(K^p, \hat{\rho})(E)\) be a classical point. Then there exists an embedding of \(\mathcal{H}(K_{\Sigma_0})\)-modules \(\otimes_{v \in \Sigma_0} \pi_{x,v} \hookrightarrow \mathcal{M}'_y\) which is an isomorphism if \(\delta\) is of non-critical slope, cf. [Em06b, Def. 4.4.3] (which is summarized below).

**Proof.** According to (0.14) in [Em06b] there is a closed embedding

\[
J_B \left( (\check{S}(K^p, E)_m[p_x]^{\text{an}})^{V,\text{alg}} \right) \hookrightarrow J_B \left( \check{S}(K^p, E)_m[p_x]^{\text{an}} \right)^{V,\text{alg}}.
\]

Note that \(V^\Sigma \simeq \delta_{\text{alg}}\) so after passing to \(\delta\)-eigenspaces we get a closed embedding

\[
J_B^\delta \left( (\check{S}(K^p, E)_m[p_x]^{\text{an}})^{V,\text{alg}} \right) \hookrightarrow J_B^\delta \left( \check{S}(K^p, E)_m[p_x]^{\text{an}} \right).
\]

The target is exactly \(\mathcal{M}'_y\) by (3.2). On the other hand

\[
(\check{S}(K^p, E)_m[p_x]^{\text{an}})^{V,\text{alg}} \simeq \bigoplus_{\sigma} (V \otimes_{E} \pi_{\sigma,p}) \otimes_{E} (\pi_{x,p}^\delta)^{K^p}
\]

with \(\pi\) running over automorphic representations of \(G(\mathbb{A})\) over \(E\) with \(\pi_\infty \simeq V\) and such that \(\theta_{r_x}\) gives the action of \(\mathcal{H}_K^{\Sigma}(\mathbb{A})\) on \((\pi_{x,p}^\delta)^{K^p}\). As noted in Lemma 3.3 there is precisely one such \(\pi\) which we will denote by \(\pi_x\) throughout this proof (consistent with the notation \(\pi_{x,v}\) introduced above). Note that \(\otimes_{v \in \Sigma} \pi_{x,v}\) is a line so

\[
(\check{S}(K^p, E)_m[p_x]^{\text{an}})^{V,\text{alg}} \simeq (V \otimes_{E} \pi_{x,p}) \otimes_{E} \left( \bigotimes_{v \in \Sigma_0} \pi_{x,v}^{K_v} \right).
\]

Since \(J_B\) is compatible with the classical Jacquet functor, cf. [Em06b, Prop. 4.3.6], we identify the source of (4.3) with

\[
(V^\Sigma \otimes_{E} \pi_{x,p})^{T=\delta} \otimes_{E} \left( \bigotimes_{v \in \Sigma_0} \pi_{x,v}^{K_v} \right).
\]

Now \(V^\Sigma \simeq \delta_{\text{alg}}\) is one-dimensional, and so is \((\pi_{x,p})_{T=\delta_{\text{sm}}}^{\delta_{\text{sm}}}\). Indeed, by Bernstein second adjointness,

\[
(\pi_{x,p})_{T=\delta_{\text{sm}}}^{\delta_{\text{sm}}} \simeq \text{Hom}_G(\text{Ind}_{B}^{G}(\delta_{\text{sm}}\delta_{B}^{-1}), \pi_{x,p}).
\]

The right-hand side is nonzero by condition (c) above, and in fact it is a line since \(\text{Ind}_{B}^{G}(\delta_{\text{sm}}\delta_{B}^{-1})\) has a unique generic constituent (namely \(\pi_{x,p}\), cf. the proof of Lemma 3.3) which occurs with multiplicity one; this follows from the theory of derivatives [BZ77, Ch. 4]. From this observation we immediately infer that \(\text{Hom}_G(\pi_{x,p}, \text{Ind}_{B}^{G}(\delta_{B}^{-1}))\) is one-dimensional. To summarize, (4.3) is an embedding \(\otimes_{v \in \Sigma_0} \pi_{x,v}^{K_v} \hookrightarrow \mathcal{M}'_y\).
Finally, since $\tilde{S}(K^p, E)_{m[p_s]}^{\text{an}}$ clearly admits a $G$-invariant norm (the sup norm), Theorem 4.4.5 in [Em06b] tells us that (4.3) is an isomorphism if $\delta$ is of non-critical slope.

To aid the reader we briefly recall the notion of non-critical slope: To each $\delta \in \hat{T}(E)$ we assign the element $\text{slp}(\delta) \in X^*(T \times Q E)$ defined as follows, cf. [Em06b, Def. 1.4.2]. First note that there is a natural surjection $T(E) \to X_*(T \times Q E)$; the cocharacter $\mu_t \in X_*(T \times Q E)$ associated with $t \in T(E)$ is given by $(\chi, \mu_t) = \text{ord}_E(\chi(t))$ for all algebraic characters $\chi$ (here $\text{ord}_E$ is the valuation on $E$ normalized such that $\text{ord}_E(\pi_F) = 1$). Then the slope of $\delta$ is the algebraic character $\text{slp}(\delta)$ satisfying $(\text{slp}(\delta), \mu_t) = \text{ord}_E(\delta(t))$ for all $t \in T$.

Let $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$. We say that $\delta = \delta_{\text{alg}} + \delta_{\text{sm}}$ is of non-critical slope if there is no simple root $\alpha$ for which the element $s_\alpha(\delta_{\text{alg}} + \rho) + \text{slp}(\delta_{\text{sm}}) + \rho$ lies in the $\mathbb{Q}_{\geq 0}$-cone generated by all simple roots.

5. Interpolation of the Weil-Deligne representations

Our goal in this section is to interpolate the Weil-Deligne representations $\text{WD}(r_\varepsilon|_{\text{Gal}_{F_\varepsilon}})$ across deformation space $X_\varepsilon$, for a fixed $\varepsilon \in \Sigma_0$. More precisely, for any (reduced) $E$-affinoid subvariety $\text{Sp}(A) \subset X_\varepsilon$ we will define a rank $n$ Weil-Deligne representation $\text{WD}_{\varepsilon, \delta}$ over $A$ such that

$$\text{WD}(r_\varepsilon|_{\text{Gal}_{F_\varepsilon}}) \simeq \text{WD}_{\varepsilon, \delta} \otimes_{A, x} \kappa(x)$$

for all points $x \in \text{Sp}(A)$. The usual proof of Grothendieck's monodromy theorem (cf. [Tat79, Cor. 4.2.2]) adapts easily to this setting, and this has already been observed by other authors. See for example [BC09, 7.8.3], [Pau11, 5.2], and [EH14, 4.1.6]. Since our setup is slightly different from those references we give the details for the convenience of the reader.

**Proposition 5.2.** Let $w$ be a place of $F$ not dividing $p$, and let $A$ be a reduced affinoid $E$-algebra. For any continuous representation $\rho : \text{Gal}_{F_w} \to \text{GL}_n(A)$ there is a unique nilpotent $N \in M_n(A)$ such that the equality $\rho(\gamma) = \exp(t_p(\gamma)N)$ holds for all $\gamma$ in an open subgroup $J \subset \text{I}_{F_w}$. (Here $t_p : I_{F_w} \to \mathbb{Z}_p$ is a choice of homomorphism as in section (4.2) of [Tat79].)

**Proof.** Choose a submultiplicative norm $\| \cdot \|$ on $A$ relative to which $A$ is complete (as $A$ is reduced one can take the spectral norm, cf. [BGR84, 6.2.4]). Let $A^\circ$ be the (closed) unit ball. Then $I + p^i M_n(A^\circ)$ is an open (normal) subgroup of $\text{GL}_n(A^\circ)$ for $i > 0$, so its inverse image $\rho^{-1}(I + p^i M_n(A^\circ)) = \text{Gal}_{F_i}$ for some finite extension $F_i$ of $F_w$. Note that $F_{i+1}/F_i$ is a Galois extension whose Galois group is killed by $p$. Let us fix an $i > 0$ and work with the restriction $\rho|_{\text{Gal}_{F_i}}$. Recall that wild inertia $P_{F_i} \subset I_{F_i}$ is the Sylow pro-$\ell$ subgroup where $\ell \neq p$ we deduce that $P_{F_i} \subset \text{Gal}_{F_i}$ for all $j \geq i$. That is $\rho$ factors through the tame quotient $I_{F_j}/P_{F_j} \simeq \prod_{q \neq p} \mathbb{Z}_q$. For the same reason $\rho$ factors further through $t_p : I_{F_i} \to \mathbb{Z}_p$. Therefore we find an element $\alpha \in I + p^i M_n(A^\circ)$ (the image of $1 \in \mathbb{Z}_p$ under $\rho$) such that $\rho(\gamma) = \alpha^{t_p(\gamma)}$ for all $\gamma \in I_{F_i}$. We let $N := \log(\alpha)$. If we choose $i$ large enough (i.e., $i > 1$ suffices, cf. the discussion in [Sch11, p. 220]) all power series converge and we arrive at $\rho(\gamma) = \exp(t_p(\gamma)N)$ for $\gamma \in I_{F_i}$. We conclude that we may take $J := I_{F_i}$. (The uniqueness of $N$ follows by taking log on both sides.)

To see that $N$ is nilpotent note the standard relation $\rho(w)N\rho(w)^{-1} = \|w\|_F N$ for $w \in W_{F_i}$. If we take $w$ to be a (geometric) Frobenius this shows that all specializations of $N^n$ at points $x \in \text{Sp}(A)$ are $0$ (by considering the eigenvalues in $\kappa(x)$ as usual). Thus all matrix entries of $N^n$ are nilpotent (by the maximum modulus principle [BGR84, 6.2.1]) hence zero since we assume $A$ is reduced. So $N^n = 0$. □
If we choose a geometric Frobenius $\Phi$ from $W_{F_v}$ (keeping the notation of the previous Proposition) we can thus define a Weil-Deligne representation $(\tilde{\rho}, N)$ on $A^n$ by the usual formula ([Tat79, 4.2.1]):

$$\rho(\Phi^{s} \gamma) = \tilde{\rho}(\Phi^{s} \gamma) \exp (t_{F}(\gamma)N)$$

where $s \in \mathbb{Z}$ and $\gamma \in I_{F_v}$. With this definition $\tilde{\rho} : W_{F_v} \rightarrow \text{GL}_n(A)$ is a representation which is trivial on the open subgroup $J \subset W_{F_v}$ (so continuous for the discrete topology on $A$).

As already hinted at above we apply this construction to $r_{\text{univ}}|_{\text{Gal}_{F_v}}$ for a fixed place $v \in \Sigma_0$, and an affinoid $\text{Sp}(A) \subset X_r$. We view the universal deformation $r_{\text{univ}} : \text{Gal}_{F} \rightarrow \text{GL}_n(R_r)$ as a representation on $A^n$ by composing with $R_r \rightarrow \mathcal{O}(X_r) \rightarrow A$. This gives a Weil-Deligne representation $\text{WD}_{r, \tilde{\rho}}$ over $A$ with the interpolative property (5.1).

6. The local Langlands correspondence for $\text{GL}_n$ after Scholze

Scholze gave a new purely local characterization of the local Langlands correspondence in [Sc13b]. His trace identity (cf. Theorem 1.2 in loc. cit.) takes the following form. Let $\Pi$ be an irreducible smooth representation of $\text{GL}_n(F_v)$, where $w$ is an arbitrary finite place of $F$. Suppose we are given $\tau = \Phi^s \gamma$ with $\gamma \in I_{F_v}$ and $s \in \mathbb{Z}_{>0}$, together with a $\mathbb{Q}$-valued "cut-off" function $h \in C^\infty_c(\text{GL}_n(O_{F_v}))$. First Scholze associates a $\mathbb{Q}$-valued function $\sigma_{\tau, h} \in C^\infty_c(\text{GL}_n(F_{w,s}))$, where $F_{w,s}$ denotes the unramified degree $s$ extension of $F_v$. The function $\sigma_{\tau, h}$ is defined by taking the trace of $\tau \times h^\gamma$ on (alternating sums of) certain formal nearby cycle sheaves à la Berkovich on deformation spaces of $\mathbb{Z}$-divisible $\mathcal{O}_{F_v}$-modules; and $h^\gamma(g) = h(tg^{-1})$. See the discussion leading up to [Sc13b, Thm. 2.6] for more details. Next one takes any function $f_{\tau, h} \in C^\infty_c(\text{GL}_n(F_v))$ whose twisted orbital integrals match those of $\sigma_{\tau, h}$. With our normalization of $\text{rec}(\cdot)$ Scholze’s trace identity reads

$$\text{tr}(f_{\tau, h} \Pi) = \text{tr}(\tau \text{rec}(\Pi \otimes | (1-n)/2) \cdot \text{tr}(h|\Pi)).$$

We will make use of a variant of $f_{\tau, h}$ which lives in the Bernstein center of $\text{GL}_n(F_v)$. We refer to section 3 of [Hai11] for a succinct review of the basic properties and different characterizations of the Bernstein center. This variant $f_{\tau}$ has the property that $\text{tr}(f_{\tau, h} \Pi) = \text{tr}(f_{\tau} \cdot h|\Pi)$ and is defined for all $\tau \in W_{F_v}$ by decreeing that $f_{\tau}$ acts on any irreducible smooth representation $\Pi$ via scaling by

$$f_{\tau} (\Pi) = \text{tr}(\tau \text{rec}(\Pi \otimes | (1-n)/2)).$$

For the existence of $f_{\tau}$ see the proof of [Sc13b, Lem. 3.2], or [Sc13a, Lem. 6.1].

We apply this construction to each of the places $i$ with $v \in \Sigma_0$. Now $\tau = (\tau_i)$ denotes a tuple of Weil elements $\tau_i \in W_{F_i}$. Via our isomorphisms $i_{\tilde{\rho}}$ we view $f_{\tau_i}$ as an element of the Bernstein center of $U(F_v^+)$, and consider $f_{\tau} := \otimes_{i \in \Sigma_0} f_{\tau_i}$.

**Lemma 6.1.** Let $x \in X_r$ be arbitrary. Then $f_{\tau}$ acts on $\otimes_{v \in \Sigma_0} \pi_{x, v}$ via scaling by

$$f_{\tau} (\otimes_{v \in \Sigma_0} \pi_{x, v}) = \prod_{v \in \Sigma_0} \text{tr}(\tau_{v} | \text{WD}(r_{x} | \text{Gal}_{F_v})).$$

**Proof.** If $\{\pi_v\}_{v \in \Sigma_0}$ is a family of irreducible smooth representations, $f_{\tau}$ acts on $\otimes_{v \in \Sigma_0} \pi_v$ via scaling by

$$f_{\tau} (\otimes_{v \in \Sigma_0} \pi_v) = \prod_{v \in \Sigma_0} \text{tr}(\tau_{v} | \text{rec}((BC_{\bar{\rho}})(v) \otimes | (1-n)/2)).$$

Now use the defining property (4.1) of the representations $\pi_{x, v}$ attached to the point $x$. 

\[\square\]
7. Interpolation of traces

Let $3(U(F_v^+))$ denote the Bernstein center of $U(F_v^+)$, and let $Z(U(F_v^+), K_v)$ be the center of the Hecke algebra $H(U(F_v^+), K_v)$. There is a canonical homomorphism $3(U(F_v^+)) \to Z(U(F_v^+), K_v)$ obtained by letting the Bernstein center act on $C_c^\infty(K_v \setminus U(F_v^+))$, cf. [Hai11, 3.2]. We let $f_{\tau_0}^{K_{\Sigma_0}}$ be the image of $f_{\tau_0}$ under this map, and consider $f_{\tau_0}^{K_{\Sigma_0}} := \oplus_{\nu \in \Sigma_0} f_{\tau_0}^{\nu}$ belonging to $Z(K_{\Sigma_0}) := \oplus_{\nu \in \Sigma_0} Z(U(F_v^+)\mathrm{k}, K_v)$ which is the center of $H(K_{\Sigma_0})$. In particular this operator $f_{\tau_0}^{K_{\Sigma_0}}$ acts on the sheaf $M$ and its stalks $M_{y}$.

If $y = (x, \delta) \in Y(K^p, \hat{r})(E)$ is a classical point of non-critical slope, and we combine Proposition 4.2 and Lemma 6.1, we deduce that $f_{\tau_0}^{K_{\Sigma_0}}$ acts on $M_y \simeq \oplus_{\nu \in \Sigma_0} K_y^{\nu}$ via scaling by

$$\prod_{\nu \in \Sigma_0} \text{tr}(\tau_{\nu}|WD(r_x|_{Gal_{F_v}})).$$

The goal of this section is to extrapolate this property to certain non-classical points $y$. As a first observation we note that the above factor can be interpolated across deformation space $X_r$. Indeed, let $Sp(A) \subset X_r$ be a reduced affinoid subvariety and let $WD_{r, 0}$ be the Weil-Deligne representation on $A^n$ constructed after Proposition 5.2.

**Lemma 7.1.** For each tuple $\tau = (\tau_{\nu}) \in \prod_{\nu \in \Sigma_0} W_{F_v}$, the element $a_\tau := \prod_{\nu \in \Sigma_0} \text{tr}(\tau_{\nu}|WD_{r, 0})$ $A$ satisfies the following interpolative property: For every point $x \in Sp(A)$ the function $a_\tau$ specializes to

$$a_\tau(x) = \prod_{\nu \in \Sigma_0} \text{tr}(\tau_{\nu}|WD(r_x|_{Gal_{F_v}})) \in \kappa(x).$$

**Proof.** This is clear from the interpolative property of $WD_{r, 0}$ by taking traces in (5.1).

Now let $\Omega \subset Sp(A) \times \hat{T}$ be an admissible open subset of $Y(K^p, \hat{r})$. Note that $M$ lives on $X_r \times \hat{T}$ but we naturally view it as a coherent sheaf on its support $Y(K^p, \hat{r})$ (cf. [BGR84, Prop. 9.5.2/4]). The module of sections $\Gamma(\Omega, M)$ carries a natural $O(\Omega)$-linear action of $H(K_{\Sigma_0})$. (We denote the structure sheaf $O_Y(K^p, \hat{r})$ by $O$ to simplify the notation here.) We let $A$ act by composing with $A \to O(\Omega)$.

We now invoke the natural weight morphism $\omega : Y(K^p, \hat{r}) \to W$ defined as the composition of maps

$$Y(K^p, \hat{r}) \hookrightarrow X_r \times \hat{T} \xrightarrow{pr} \hat{T} \xrightarrow{\text{can}} W.$$

In general $Y(K^p, \hat{r})$ is admissibly covered by open affinoids $\Omega$ such that $\omega(\Omega) \subset W$ is an open affinoid, $\omega : \Omega \to \omega(\Omega)$ is finite and surjective when restricted to any irreducible component of $\Omega$. We will assume we are in the étale situation, i.e. $\Omega$ is chosen such that $\omega$ restricts to an isomorphism $\omega : \Omega \xrightarrow{\text{can}} \omega(\Omega)$. By shrinking $\Omega$ if necessary we can arrange for the following to hold.

**Proposition 7.2.** The operator $f_{\tau_0}^{K_{\Sigma_0}}$ acts on $\Gamma(\Omega, M)$ via scaling by $a_{\tau_0}$, for every $\tau \in \prod_{\nu \in \Sigma_0} W_{F_v}$.

**Proof.** We fix a $\tau$ throughout the proof and introduce the operator $\varphi := f_{\tau_0}^{K_{\Sigma_0}} - a_{\tau_0} \cdot \text{Id}$ on $\Gamma(\Omega, M)$. For each point $y \in \Omega$ there is a compatible operator $\varphi_y$ on the stalk $M_y$. By our preliminary remarks above we know that $\varphi_y = 0$ if $y$ belongs to the set $D \subset \Omega$ of classical points of non-critical slope. As is well-known, $D$ is dense (cf. [BC09, Thm. 7.3.1] or the proof of [BHS16, Thm. 3.18]). I.e. $O(\Omega) \to \prod_{y \in D} O_y$ is injective, cf. [Che04, 6.4.5]. To show $\varphi = 0$ it suffices to show $\Gamma(\Omega, M) \to \prod_{y \in D} M_y$ is injective. This follows immediately if $\Gamma(\Omega, M)$ is finite type projective over $O(\Omega)$ (by viewing $\Gamma(\Omega, M)$ as a direct summand of $O(\Omega)^{\oplus d}$ for some $d$).
From a well-known result in the theory of eigenvarieties (cf. [Buz07, Thm. 3.3]) one deduces as in the proof of [BHS16, Prop. 3.10] that $Y(K^p, \bar{\rho})$ has an admissible covering $\{\Omega_i\}_{i \in I}$ with the following property: For each $i \in I$ there is an admissible open subset $W_i \subset W$ such that $\Gamma(\Omega_i, \mathcal{M})$ is a finite type projective $\mathcal{O}_W(W_i)$-module over $\omega$. In fact we may take $W_i := \omega(\Omega_i)$, cf. [Che04, Thm. C].

By shrinking $\Omega$ we may assume $\Omega \subset \Omega_i$ for some $i$. We conclude that $\Gamma(\Omega, \mathcal{M})$ is a finite type projective module over $\mathcal{O}_W(\omega(\Omega)) \to \mathcal{O}(\Omega)$. Now the previous paragraph of the proof applies.

By specialization at any point $y = (x, \delta) \in \Omega$ we immediately find that $f_{x}^{K_{\Sigma_0}}$ acts on $\mathcal{M}_y$ (and hence its dual $\mathcal{M}'_y$) via scaling by $a_\tau(x)$. We summarize this below.

**Corollary 7.3.** Let $y \in Y(K^p, \bar{\rho})$ be any point at which $\omega$ is étale. Then $f_{x}^{K_{\Sigma_0}}$ acts on $\mathcal{M}_y$ via scaling by

$$\prod_{v \in \Sigma_0} \text{tr}(\tau_v|\text{WD}(r_x|_{\text{Gal}(\mathbb{F}_v)})).$$

**Proof.** Pick a small enough neighborhood $\Omega$ around $y$ such that $\omega : \Omega \to \omega(\Omega)$ and use the previous Proposition 7.2. \hfill \Box

**8. PROOF OF THE MAIN RESULT**

We now vary $K_{\Sigma_0}$ and reinstate the notation $\mathcal{M}_{K^p}$ (instead of just writing $\mathcal{M}$) to stress the dependence on $K^p = K_{\Sigma_0}K$. Suppose $K_{\Sigma_0}^x \subset K_{\Sigma_0}$ is a compact open subgroup, and let $K_{\Sigma}' = K_{\Sigma_0}^xK$. Recall that the global sections of $\mathcal{M}_{K_{\Sigma}'}$ is the dual of $J_{B}(\hat{S}(K^p), E)_{m}$. Thus we find a natural transition map $\mathcal{M}_{K_{\Sigma}'} \to \mathcal{M}_{K_{\Sigma}'}$ of sheaves on $X_{\Sigma} \times \hat{T}$. Taking their support we find that $Y(K^p, \bar{\rho}) \to Y(K^p, \bar{\rho})$. Passing to the dual stalks at a point $y \in Y(K^p, \bar{\rho})$ yields an embedding $\mathcal{M}'_{K_{\Sigma}^x, y} \to \mathcal{M}'_{K_{\Sigma}^x, y}$ which is equivariant for the $\mathcal{H}(K_{\Sigma_0})$-action. The limit $\lim_{\rightarrow K_{\Sigma_0}} \mathcal{M}'_{K_{\Sigma}^x, y}$ thus becomes an admissible representation of $U(F_{\Sigma_0}^+) \to \prod_{v \in \Sigma_0} \text{GL}_n(F_v)$ over $\kappa(y)$.

**Lemma 8.1.** Let $y \in Y(K^p, \bar{\rho})$ be any point étale over $W$. Let $\otimes_{v \in \Sigma_0} \pi_v$ be an arbitrary irreducible subquotient of $\lim_{\rightarrow K_{\Sigma_0}} \mathcal{M}'_{K_{\Sigma}^x, y}$. Then for all places $v \in \Sigma_0$ we have an isomorphism

$$\text{WD}(r_x|_{\text{Gal}(\mathbb{F}_v)})^{ss} \simeq \text{rec}(BC_{v}(\pi_v) \otimes |\text{det}|^{(1-n)/2})^{ss}.$$

(Here ss means semisimplification of the underlying representation $\tilde{\rho}$ of $W_{F_v}$, and setting $N = 0$.)

**Proof.** By Lemma 7.3 we know that $f_{x}$ acts on $\lim_{\rightarrow K_{\Sigma_0}} \mathcal{M}'_{K_{\Sigma}^x, y}$ via scaling by $a_\tau(x)$. On the other hand, by the proof of Lemma 6.1 we know that $f_{x}(\otimes_{v \in \Sigma_0} \pi_v)$. By comparing the two expressions we find that

$$\prod_{v \in \Sigma_0} \text{tr}(\tau_v|\text{WD}(r_x|_{\text{Gal}(\mathbb{F}_v)})) \equiv \prod_{v \in \Sigma_0} \text{tr}(\tau_v|\text{rec}(BC_{v}(\pi_v) \otimes |\text{det}|^{(1-n)/2}))$$

for all tuples $\tau$. This shows that $\text{WD}(r_x|_{\text{Gal}(\mathbb{F}_v)})$ and $\text{rec}(BC_{v}(\pi_v) \otimes |\text{det}|^{(1-n)/2})$ have the same semisimplification for all $v \in \Sigma_0$ by "linear independence of characters". \hfill \Box

We use Lemma 8.1 to show that $\lim_{\rightarrow K_{\Sigma_0}} \mathcal{M}'_{K_{\Sigma}^x, y}$ has finite length (which for an admissible representation is equivalent to being finitely generated by Howe’s Theorem, cf. [BZ76, 4.1]).

**Lemma 8.2.** The length of $\lim_{\rightarrow K_{\Sigma_0}} \mathcal{M}'_{K_{\Sigma}^x, y}$ as a $U(F_{\Sigma_0}^+)$-representation is finite, and uniformly bounded in $y \in Y(K^p, \bar{\rho})(E)$.
Proof. We first show finiteness. Suppose the direct limit is of infinite length, and choose an infinite proper chain of $U(F_{\Sigma_0}^+)$-invariant subspaces $(V_i \neq V_{i+1})$

$$\lim_{K \rightarrow K_{\Sigma_0}} M_{\rho, y}^i =: V_0 \supset V_1 \supset V_2 \supset V_3 \supset \cdots .$$

Taking $K_{\Sigma_0}$-invariants (which is exact as $\text{char}(E) = 0$) we find a chain of $H(K_{\Sigma_0})$-submodules $V_i^{K_{\Sigma_0}} \subset M_{\rho, y}^i$. Now the latter is finite-dimensional so this chain must become stationary. I.e., $V_i / V_{i+1} has no nonzero $K_{\Sigma_0}$-invariants for $i$ large enough. If we can show that every irreducible subquotient (such exist by Zorn!) $\otimes_{v \in \Sigma_0} \pi_v$ of $\lim_{K \rightarrow K_{\Sigma_0}} M_{\rho, y}^i$ has nonzero $K_{\Sigma_0}$-invariants, we are done. We will show that we can find a $K_{\Sigma_0}$ with this last property.

The local Langlands correspondence preserves $\epsilon$-factors, and hence conductors. (See [JPSS] for the definition of conductors in the $GL_n$-case, and [Tat79, p. 21] for the Artin conductor of a Weil-Deligne representation.) Therefore, for every place $v \in \Sigma_0$ we have the bound on the conductor of $BC_{\pi_v}(\pi_v)$,

$$c(\pi_v) := c(BC_{\pi_v}(\pi_v))$$

$$= c(\det(BC_{\pi_v}(\pi_v))) \leq c(\det(BC_{\pi_v}(\pi_v)) \cdot \det(|(1-n)/2|^{\ast})) + n.$$  

We will impose a mild condition on $x$ and $\rho$ in order to get uniformity. Since $\sum_{x \in \Sigma_0} \pi_v$ as above satisfies $\pi_v^{K_{\rho}} \neq 0$ as desired. This shows $c(\pi_v)$ is bounded in terms of $x$. If we take $K_{\Sigma_0}$ small enough, say $K_{\Sigma_0} \subset \prod_{v \in \Sigma_0} K_v$ where

$$i_v(K_v) \subset \{ g \in GL_n(O_{F_v}) : (g_{v1}, \ldots, g_{vn}) \equiv (0, \ldots, 1) \mod \mathfrak{p}_{F_v}^N \}$$

with $N$ greater than the right-hand side of the inequality (8.3), then every constituent $\otimes_{v \in \Sigma_0} \pi_v$ as above satisfies $\pi_v^{K_v} \neq 0$ as desired. This shows the length is finite.

To get a uniform bound in $K_{\rho}$ and $\rho$ we improve on the bound (8.3) using [Liv89, Prop. 1.1]: Since $r_x|_{\text{Gal}_{F_v}}$ is a lift of $\bar{\rho}|_{\text{Gal}_{F_v}}$ the afore quoted Proposition implies that

$$c(\det(BC_{\pi_v}(\pi_v))) \leq c(\bar{\rho}|_{\text{Gal}_{F_v}}) + n.$$  

One can improve this bound but the point here is to get uniformity. Taking $K_{\Sigma_0}$ as above with $N$ greater than $c(\bar{\rho}|_{\text{Gal}_{F_v}}) + 2n$ the above argument guarantees that the $U(F_{\Sigma_0}^+)$-length of $\lim_{K \rightarrow K_{\Sigma_0}} M_{\rho, y}^i$ is the same as the $H(K_{\Sigma_0})$-length of $M_{\rho, y}^i$ which is certainly at most $\dim_E M_{\rho, y}^i$. This dimension is uniformly bounded in $y$ since $M_{\rho, y}$ is coherent. $\square$

8.1. Generic representations. Recall the definition of $\pi_{x,v}$ in (4.1). We will impose a mild condition on $r_x|_{\text{Gal}_{F_v}}$ which ensures that $\pi_{x,v}$ is fully induced from a supercuspidal representation of a Levi subgroup (thus in particular is generic, cf. [BZ77]).

Definition 8.4. Decompose $WD(r_x|_{\text{Gal}_{F_v}})^{ss} \simeq \bar{\rho}_1 \oplus \cdots \oplus \bar{\rho}_l$ into a sum of irreducible representations $\bar{\rho}_i : W_{F_v} \rightarrow GL_{n_i}(\mathbb{Q}_p)$. We say $r_x|_{\text{Gal}_{F_v}}$ is generic if $\bar{\rho}_i \ncong \bar{\rho}_j \otimes \epsilon$ for all $i \neq j$. 

For the rest of this section we will assume \( r_x \) is generic at each \( v \in \Sigma_0 \). In the notation of Definition 8.4, each \( \tilde{\rho}_i \) corresponds to a supercuspidal representation \( \tilde{\pi}_i \) of \( \text{GL}_{n_i}(F_v) \) and

\[
\pi_{x,v} \simeq \text{Ind}_{P_{n_1,\ldots,n_t}}^{\text{GL}_{n_1}(F_{\tilde{v}})}(\tilde{\pi}_1 \otimes \cdots \otimes \tilde{\pi}_t)
\]

since the induced representation is irreducible, cf. [BZ77]. Indeed \( \tilde{\pi}_i \ncong \tilde{\pi}_j(1) \) for all \( i \neq j \). (The twiddles above \( \rho_i \) and \( \pi_i \) should not be confused with taking the contragredient.)

By Lemma 8.1 the factor \( \pi_v \) of any irreducible subquotient \( \otimes_{v \in \Sigma_0} \pi_v \) of \( \lim_{\rightarrow} K_{\Sigma_0} M'_{K^F,y} \) has the same supercuspidal support as \( \pi_{x,v} \). Since the latter is fully induced from \( P_{n_1,\ldots,n_t} \) they must be isomorphic.

**Corollary 8.5.** Let \( y = (x, \delta) \in Y(K^F, \tilde{r}) \) be a point étale over \( W \) for which \( r_x \) is generic at every \( v \in \Sigma_0 \). Then \( \lim_{\rightarrow} K_{\Sigma_0} M'_{K^F,y} \) has finite length, and every irreducible subquotient is isomorphic to \( \otimes_{v \in \Sigma_0} \pi_{x,v} \).

Altogether this proves Theorem 1.1 in the Introduction.

**References**


[BZ76] I. Bernstein and A. Zelevinsky, *Representations of the group \( GL(n, F) \), where \( F \) is a local non-Archimedean field*. Uspehi Mat. Nauk 31 (1976), no. 3(189), 5–70.


E-mail address: csorensen@ucsd.edu
Department of Mathematics, UCSD, La Jolla, CA, USA.