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Journal of Number Theory

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The local Langlands correspondence in families and Ihara's lemma for $U(n)$



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ARTICLE INFO

Article history:

Received 27 August 2014

Received in revised form 3 December 2015

Accepted 10 December 2015

Available online 3 February 2016

Communicated by Francesco Calegari

MSC:

11F33

Keywords:

Galois representations

Local Langlands in families

Ihara's lemma

ABSTRACT

The goal of this paper is to reformulate the conjectural Ihara lemma for $U(n)$ in terms of the local Langlands correspondence in families $\tilde{\pi}_\Sigma(\cdot)$, as developed by Emerton and Helm. The reformulation roughly takes the following form. Suppose we are given an irreducible mod ℓ Galois representation $\bar{r} = \bar{r}_m$, which is modular of full level (and small weight), and a finite set of places Σ , none of which divides ℓ . Then $\tilde{\pi}_\Sigma(r_m)$ exists, and has a global realization as a localized module of ℓ -integral algebraic modular forms, where r_m is the universal deformation of \bar{r} of type Σ . This is unconditional for $n = 2$, where Ihara's lemma is an almost trivial consequence of the strong approximation theorem. We emphasize that throughout we will work with *banal* primes $\ell > n$. That is, those for which $\#k(v)^i \not\equiv 1 \pmod{\ell}$, where $1 \leq i \leq n$, for $v \in \Sigma$. A weakening of this assumption (quasi-banal) is ubiquitous in [CHT] (particularly in their Chapter 5 on the non-minimal case and Ihara), where the mod ℓ representation theory of $GL_n(F_{\bar{v}})$ plays a pivotal role. We will need actual banality, to ensure the functors we define are exact. Banality also removes many of the subtleties in the Emerton–Helm correspondence. For example, in yet unpublished work [Hel13], Helm shows that local Langlands in families do exist in the banal case, although we do not use his result. We should stress that our main result has been

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<http://dx.doi.org/10.1016/j.jnt.2015.12.012>

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common knowledge among experts for some time, but its proof has never appeared in print.

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1. Introduction

The modularity of elliptic curves E/\mathbb{Q} (and hence Fermat’s Last Theorem) was proved by means of the Galois representation $r_{E,\ell}$ of $\Gamma_{\mathbb{Q}} = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the ℓ -adic Tate module $T_{\ell}(E) \simeq \mathbb{Z}_{\ell}^{\oplus 2}$, for various primes ℓ . The key step was to prove a modularity lifting theorem, roughly saying that $r_{E,\ell}$ arises from a modular form if its reduction $\bar{r}_{E,\ell}$ does. In a fancier language, one sought to identify a universal deformation ring R_{Σ}^{univ} for $\bar{r}_{E,\ell}$ with a corresponding (localized) Hecke algebra \mathbb{T}_{Σ} – which parametrizes those deformations of $\bar{r}_{E,\ell}$ coming from modular forms. The set of primes Σ is where one allows ramification. Taylor and Wiles found an ingenious method to prove $R_{\emptyset}^{univ} \simeq \mathbb{T}_{\emptyset}$, by an intricate patching argument, which is the so-called minimal case. The general case, where $\Sigma \neq \emptyset$, is then reduced to the minimal case by verifying that both sides of the isomorphism $R_{\Sigma}^{univ} \simeq \mathbb{T}_{\Sigma}$ grow in precisely the same way when one augments Σ . This reduction step requires one to control “level-raising” congruences on the Hecke side. The main ingredient which makes this work is a classical lemma of Ihara, which states that $\ker(J_0(N)^{\oplus 2} \rightarrow J_0(Np))$ is Eisenstein whenever $p \nmid N$.

In [CHT], Clozel, Harris, and Taylor, almost succeeded in extending this approach to GL_n , or more accurately for definite unitary groups in n variables, say $G = U(B, \star)_{/F^+}$, which becomes an inner form B^{\times} of GL_n over an imaginary CM-field $F = EF^+$ (assumed to be unramified everywhere over F^+). Fix a prime $\ell > n$, which splits in E , is unramified in F^+ , and such that $S_{\ell} \cap S(B) = \emptyset$. Start off with an absolutely irreducible mod ℓ Galois representation,

$$\bar{r} : \Gamma_F = \text{Gal}(\bar{F}/F) \longrightarrow \text{GL}_n(k).$$

Here k/\mathbb{F}_{ℓ} is a finite extension, which we think of as the residue field $k = \mathcal{O}/\lambda$ of a large enough coefficient field K/\mathbb{Q}_{ℓ} . To make the modularity lifting machinery work, we need to assume \bar{r} has large image – more precisely, that $\bar{r}(\Gamma_{F^+(\zeta_{\ell})})$ is “big”, in a formal sense (see Definition 2.5.1, p. 55 in [CHT]). The fundamental hypothesis is of course that \bar{r} is modular, by which we mean it comes from a modular Hecke eigensystem for G . That is, $\bar{r} \simeq \bar{r}_{\mathfrak{m}}$ for a maximal ideal $\mathfrak{m} \subset \mathbb{T}_{a, \{\rho_v\}, \emptyset}^T(U)$ of the Hecke algebra acting faithfully on the module of algebraic modular forms, $S_{a, \{\rho_v\}, \emptyset}(U, \mathcal{O})$. We refer to the main text for any unexplained notation (see sections 3.2 and 3.3 below); here we only point out that \mathbb{T}^T is the Hecke algebra generated by operators away from the finite set of split places

$$T := S \cup S_{\ell} \cup S(B),$$

for a collection of **banal**¹ places S (recall that v is banal if $\#\mathrm{GL}_n(k(v))$ is not divisible by ℓ , and quasi-banal if it is banal or $\ell > n$ and $q_v := \#k(v) \equiv 1 \pmod{\ell}$, see Definition 5.1.1 in [CHT]). Moreover, for each $v \in T$ we choose a place \tilde{v} of F above it, which in particular pins down an identification $G(F_v^+) \simeq \mathrm{GL}_n(F_{\tilde{v}})$, well-defined up to conjugacy – for $v \notin S(B)$. We must impose two rather strong conditions on these data (which can be traced back to [CHT], linked to the modularity lifting techniques available at the time it was written):

- $U = U(\emptyset) \subset G(\mathbb{A}_{F^+}^\infty)$ is maximal compact (“level one”),
- $a = (a_{\tau,i})$ is in the Fontaine–Laffaille range ($0 \leq a_{\tau,i} < \ell - n$).

There are additional mild hypotheses on the types $\{\rho_v\}$, which we suppress for now (see section 4.2). The goal of [CHT] was to identify the universal deformation ring R_Σ^{univ} of \bar{r} with a certain localized Hecke algebra \mathbb{T}_Σ , following the strategy for GL_2 outlined in the first paragraph. The minimal case ($\Sigma = \emptyset$) did not cause much trouble. The original approach goes through. Not surprisingly, when $\Sigma \subset S$ is enlarged, the main obstacle was to somehow measure the presence of congruences between modular forms of different levels $U(\Sigma)$. The latter group coincides with U at all $v \notin \Sigma$, but at places $v \in \Sigma$ one replaces the maximal compact U_v with the mirahoric subgroup $U_1(\tilde{v}^n)$ of depth n – with the *same* n as in GL_n . Pulling back \mathfrak{m} to a maximal ideal of level $U(\Sigma)$, and looking at the universal modular deformation of \bar{r} of type Σ , we arrive at

$$r_{\mathfrak{m}} : \Gamma_F \longrightarrow \mathrm{GL}_n(\mathbb{T}_\Sigma), \quad \mathbb{T}_\Sigma = \mathbb{T}_{a, \{\rho_v\}, \emptyset}^T(U(\Sigma))_{\mathfrak{m}}.$$

Here \mathbb{T}_Σ is a *reduced* complete local Noetherian \mathcal{O} -algebra. The main result of [CHT], which involves ideas from Mann’s Harvard thesis [Man01], is an isomorphism $R_\Sigma^{univ} \simeq \mathbb{T}_\Sigma$ contingent on a conjectural (for $n > 2$) analogue of Ihara’s lemma. Namely, Conjecture B in [CHT] which we quote here.

Conjecture 1. *Let $\mathfrak{m} \subset \mathbb{T}_{a, \{\rho_v\}, \emptyset}^T(U)$ be a non-Eisenstein maximal ideal (that is, for which $\bar{r}_{\mathfrak{m}}$ is absolutely irreducible). Let $x \in S$ be a banal place of F^+ . Then every irreducible $\bar{k}[G(F_x^+)]$ -submodule of $S_{a, \{\rho_v\}, \emptyset}(U^x, \bar{k})_{\mathfrak{m}}$ is generic.*

For $n = 2$ this is almost trivially true, since non-generic representations of GL_2 are characters, but for $n > 2$ it is an important open problem whose arithmetic potential

¹ The assumption that ℓ is banal for $v \in S$ is to say that $\#k(v)^i \not\equiv 1 \pmod{\ell}$ for $i = 1, \dots, n$. Assuming this, many of the subtleties in Emerton–Helm’s local Langlands in families disappear. For example, its very existence is already known – by a purely local argument of Helm, see Remark 7.9, p. 16 in [Hel13] – which we do not use, though. In [CHT], they operate with a weaker notion called *quasi-banal* (see their Definition 5.1.1), which includes the *limit case* where $\ell > n$ and $q_v := \#k(v) \equiv 1 \pmod{\ell}$. This hypothesis permeates Chapter 5 in [CHT], in which results of Vigneras on the mod ℓ representation theory of $\mathrm{GL}_n(F_{\tilde{v}})$ are employed – to be precise, that taking Iwahori-invariants is an equivalence of categories (see Lemma 5.1.3 in [CHT]), among other things. It would be of interest to extend the results of our paper to the quasi-banal case, where local Langlands in families is *not* yet known to exist in general (to the best of our knowledge). However, it seems we really need actual banality to guarantee the functors \mathcal{L}_r we define below are exact.

is not limited to $R = \mathbb{T}$ theorems, but conceivably has applications to special values of adjoint L -functions. It should be emphasized that Taylor has since shown that at least the reduced quotient $(R_{\Sigma}^{univ})^{red}$ is isomorphic to \mathbb{T}_{Σ} , by employing a variant of the patching argument due to Kisin, which somehow does not distinguish between the minimal and non-minimal cases but treats both at once. However, this is a priori a weaker statement.

The point of this paper is to revisit Ihara’s lemma for G , and recast it in a more robust contemporary framework, invoking the local Langlands correspondence in families, and its mod ℓ variant, as developed recently by Emerton and Helm. We briefly give the gist of it. The question is whether one can interpolate the local Langlands correspondence in families, and somehow make sense of it over more general coefficient rings A (as opposed to just fields of characteristic zero). To fix ideas, we take A to be a complete *reduced* Noetherian local \mathcal{O} -algebra. Suppose we have a collection of Galois representations $r_w : \Gamma_{F_w} \rightarrow \mathrm{GL}_n(A)$; one for each place $w \in \Delta$, where Δ is a finite set of finite places of a number field F , none of which lie above the residue characteristic ℓ . For each minimal prime $\mathfrak{p} \subset A$, one can look at the specializations $r_w \otimes_A \kappa(\mathfrak{p})$, over the residue field $\kappa(\mathfrak{p})$ (which is a finite extension of K), and via local Langlands attach a smooth $\kappa(\mathfrak{p})$ -representation $\tilde{\pi}(r_w \otimes_A \kappa(\mathfrak{p}))$ of $\mathrm{GL}_n(F_w)$. Examples show that it is more natural to take $\pi(\cdot)$ to be the *generic* local Langlands correspondence of Breuil and Schneider. Thus the representations are always generic, but possibly reducible. The tilde in $\tilde{\pi}(\cdot)$ signifies taking the smooth $\kappa(\mathfrak{p})$ -dual, which will be a convenient normalization throughout. Now, it is natural to ask for a smooth $A[\mathrm{GL}_n(F_{\Delta})]$ -module H which *interpolates* these representations, as \mathfrak{p} varies (in the sense of (2) below). Theorem 6.2.1 in [EH14] shows that there can be at *most* one such H satisfying two additional technical hypotheses; (1) and (3) below.

- (1) H is A -torsion free (only zerodivisors can annihilate a nonzero $h \in H$).
- (2) For each minimal prime $\mathfrak{p} \subset A$, there is a $\mathrm{GL}_n(F_{\Delta})$ -equivariant isomorphism,

$$\otimes_{w \in \Delta} \tilde{\pi}(r_w \otimes_A \kappa(\mathfrak{p})) \xrightarrow{\sim} \kappa(\mathfrak{p}) \otimes_A H.$$

- (3) The smooth dual $(H/\mathfrak{m}H)^{\vee}$ is *essentially AIG* (absolutely irreducible and generic) which means its socle is absolutely irreducible, generic, and all other constituents are non-generic.

If such an H exists, one denotes it $\tilde{\pi}(\{r_w\}_{w \in \Delta})$. In ongoing work of Helm, he shows the existence of H when Δ consists of banal places by relating deformation rings to his theory of the integral Bernstein center. See Theorem 7.8 and Example 7.10 of [Hel13].

In our setup we will take $A = \mathbb{T}_{\Sigma}$, and $\Delta = \tilde{\Sigma}$ (the collection of all choices \tilde{v} , for $v \in \Sigma$). We then seek a global realization of the module $\tilde{\pi}(\{r_{\mathfrak{m}, \tilde{v}}\}_{v \in \Sigma})$. Our main result exhibits it as a module of algebraic modular forms, granting Conjecture 1 (and a minor multiplicity one assumption):

Theorem 1. *Admit Ihara’s lemma, and multiplicity one for the automorphic spectrum of G (both hold for $n = 2$). Let Σ be a set of **banal** places for ℓ , as above. Then there are $G(F_\Sigma^+)$ -equivariant \mathbb{T}_Σ -linear isomorphisms, unique modulo \mathbb{T}_Σ^\times (by 6.2.1 (5) in [EH14]),*

$$\tilde{\pi}(\{r_{\mathfrak{m},\tilde{v}}\}_{v \in \Sigma}) \simeq (\otimes_{v \in \Sigma} \tilde{\pi}(r_{\mathfrak{m},\tilde{v}}))^{tf} \simeq S_{a,\{\rho_v\},\emptyset}(U^\Sigma, \mathcal{O})_{\mathfrak{m}},$$

where tf denotes the maximal \mathbb{T}_Σ -torsionfree quotient (and $U^\Sigma = \prod_{v \notin \Sigma} U_v$ is a product of hyperspecial maximal compact subgroups, away from Σ).

The first isomorphism in Theorem 1 is Proposition 6.2.4 in [EH14]. The point of our paper is to give a concrete global realization of $\tilde{\pi}(\{r_{\mathfrak{m},\tilde{v}}\}_{v \in \Sigma})$, in the banal case, contingent on Ihara’s lemma (and multiplicity one, which should be a minor issue in comparison. In fact Mok has informed me it is no issue at all. It is known, and follows from the results of [Mok15]). In fact, the second isomorphism in Theorem 1 is equivalent to Ihara’s lemma, as we will now explain.²

From now on, for brevity, we will use the notation

$$H := S_{a,\{\rho_v\},\emptyset}(U^\Sigma, \mathcal{O})_{\mathfrak{m}}.$$

It becomes a \mathbb{T}_Σ -module in a natural way (see section 5.2). Thus the content of Theorem 1 is that H satisfies the desiderata (1)–(3) above. We first show that H^{tf} satisfies these criteria, and only at the very end we show that in fact $H = H^{tf}$ is torsion-free (over \mathbb{T}_Σ). It is fairly easy to show H satisfies (2), and deduce that H^{tf} does too. What requires some more serious thought is to show that $(H/\mathfrak{m}H)^\vee$ is essentially AIG, and this is where Ihara’s lemma comes in. Indeed, it implies Ihara’s lemma: Petersson duality identifies $(H/\mathfrak{m}H)^\vee$ with a space of mod ℓ modular forms on a closely related unitary group $G' = U(B^{op}, \star)$ (see Theorem 4) and all simple submodules hereof must be generic, because the socle is. Conversely, we must show $(H/\mathfrak{m}H)^\vee$ has a unique generic constituent which must sit as a submodule if we believe Ihara. To do that, we introduce a certain (exact) localization functor $\mathcal{L} = \mathcal{L}_{n,\Sigma}$, which commutes with base change: Take $U_1(\tilde{v}^n)$ -invariants, for all $v \in \Sigma$, and localize at the ideal generated by Atkin–Lehner operators $U^{(j)}$. We study its local properties in the first section, building on Mann’s thesis [Man01]; with focus on how it detects generic representations. Applying \mathcal{L} to $H/\mathfrak{m}H$, we essentially reduce the problem to verifying that $\mathcal{L}(H)$ is free of rank one over \mathbb{T}_Σ . The freeness is a byproduct of the proof of the isomorphism $R_\Sigma^{univ} \xrightarrow{\sim} \mathbb{T}_\Sigma$, along the lines of [Dia97], by a second application of Ihara’s lemma. (This freeness is what used to be called mod ℓ multiplicity one.) The fact that the rank is one obviously uses the $m_\pi = 1$ hypothesis for G , in conjunction with our local results on \mathcal{L} combined with

² Experts have informed us that the relation between Ihara and local Langlands in families, expounded upon in this paper, has been well-known among them for quite some time, although never written down in detail anywhere (albeit the connection is briefly hinted at in the first line on p. 3 of [EH14]). Our paper fills that gap.

known properties of the mod ℓ local Langlands correspondence of Emerton and Helm, [EH14].

In the last section of this paper we give a mod ℓ reformulation of Ihara’s lemma, in terms of the mod ℓ local Langlands correspondence $\bar{\pi}(\cdot)$ of Emerton–Helm. As a Corollary of Theorem 1, we show that (admitting Ihara) $\otimes_{v \in \Sigma} \bar{\pi}(\bar{r}_{\mathfrak{m}, \bar{v}})$ maps onto $H/\mathfrak{m}H$. See Corollary 4 in section 6 for more details.

We anticipate that Theorem 1 will have applications to strong local–global compatibility for unitary groups G , in two variables, in the vein of [Eme11], which is concerned with the case of $\mathrm{GL}(2)/\mathbb{Q}$. The key object is the completed cohomology $\hat{H}^0(U^{\Sigma \cup S_\ell})_{\mathcal{O}, \mathfrak{m}}$, where as indicated we shrink the level U_ℓ . It becomes a module for the “big” Hecke algebra $\mathbb{T}_\Sigma^{\text{big}}$; the corresponding inverse limit over shrinking U_ℓ . The goal is to somehow factor the two natural actions of $G(F_\Sigma^+)$ and $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$ as a (completed) tensor product,

$$\hat{H}^0(U^{\Sigma \cup S_\ell})_{\mathcal{O}, \mathfrak{m}} \simeq \pi(\{r_{\mathfrak{m}, \bar{v}}\}_{v \in S_\ell}) \hat{\otimes}_{\mathbb{T}_\Sigma^{\text{big}}} \pi(\{r_{\mathfrak{m}, \bar{v}}\}_{v \in \Sigma}).$$

(Here $\hat{\otimes}$ is a direct limit of ϖ -adically completed tensor products; see Definition C.43 on p. 114 in [Eme11].) The first factor $\pi(\{r_{\mathfrak{m}, \bar{v}}\}_{v \in S_\ell})$ arises from the “ ℓ ”-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_\ell)$ – of Berger, Breuil, Colmez, Kisin, Paskunas and others. The other factor $\pi(\{r_{\mathfrak{m}, \bar{v}}\}_{v \in \Sigma})$ should be the smooth \mathcal{O} -dual of local Langlands in families over $\mathbb{T}_\Sigma^{\text{big}}$ (a codmissible module as in Emerton’s Appendix C of [Eme11]). To show this, one would introduce the module

$$X := \mathrm{Hom}_{\mathbb{T}_\Sigma^{\text{big}}[G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)]}(\pi(\{r_{\mathfrak{m}, \bar{v}}\}_{v \in S_\ell}), \hat{H}^0(U^{\Sigma \cup S_\ell})_{\mathcal{O}, \mathfrak{m}})$$

and try to identify its maximal cotorsion-free submodule X_{ctf} with $\pi(\{r_{\mathfrak{m}, \bar{v}}\}_{v \in \Sigma})$. Among other things, the key property one would have to verify is that the representation $(X/\varpi X)[\mathfrak{m}]$ is essentially AIG. However, the latter embeds into

$$\mathrm{Hom}_{k[G(\mathcal{O}_{F^+} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)]}(W_a, H^0(U^{\Sigma \cup S_\ell})_{k, \mathfrak{m}}[\mathfrak{m}]) \simeq S_{a, \emptyset}(U^\Sigma, k)_{k, \mathfrak{m}}[\mathfrak{m}]$$

for suitable Serre weights a , and our paper shows this last representation on mod ℓ modular forms is indeed essentially AIG (hence so is any submodule). The details have been worked out in a separate paper with P. Chojecki, in natural continuation of [CS1]. A preprint [CS2] is now available, in which we prove strong local–global compatibility when \bar{r} is irreducible at all places above ℓ . This is admittedly a rather strong assumption, which we expect can be weakened significantly.

On that note, some of the initial motivation for writing this paper was to ease the transition between [Eme11] and [EH14]: In part (1) of Theorem 4.1.2 on p. 41 of [Eme11], the Kirillov functor \mathcal{F}_∞ (in the notation of 2.4 below) is claimed to be exact, without proof. It turns out, for $n = 2$ it is indeed exact – see Corollary 2 in section 2.5 below – but this is not immediate, and it is not obvious at all whether the exactness of \mathcal{F}_∞ continues to hold for $n > 2$ (left-exactness is OK). It is also unclear to what extent it commutes

with extension of scalars. This leads to an ad hoc notion of generic in Definition 4.1.3 of [Eme11], used in his Theorems 4.4.1 and 5.7.7, which a priori could be different from the standard notion (existence of Whittaker models) used in [EH14]. Hence the connection to local Langlands in families appears to be a bit blurry (at least to the untrained eye). One of the points of this paper is to replace \mathcal{F}_∞ by a certain localization functor \mathcal{L}_n , which enjoys all the desired properties. To verify these we rely on Mann’s Harvard thesis [Man01], and extend some of Mann’s results.

Recently some very interesting progress has been made towards Conjecture 1, by independent work of Thorne and Boyer – see [Tho14] and [Boy15]. In his work which was posted on arXiv in November 2015, Boyer proves Conjecture 1 under three assumptions (H1)–(H3) on \mathfrak{m} (listed on p. 3 of [Boy15]). Essentially (H1) says that $\bar{r}_\mathfrak{m}|_{\Gamma_{F_w}}$ is irreducible at a suitable place w ; (2) says that the Satake parameters at a certain place w does not contain a subset of the form $\{\alpha, q_w\alpha\}$; (3) is a multiplicity one assumption on the image $\bar{r}_\mathfrak{m}(\Gamma_{F_{w_0}})$, for a specific place w_0 . The overall idea (also pursued elsewhere by other people, cf. the acknowledgments in 1.1 of [Tho14]) is to pass to the cohomology of Shimura varieties of Kottwitz–Harris–Taylor type.

2. Local preliminaries

Throughout this section, we work with two local fields. The base field is a finite extension F/\mathbb{Q}_p , with integers \mathcal{O}_F , uniformizer $\varpi = \varpi_F$, and residue field k_F . The coefficient field is a finite extension K/\mathbb{Q}_ℓ , with integers \mathcal{O} , maximal ideal $\lambda = (\varpi_K)$, and residue field k . Here $p \neq \ell$. Later on we will always assume K is large enough to contain the image of every embedding $F \hookrightarrow \bar{K}$. We consider smooth representations of $G = \mathrm{GL}_n(F)$ on A -modules, where A is an \mathcal{O} -algebra. Eventually we will take A to be a complete local Noetherian ring (specifically a localized Hecke algebra), but we will not need this assumption in the beginning. We will always assume we are in the *banal* case, which is to say $\#\mathrm{GL}_n(k_F)$ is prime to ℓ . Under this assumption, the mod ℓ representation theory of $\mathrm{GL}_n(F)$ is completely analogous the theory for characteristic zero coefficients (cf. the book [Vig96] which works this out in detail).

2.1. Smooth representations in families

Let V be a smooth $A[G]$ -module; which means V is the union of all U -invariants V^U , where U ranges over the compact open subgroups of G . Each V^U is an A -submodule, with an action of the Hecke algebra \mathcal{H}_U of compactly supported U -biinvariant functions $G \rightarrow \mathcal{O}$ (equipped with convolution). As an \mathcal{O} -module, \mathcal{H}_U is generated by characteristic functions of double cosets, $[U\alpha U]$ with $\alpha \in G$. The inclusion $V^U \subset V$ has a natural splitting: Define an operator e_U on V by

$$e_U(x) = \frac{1}{[U : U_x]} \cdot \sum_{u \in U/U_x} ux,$$

where $U_x \subset U$ is an open subgroup fixing x . This is well-defined since the pro-order $|U|$ is prime to ℓ , and therefore the index $[U : U_x]$ is invertible in \mathcal{O} . Clearly this defines a retraction $e_U : V \rightarrow V^U$, and therefore $V = V^U \oplus \ker(e_U)$ as A -modules.

Lemma 1. *The functor $V \mapsto V^U$ is exact³, and commutes with base change.*

Proof. Exactness is clear (make use of e_U). Now, suppose B is an A -algebra, and let us extend scalars to B . On the one hand,

$$V = V^U \oplus \ker(e_U) \implies V \otimes_A B = (V^U \otimes_A B) \oplus (\ker(e_U) \otimes_A B).$$

On the other hand, letting e_U act on $V \otimes_A B$, we have

$$V \otimes_A B = (V \otimes_A B)^U \oplus \ker(V \otimes_A B \xrightarrow{e_U} V \otimes_A B).$$

There are obvious inclusions among the summands in these decompositions. Those inclusions must be equalities. \square

2.2. Mirahoric subgroups and their invariants

We will now take U to be one of the mirahoric subgroups $U_1(\varpi^r)$, which appear in the definition of conductors for GL_n . They are defined as

$$U_1(\varpi^r) = \{u \in GL_n(\mathcal{O}_F) : u \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{\varpi^r}\}.$$

More precisely, the last row of u is congruent to $(0, \dots, 0, 1)$ modulo ϖ^r . We extend the definition to $r = \infty$ by taking $U_1(\varpi^\infty)$ to be the mirabolic subgroup $P(\mathcal{O}_F)$, which is the intersection of all the $U_1(\varpi^r)$. For later use, observe that $U_1(\varpi^r)$ is generated by $P(\mathcal{O}_F)$ and the principal congruence subgroup $U(\varpi^r)$ consisting of all $u \equiv 1 \pmod{\varpi^r}$. For V as above, we get an increasing sequence of A -modules,

$$V^{U_1(\varpi)} \subset V^{U_1(\varpi^2)} \subset \dots \subset V^{U_1(\varpi^r)} \subset \dots \subset V^{P(\mathcal{O}_F)} = \bigcup_{r=1}^\infty V^{U_1(\varpi^r)}.$$

Here $V^{U_1(\varpi^r)}$ has an A -linear action of the Hecke algebra $\mathcal{H}_{U_1(\varpi^r)}$. It contains operators $U^{(j)}$, for $j = 1, \dots, n - 1$, which are analogs of the classical Atkin–Lehner operator U_p on elliptic modular forms of level $\Gamma_0(p)$.

Definition 1. $U^{(j)} = [U_1(\varpi^r)\alpha_j U_1(\varpi^r)]$, where $\alpha_j = \begin{pmatrix} \varpi \cdot 1_j & \\ & 1_{n-j} \end{pmatrix}$.

³ This is where we use that ℓ is banal for $GL_n(F)$, and not just quasi-banal (as defined in the Introduction). We could slightly weaken the assumption to $\ell \nmid |U|$, but for the mirahoric subgroups $U = U_1(\varpi^r)$ we will eventually look at, this just boils down to banality.

Their action can be made explicit, as was done in Mann’s Harvard PhD thesis (among other things). Unfortunately, this is neither published nor widely available. (The author is in possession of a copy.) Many parts of it are reviewed or reproved in [CHT]. One of Mann’s key findings is a set of coset representatives, which is independent of r . We recall this set here for future reference.

Proposition 1. *Let $r > 0$ and $j < n$. Then there is a partition*

$$U_1(\varpi^r)\alpha_j U_1(\varpi^r) = \bigsqcup_{I,b} bU_1(\varpi^r),$$

where $I \subset \{1, \dots, n-1\}$ is a subset of cardinality $|I| = j$, and $b \in B$ ranges over the set B_I of upper-triangular matrices in G satisfying the conditions below:

- $b_{ii} = \varpi$ for $i \in I$, and $b_{ii} = 1$ otherwise.
- For a pair $i < j$ such that $i \in I$ and $j \notin I$, allow arbitrary $b_{ij} \in Z$. Let $b_{ij} = 0$ for any other pair $i < j$. (That is, such that $i \notin I$ or $j \in I$.)

Here $Z \subset \mathcal{O}_F$ is a complete set of representatives for k_F containing 0.

Proof. This is Proposition 4.1 on p. 10 in [Man01]. See also p. 82 in [CHT]. \square

We allow $r = \infty$ in Proposition 1. For $r = 0$ Mann gives a similar set of representatives; the only difference is that I runs over subsets of $\{1, \dots, n\}$. Using this, he deduces the following from the fact that $\mathcal{H}_{\text{GL}_n(\mathcal{O}_F)}$ is commutative.

Corollary 1. *The operators $U^{(j)}$, for $j = 1, \dots, n-1$, generate a commutative subalgebra of $\mathcal{H}_{U_1(\varpi^r)}$.*

Proof. This is Corollary 4.2 on p. 11 in [Man01]. \square

This subalgebra is a quotient of the polynomial algebra $\mathcal{O}[X_1, \dots, X_{n-1}]$, which we will just denote $\mathcal{O}[X]$ in what follows. In the notation above, each $V^{U_1(\varpi^r)}$ thus becomes a module over $A[X] = \mathcal{O}[X] \otimes_{\mathcal{O}} A$, by sending $X_j \mapsto U^{(j)}$.

Lemma 2. *For $0 < r \leq s$, the inclusion $V^{U_1(\varpi^r)} \subset V^{U_1(\varpi^s)}$ is $A[X]$ -linear. The corresponding quotient module is A -torsion free if V is.*

Proof. The inclusion is clearly $A[X]$ -linear by Mann’s Proposition (the independence of r). The torsion freeness is easily checked: If ax is fixed by $U_1(\varpi^r)$, we must have $a(ux - x) = 0$ for all $u \in U_1(\varpi^r)$. If V is A -torsion free we deduce $ux = x$, unless a is a zero-divisor. \square

2.3. Localization at the Atkin–Lehner ideal

We will view the various $A[X]$ -modules $V^{U_1(\varpi^r)}$, from the previous section, simply as $\mathcal{O}[X]$ -modules (via $\mathcal{O} \rightarrow A$) and localize them at the maximal ideal

$$\mathfrak{n} = (\lambda, X_1, \dots, X_{n-1}) \subset \mathcal{O}[X].$$

(Recall that $\lambda \subset \mathcal{O}$ is the maximal ideal, and $k = \mathcal{O}/\lambda$.) We shall refer to \mathfrak{n} as the Atkin–Lehner ideal, again by analogy with the classical theory. More formally, we introduce a sequence of functors \mathcal{L}_r from smooth $A[G]$ -modules to the category of A -modules with $\mathcal{O}[X]_{\mathfrak{n}}$ -action, as follows:

Definition 2. $\mathcal{L}_r(V) = (V^{U_1(\varpi^r)})_{\mathfrak{n}} = V^{U_1(\varpi^r)} \otimes_{\mathcal{O}[X]} \mathcal{O}[X]_{\mathfrak{n}}$.

This is motivated by the definition of X_S at the bottom of p. 147 in [CHT]. Let us summarize some of the main properties of \mathcal{L}_r , which will be used below.

Lemma 3.

- (1) \mathcal{L}_r is exact, and commutes with base change.
- (2) For $0 < r \leq s$, the inclusion $\mathcal{L}_r(V) \subset \mathcal{L}_s(V)$ is $\mathcal{O}[X]_{\mathfrak{n}}$ -linear, and the corresponding quotient module is A -torsion free if V is.

Proof. For (1), note that \mathcal{L}_r is a composition of two functors with those properties; taking $U_1(\varpi^r)$ -invariants and then localizing at \mathfrak{n} . Therefore,

$$\mathcal{L}_r(V \otimes_A B) = \mathcal{O}[X]_{\mathfrak{n}} \otimes_{\mathcal{O}[X]} (V^{U_1(\varpi^r)} \otimes_A B) = \mathcal{L}_r(V) \otimes_A B,$$

and exactness is immediate. For (2), note that the quotient in question is

$$\mathcal{L}_s(V)/\mathcal{L}_r(V) \simeq (V^{U_1(\varpi^s)}/V^{U_1(\varpi^r)})_{\mathfrak{n}} = M_{\mathfrak{n}},$$

and M is known to be A -torsion free, if V is. Then so is $M_{\mathfrak{n}}$, as is easily verified. (Indeed, if $\frac{m}{s} \in M_{\mathfrak{n}}$ is annihilated by $a \in A$, then $s'am = 0$, for some $s' \notin \mathfrak{n}$. Hence $a|0$ or $s'm = 0$. The latter implies $\frac{m}{s} = 0$ in $M_{\mathfrak{n}}$.) \square

Over \bar{K} there is the following more concrete description: If V is an admissible $\mathcal{O}[G]$ -module (which means the U -invariants are finitely generated over \mathcal{O}), and K is large enough, $\mathcal{L}_r(V) \otimes_{\mathcal{O}} \bar{K}$ breaks up into generalized eigenspaces in $V^{U_1(\varpi^r)} \otimes_{\mathcal{O}} \bar{K}$ for eigensystems $\mathcal{O}[X] \rightarrow \mathcal{O}$ sending each X_j to an eigenvalue in $\lambda \subset \mathcal{O}$. (That is, systems lifting the natural mod \mathfrak{n} reduction map $\mathcal{O}[X] \rightarrow k$.)

2.4. The common kernel of the U -operators

We introduce variants of the functors \mathcal{L}_r , which are easier to compute, but they may not share the same nice properties (they turn out to be left-exact, but in general, for $n > 2$, it is not clear whether they are exact, or whether they commute with base change).

Definition 3. $\mathcal{F}_r(V) = \{x \in V^{U_1(\varpi^r)} : U^{(1)}x = \dots = U^{(n-1)}x = 0\}$.

This defines a functor from smooth $A[G]$ -modules to A -modules. Again, we include the case $r = \infty$, where one takes vectors $x \in V^{P(\mathcal{O}_F)}$ annihilated by all the $U^{(j)}$. We note that \mathcal{F}_∞ generalizes what Emerton calls the Kirillov functor (for $n = 2$) in section 4.1 of [Eme11]. Clearly, as r varies, we have a chain

$$\mathcal{F}_1(V) \subset \mathcal{F}_2(V) \subset \dots \subset \mathcal{F}_r(V) \subset \dots \subset \mathcal{F}_\infty(V) = \bigcup_{r=1}^\infty \mathcal{F}_r(V),$$

and the subquotients are A -torsion free if V is.

Lemma 4. *The canonical map $\mathcal{F}_r(V) \rightarrow \mathcal{L}_r(V)$ is injective.*

Proof. Suppose $x \in \mathcal{F}_r(V)$ maps to 0 in $\mathcal{L}_r(V)$. That is, $sx = 0$ for some $s \notin \mathfrak{n}$. Since x is annihilated by every $U^{(j)}$, the algebra $\mathcal{O}[X]$ acts on x via the constant-term map $\gamma : \mathcal{O}[X] \rightarrow \mathcal{O}$ (which sends $X_j \mapsto 0$). In particular, $\gamma(s)x = 0$; from which we deduce that $x = 0$ or $\gamma(s) \in \lambda$ (that is, is a non-unit in \mathcal{O}). The latter violates the fact that $s \notin \mathfrak{n}$. \square

At the end of the next subsection (Corollary 3) we give an example showing that the inclusion $\mathcal{F}_r(V) \hookrightarrow \mathcal{L}_r(V)$ is usually *not* surjective: If V is an unramified principal series for $\mathrm{GL}(2)$ over \bar{K} , the eigenspace $\mathcal{F}_r(V)$ is one-dimensional, whereas the *generalized* eigenspace $\mathcal{L}_r(V)$ has dimension $r - 1$.

2.5. A refinement of a result of Mann

To start off, let us get our definition of generic in place. We follow [JS83], which works with complex coefficients. However, the definition in the banal case (that is, with k -coefficients) follows the same lines (cf. [Vig96]).

Definition 4. A (possibly reducible) smooth admissible G -representation V (over K say) is said to be generic if there is a G -equivariant embedding $V \hookrightarrow \mathrm{Ind}_N^G(\psi) = \{W : G \rightarrow K \text{ such that } W(n g) = \psi(n)W(g)\}$, for some (hence any) non-trivial character $\psi : F \rightarrow K^\times$ viewed as a character of the unipotent radical $N \subset B$ in the usual manner.

The main result of [JS83] (their Proposition 3.2) states that certain parabolically induced representations $V = \mathrm{Ind}_Q^G(\tau_1 \otimes \dots \otimes \tau_\nu)$ are generic – even when they are

reducible. Here $Q \subset G$ is the standard parabolic subgroup with Levi factor $GL_{n_1} \times \cdots \times GL_{n_\nu}$, and the τ_i are essentially tempered irreducible representations of the GL_{n_i} with decreasing exponents. This means $\tau_i = \tau_{i,0} \otimes |\cdot|_F^{e_i}$, where $\tau_{i,0}$ is *unitary* and tempered, and $e_1 > \cdots > e_\nu$. (These notions only make sense over \mathbb{C} .) In fact, [JS83] proves the existence of a Kirillov model: For Whittaker functions W in the image of $V \hookrightarrow \text{Ind}_N^G(\psi)$, the mirabolic restriction map $W \mapsto W|_P$ is injective. This was announced as Theorem 5.20, p. 50 in [BZ76], for any irreducible representation V whatsoever (which was an earlier conjecture of Gelfand and Kazhdan). As pointed out in [JS83], when all the $\tau_{i,0}$ are supercuspidal, the injectivity of $W \mapsto W|_P$ in the reducible case can also be deduced from Theorem 4.11 on p. 456 of [BZ77], whose proof carries over verbatim to the banal setting (cf. Chapter III of [Vig96]; more precisely section III.1 on “Dérivées” and especially its subsection 1.7). Below we will only use this result for (unramified) principal series $V = \text{Ind}_B^G(\tau_1 \otimes \cdots \otimes \tau_n)$ over \bar{K} and \bar{k} , where the condition on the exponents is replaced by $\tau_i \tau_j^{-1} \neq |\cdot|_F^{-1}$ for $i < j$, which makes sense over any coefficient field with characteristic not p , and which (at least in the banal case) guarantees that the unique generic constituent of V coincides with the unique irreducible subrepresentation $V_+ \subset V$.

The following will be the crucial local input used later in our paper.

Theorem 2. *Allow $A = \bar{K}$ or $A = \bar{k}$ in part (1), and let $A = \bar{k}$ in parts (2)–(4):*

- (1) *Let V be an irreducible generic smooth admissible representation of G over A . Then $\mathcal{F}_r(V) = \mathcal{F}_\infty(V)$ is one-dimensional over A for r sufficiently large. The same result holds for (possibly reducible) principal series representations $V = \text{Ind}_B^G(\tau_1 \otimes \cdots \otimes \tau_n)$ as above with generic socle $V_+ \subset V$.*
- (2) *(Mann) Suppose $V = \text{Ind}_B^G(\bar{\chi}_1 \otimes \cdots \otimes \bar{\chi}_n)$ is an unramified principal series representation of G over \bar{k} , possibly reducible. Then $\mathcal{F}_n(V) = \mathcal{L}_n(V)$ is one-dimensional over \bar{k} (where the subscript n is the same n as in GL_n).*
- (3) *Suppose $V = \text{Ind}_B^G(\bar{\chi}_1 \otimes \cdots \otimes \bar{\chi}_n)$ is an unramified principal series representation of G over \bar{k} , and let V_+ be its unique generic constituent. Then $\mathcal{F}_n(V_+) = \mathcal{L}_n(V_+)$ is one-dimensional over \bar{k} .*
- (4) *Suppose $V = \text{Ind}_B^G(\bar{\chi}_1 \otimes \cdots \otimes \bar{\chi}_n)$ is an unramified principal series representation of G over \bar{k} , and let V' be a non-generic constituent. Then $\mathcal{F}_n(V') = \mathcal{L}_n(V')$ is trivial.*

Proof. Part (1) is a computation in the Kirillov model. Pick a non-trivial additive character $\psi : F \rightarrow A^\times$, with kernel \mathcal{O}_F , and view it as a character of the unipotent radical $N \subset B$ in the usual fashion (via $N^{ab} \simeq F^{n-1}$ composed with summation). Let $\text{Ind}_N^G(\psi)$ be its smooth induction to G , and $\text{ind}_N^G(\psi)$ its compact induction. By our genericity assumption, $V \hookrightarrow \text{Ind}_N^G(\psi)$, but V need not be irreducible. Thus we identify V with a space of Whittaker functions, \mathcal{W}_V . We restrict each $W \in \mathcal{W}_V$ to the mirabolic subgroup $P = P(F)$. As discussed in the paragraph leading up to the Theorem, the map $W \mapsto W|_P$ is injective – by the result of Bernstein–Zelevinsky (Theorem 4.11 in [BZ77]).

So $V \hookrightarrow \text{Ind}_N^P(\psi)$. Writing down the Bernstein–Zelevinsky filtration of $V|_P$ (as in the proof of [Corollary 2](#) below) we find that $V|_P$ also contains a copy of the (irreducible) Kirillov representation, $\text{ind}_N^P(\psi) \hookrightarrow V$. We will prove (1) by verifying that $\mathcal{F}_\infty(\text{Ind}_N^P(\psi))$ is spanned by one nonzero function W_∞ , which is compactly supported modulo N . This proves (1) since any P -map $\text{ind}_N^P(\psi) \hookrightarrow \text{Ind}_N^P(\psi)$ is proportional to the standard embedding (this is part (3) of the Proposition on p. 49 in [\[BZ76\]](#)).

With every integer tuple $m = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$, we associate the diagonal matrix

$$\rho_m = \text{diag}(\varpi^{m_1}, \dots, \varpi^{m_{n-1}}, 1) \in P.$$

Using the Iwasawa decomposition for GL_{n-1} , it is not hard to check that these ρ_m form a complete set of inequivalent representatives for the double cosets

$$N \backslash P / P(\mathcal{O}_F) \leftrightarrow \{\rho_m\}.$$

Consequently, every $P(\mathcal{O}_F)$ -invariant $W \in \text{Ind}_N^P(\psi)$ is completely determined by its values $W(\rho_m)$. Furthermore, as an easy calculation shows,

$$W(\rho_m) \neq 0 \implies \psi(N \cap \rho_m P(\mathcal{O}_F) \rho_m^{-1}) = 1 \implies m_1 \geq \dots \geq m_{n-1} \geq 0.$$

That is, $m \in \mathbb{Z}_+^{n-1}$ must be dominant (here we use that ψ was chosen to have conductor \mathcal{O}_F). This sets up an isomorphism of A -vector spaces,

$$\text{Ind}_N^P(\psi)^{P(\mathcal{O}_F)} \xrightarrow{\sim} A^{\mathbb{Z}_+^{n-1}}, \quad W \mapsto (W(\rho_m))_{m \in \mathbb{Z}_+^{n-1}},$$

which identifies $\text{ind}_N^P(\psi)^{P(\mathcal{O}_F)}$ with finitely supported sequences, $\oplus_{\mathbb{Z}_+^{n-1}} A$.

We will transfer the $U^{(j)}$ -action across this isomorphism, and explicitly write down the corresponding operator on tuples. Say $a = (a_m) \in A^{\mathbb{Z}_+^{n-1}}$. Then, using Mann’s double coset representatives given in [Proposition 1](#),

$$(U^{(j)}a)_m = \sum_{I \subset \{1, \dots, n-1\}; |I|=j} |B_I| \cdot a_{m+e_I},$$

where $a_{m+e_I} = 0$ if $m + e_I$ is not dominant. In this expansion $|B_I|$ is a non-negative power of q_F , and e_I is the characteristic function of I . It follows from [Proposition 1](#), with $r = \infty$: Indeed, for any $P(\mathcal{O}_F)$ -invariant $W \in \text{Ind}_N^P(\psi)$,

$$(U^{(j)}W)(\rho_m) = \sum_{I,b} W(\rho_m b) = \sum_{I,b} W(\rho_m b \rho_m^{-1} \rho_m),$$

and we factor $\rho_m b \rho_m^{-1} = ut$, where $u \in N$, and $t = \text{diag}(b_{11}, \dots, b_{nn}) \in T$. Then $t\rho_m = \rho_{m+e_I}$, where e_I is the characteristic function of I . Moreover, we have

$$\psi(u) = \psi(\varpi^{m_1-m_2} b_{12} b_{22}^{-1} + \dots + \varpi^{m_{n-1}-0} b_{n-1,n} b_{nn}^{-1}).$$

By the second bullet in Proposition 1, $b_{i-1,i} = 0$ unless $i \notin I$ and $i - 1 \in I$. In particular, $b_{i-1,i} \neq 0$ implies $b_{ii} = 1$. Since m is dominant, the sum lies in \mathcal{O}_F , so that $\psi(u) = 1$. From this, it is straightforward to deduce the expression for $U^{(j)}a$. We proceed to compute the common kernel of the $U^{(j)}$ -operators on tuples. We start with the last operator, $U^{(n-1)}$. Here I would have to be the whole set $\{1, \dots, n - 1\}$, and

$$(U^{(n-1)}a)_m = |B_{\{1, \dots, n-1\}}| \cdot a_{m+(1, \dots, 1)}.$$

We read off that $\ker U^{(n-1)}$ consists of a such that $a_m = 0$ whenever $m_{n-1} > 0$. Move on to $U^{(n-2)}$, and intersect its kernel with $\ker U^{(n-1)}$. For a tuple a in the latter, only $I = \{1, \dots, n - 2\}$ contributes to the sum defining $U^{(n-2)}a$, so

$$(U^{(n-2)}a)_m = |B_{\{1, \dots, n-2\}}| \cdot a_{m+(1, \dots, 1, 0)}.$$

Again, we read off that $\ker U^{(n-2)} \cap \ker U^{(n-1)}$ consists of a such that $a_m = 0$ whenever $m_{n-1} > 0$ or $m_{n-2} > m_{n-1}$. Continuing this way, eventually we find that the common kernel of all the $U^{(j)}$, as $j = 1, \dots, n - 1$, consists of a such that $a_m = 0$ whenever at least one of the inequalities $m_{i-1} \geq m_i$ is strict (with the convention $m_n = 0$). In other words, $m \neq 0$. Altogether, this shows that $\mathcal{F}_\infty(\text{Ind}_N^P(\psi))$ is spanned by the function W_∞ with $W_\infty(\rho_m) = \delta_{m,0}$, which clearly is compactly supported modulo N , and therefore lies in $\text{ind}_N^P(\psi)$, as desired:

$$\langle W_\infty \rangle = \mathcal{F}_\infty(\text{ind}_N^P(\psi)) \xrightarrow{\sim} \mathcal{F}_\infty(V) \xrightarrow{\sim} \mathcal{F}_\infty(\text{Ind}_N^P(\psi)) = \langle W_\infty \rangle$$

Finally we take r large enough such that $W_\infty \in \mathcal{F}_\infty(V)$ lies in $\mathcal{F}_r(V)$.

Part (2) is due to Mann: Since $A = \bar{k}$, we can think of $\mathcal{L}_n(V)$ as just being the simultaneous generalized eigenspace for the $U^{(j)}$ -operators on $V^{U_1(\varpi^n)}$, with eigenvalues 0. By Proposition 4.4 on p. 12 in [Man01], these e -spaces are parametrized by proper subsets $\mathcal{T} \subset \{1, \dots, n\}$. The corresponding eigenvalue of $U^{(|\mathcal{T}|)}$ is nonzero unless $\mathcal{T} = \emptyset$ (by the explicit eigenvalue formula in [Man01]). Thus $\mathcal{L}_n(V)$ corresponds to $\mathcal{T} = \emptyset$, and therefore has dimension one – again, using the dimension formula in Mann’s Proposition 4.4. (See also the proof of Corollary 4.5 in [Man01].) In particular, being one-dimensional, the generalized eigenspace $\mathcal{L}_n(V)$ coincides with the actual eigenspace $\mathcal{F}_n(V)$.

For part (3), first observe that since \mathcal{L}_n is exact, $\dim \mathcal{L}_n(V_+)$ can be at most one – using that $\mathcal{L}_n(V)$ is one-dimensional, as explained in the previous paragraph. Moreover, by Lemma 4,

$$\dim \mathcal{F}_n(V_+) \leq \dim \mathcal{L}_n(V_+) \leq 1.$$

It remains to show $\mathcal{F}_n(V_+)$ is nonzero. Since we are in the banal case, we may permute the $\bar{\chi}_i$ by some Weyl element $w \in S_n$ to ensure that $V_+ \subset \text{Ind}_B^G(\bar{\chi}_{w(1)} \otimes \dots \otimes \bar{\chi}_{w(n)})$, cf. [Vig01]. Then by parts (1) and (2), we know that

$$\mathcal{F}_\infty(V_+) = \mathcal{F}_\infty(\text{Ind}_B^G(\bar{\chi}_{w(1)} \otimes \cdots \otimes \bar{\chi}_{w(n)})) = \mathcal{F}_n(\text{Ind}_B^G(\bar{\chi}_{w(1)} \otimes \cdots \otimes \bar{\chi}_{w(n)})) = \langle \xi \rangle$$

is one-dimensional, say spanned by ξ . This ξ lies in V_+ and is $U_1(\varpi^n)$ -invariant. I.e., $\xi \in \mathcal{F}_n(V_+)$.

Part (4) now follows more or less immediately from the exactness of \mathcal{L}_n . Indeed, again from (2) we get the estimate

$$1 = \dim \mathcal{L}_n(V) \geq \dim \mathcal{L}_n(V_+) + \dim \mathcal{L}_n(V'),$$

and we know $\dim \mathcal{L}_n(V_+) = 1$ from part (3). We conclude that $\mathcal{L}_n(V') = 0$. \square

A slight variant of these methods gives a lower bound on $\dim \mathcal{F}_\infty(V)$.

Corollary 2. *Let $A = \bar{K}$ or $A = \bar{k}$, as in the previous Theorem. Let V be any smooth representation of G over A , of finite length. Then*

$$\dim \mathcal{F}_\infty(V) \geq \dim V_{N,\psi},$$

where $V_{N,\psi}$ denotes the (N, ψ) -coinvariants of V , with ψ an arbitrary generic character of $N^{ab} \simeq F^{n-1}$ (the Bernstein–Zelevinsky “top derivative” $V^{(n)}$). Equality holds for $n = 2$. In particular, \mathcal{F}_∞ is exact on the category of finite length smooth representations of $GL(2)$.

Proof. Consider the Bernstein–Zelevinsky filtration of $V|_P$, as introduced at the very bottom of p. 48 in [BZ76],

$$V_{n-1} \subset V_{n-2} \subset \cdots \subset V_0 = V$$

The completely non-degenerate part $V^{nd} = V_{n-1}$ is a direct sum of $\dim V_{N,\psi}$ copies of the Kirillov representation $\text{ind}_N^P(\psi)$, and V/V^{nd} is degenerate. As just shown during the proof of Theorem 2, part (1), $\mathcal{F}_\infty(\text{ind}_N^P(\psi))$ is one-dimensional, and therefore $\mathcal{F}_\infty(V^{nd})$ has dimension $\dim V_{N,\psi}$. Since \mathcal{F}_∞ certainly is at least left-exact, the inequality follows.

Now suppose $n = 2$. The above filtration has length two, so V/V^{nd} can be assumed to be given by a character χ . Now, $\chi^{P(\mathcal{O})} = 0$ unless χ is unramified, in which case $U^{(1)}$ acts on $\chi^{P(\mathcal{O})} = A$ by multiplication by $\chi(\varpi) \neq 0$. Ergo $\mathcal{F}_\infty(\chi) = 0$, so that $\mathcal{F}_\infty(V)$ and $V_{N,\psi}$ have the same dimension. Since the functor $(-)_N,\psi$ is known to be exact (Proposition 2.35 in [BZ76]), by a dimension-count we deduce that \mathcal{F}_∞ is exact too. Note that $V_{N,\psi}$ is finite-dimensional if V has finite G -length (Corollary 5.22 on p. 51 in [BZ76]). \square

For $n > 2$ there are more complicated non-generic representations V of $GL(n)$ than just characters, and it seems difficult to show that $\mathcal{F}_\infty(V) = 0$ for those.

As another application in the $GL(2)$ -case, and as a curiosum, we mention:

Corollary 3. *Let V be an irreducible unramified principal series representation of $GL_2(F)$ over $A = \bar{K}$ or $A = \bar{k}$, with Satake parameters $\{\alpha, \beta\}$, and consider the operator $U^{(1)} : V^{U_1(\varpi^r)} \rightarrow V^{U_1(\varpi^r)}$ for $r \geq 2$. Its Jordan normal form is*

$$U^{(1)} \sim J_{r-1}(0) \oplus J_1(q^{\frac{1}{2}}\alpha) \oplus J_1(q^{\frac{1}{2}}\beta),$$

where $J_t(\lambda)$ is the $t \times t$ Jordan block with eigenvalue λ .

Proof. From Proposition 4.4 in [Man01], we know that the generalized eigenspaces for $U^{(1)}$ on $V^{U_1(\varpi^r)}$ correspond to proper subsets $\mathcal{T} \subset \{1, 2\}$. The dimension is $\binom{r-1}{1-|\mathcal{T}|}$, which is $r-1$ or 1 according to whether \mathcal{T} is empty or not, and the pertaining eigenvalues are

$$\lambda_\emptyset = 0, \quad \lambda_{\{1\}} = q^{\frac{1}{2}}\alpha, \quad \lambda_{\{2\}} = q^{\frac{1}{2}}\beta.$$

To see that $\lambda_\emptyset = 0$ has only one Jordan block, we are to check the actual eigenspace is one-dimensional. But this is exactly $\mathcal{F}_r(V)$, which we know has dimension one by part (1) of Theorem 2. \square

For instance this shows that $U^{(1)}$ does *not* act semisimply on $V^{U_1(\varpi^r)}$ unless $r = 2$ (a question left open in [Man01], p. 12). For $GL(n)$ we have several operators $U^{(j)}$, and the *simultaneous* generalized eigenspaces are indexed by subsets \mathcal{T} , as in the previous proof. This makes it difficult to pin down the Jordan normal form of a specific operator $U^{(j)}$ when $n > 2$.

3. The global setup

Throughout we will adopt the notation used in [CHT]. Here we briefly recall a few essential definitions from their subsection 3.3 which will remain in force.

3.1. Definite unitary groups

Let $\ell > n > 1$ be a prime, which splits in the imaginary quadratic field E . Fix a totally real field F^+ and introduce the CM-field $F = EF^+$. We will always assume F/F^+ is unramified everywhere, although this assumption may be removable. Choose a central division F -algebra B , of dimension n^2 over F , which satisfies the conditions:

- $B^{op} \simeq B \otimes_{E,c} E$,
- $B_w \simeq M_n(F_w)$ at places w above $w|_{F^+} \notin S(B)$,
- B_w is a *division algebra* whenever $w|_{F^+} \in S(B)$.

Here $S(B) \neq \emptyset$ is a finite set of places of F^+ which split in F . We assume $S(B)$ is disjoint from S_ℓ (the places of F^+ above ℓ), and in addition require that $\#S(B)$ has the same parity as $\frac{n}{2}[F^+ : \mathbb{Q}]$ when n is even. We endow B with an F^+ -linear anti-involution

\star of the second kind ($\star|_F = c$), and let $G = U(B, \star)_{/F^+}$ be the associated unitary group. We will always assume the pair (B, \star) is chosen such that the following two bullets are fulfilled.

- $G(F_v^+) \simeq U(n)$ is compact, for all $v \in S_\infty$.
- G is quasi-split over F_v^+ , for all $v \notin S(B)$.

By choosing an order \mathcal{O}_B we may even define a model of G over \mathcal{O}_{F^+} , which will still be denoted by G . Suppose v splits in F , say $v = ww^c$. The choice of a place $w|v$ above each v determines isomorphisms (unique up to conjugacy)

$$i_w : G(F_v^+) \xrightarrow{\sim} \mathrm{GL}_n(F_w), \quad i_w : G(F_v^+) \xrightarrow{\sim} B_w^\times,$$

according to whether v lies outside $S(B)$ or not. We can (and will) even arrange that i_w identifies $G(\mathcal{O}_{F_v^+})$ with $\mathrm{GL}_n(\mathcal{O}_{F_w})$ and $\mathcal{O}_{B,w}^\times$ respectively.

We will always choose our coefficient field K large enough. That is, K/\mathbb{Q}_ℓ is a finite extension such that every embedding $F \hookrightarrow \bar{K}$ actually maps into K . We let $\mathcal{O} = \mathcal{O}_K$ be its valuation ring, and $k = \mathcal{O}/\lambda$ its residue field. Note,

$$I_\ell = \mathrm{Hom}(F^+, K) \xrightarrow{\sim} \bigsqcup_{v \in S_\ell} \mathrm{Hom}_{cts.}(F_v^+, K),$$

and similarly for F . If for each $v \in S_\ell$ we have marked a place \tilde{v} of F above it, this collection $\{\tilde{v}\}$ defines a subset $\tilde{I}_\ell \subset \mathrm{Hom}(F, K)$ such that $\tilde{I}_\ell \xrightarrow{\sim} I_\ell$ via the obvious restriction map.

3.2. Algebraic modular forms

3.2.1. Weights

Let \mathbb{Z}_+^n denote all n -tuples of integers $a = (a_1 \geq \dots \geq a_n)$. Each such a is the highest weight of a unique irreducible algebraic representation $\xi_a : \mathrm{GL}_n \rightarrow \mathrm{GL}(W_a)$ over \mathbb{Q} , which can even be defined over \mathbb{Z} . Let $M_a \subset W_a$ be a \mathbb{Z} -lattice. Wt_n is the subset of $(\mathbb{Z}_+^n)^{\mathrm{Hom}(F, K)}$ consisting of $a = (a_\tau)_{\tau: F \rightarrow K}$ which satisfy the polarization condition $a_{\tau c, i} = -a_{\tau, n+1-i}$. With each such a , we associate an irreducible representation

$$\xi_a : G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \hookrightarrow \prod_{\tau \in \tilde{I}_\ell} \mathrm{GL}_n(K) \longrightarrow \mathrm{GL}(W_a) = \mathrm{GL}(\otimes_{\tau \in \tilde{I}_\ell} W_{a_\tau}).$$

(Here we abuse notation, and view W_{a_τ} over K .) There is a natural \mathcal{O} -lattice $M_a \subset W_a$, stable under the action of the group $G(\mathcal{O}_{F^+} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)$.

3.2.2. Types

For each place $v = ww^c \in S(B)$, suppose we are given an absolutely irreducible representation $\rho_v : G(F_v^+) \rightarrow \text{GL}(M_{\rho_v})$, with open kernel, on a finite free \mathcal{O} -module M_{ρ_v} . Via Jacquet–Langlands, $\rho_v \circ i_w^{-1}$ corresponds to a generalized Steinberg representation $\text{Sp}_{n/m_v}(\pi_w)$ of $\text{GL}_n(F_w)$; here π_w is a supercuspidal representation of $\text{GL}_{n/m_v}(F_w)$, which in turn corresponds to $\tilde{r}_w : \Gamma_{F_w} \rightarrow \text{GL}_{n/m_v}(\bar{K})$ under local Langlands (suitably normalized). We will always assume K is large enough, so that \tilde{r}_w can be conjugated into $\text{GL}_{n/m_v}(\mathcal{O})$. A priori there could be several inequivalent Γ_{F_w} -stable lattices in K^{n/m_v} . We will avoid this potential ambiguity by imposing extra conditions on the \tilde{r}_w . Namely, $\tilde{r}_w \otimes_{\mathcal{O}} k$ should be absolutely irreducible, and not isomorphic to any cyclotomic twist $(\tilde{r}_w \otimes_{\mathcal{O}} k)(i)$ for $i = 1, \dots, m_v$.

Remark. In [CHT] they also operate with a set of places R , and for each $w|v$ with $v \in R$, specify characters $\chi_w : k(w)^{\times n} \rightarrow \mathcal{O}^{\times}$, which play the role of eigen-characters for the action of Iwahori $\text{Iw}(w)$ on the $\text{Iw}_1(w)$ -invariants. However, in their modularity lifting theorems (such as their Theorem 5.4.1) they eventually take $R = \emptyset$. For simplicity we will take R to be empty throughout.

3.2.3. Automorphic forms

Choose a compact open subgroup $U \subset G(\mathbb{A}_{F^+}^{\infty})$, which we assume factors as $U = \prod_v U_v$. We often assume U is sufficiently small; by which we mean some U_v contains no non-trivial elements of finite order. Moreover, we assume $U_v \subset G(\mathcal{O}_{F_v^+})$ for every $v \in S_{\ell}$, so that U_v acts on M_a . Introduce

$$M_{a, \{\rho_v\}, \emptyset} := M_a \otimes_{\mathcal{O}} \left(\bigotimes_{v \in S(B)} M_{\rho_v} \right),$$

which is an \mathcal{O} -module with commuting actions of $G(\mathcal{O}_{F^+} \times \mathbb{Z}_{\ell})$ and $G(F_{S(B)}^+)$. We include the subscript \emptyset to emphasize we take R empty (cf. the remark above); to stick with the notation used in [CHT]. Now, for any \mathcal{O} -algebra A , let $S_{a, \{\rho_v\}, \emptyset}(U, A)$ be the set of functions,

$$f : G(F^+) \backslash G(\mathbb{A}_{F^+}^{\infty}) \rightarrow M_{a, \{\rho_v\}, \emptyset} \otimes_{\mathcal{O}} A, \quad f(gu) = u_{S_{\ell} \cup S(B)}^{-1} f(g),$$

for all $u \in U$. When U is sufficiently small this is a finite free A -module. We will make frequent use of its automorphic description over $\bar{K} \simeq \mathbb{C}$,

$$S_{a, \{\rho_v\}, \emptyset}(U, \mathcal{O}) \otimes \bar{K} \simeq \bigoplus_{\pi} m_{\pi} \cdot \left(\bigotimes_{v \in S(B)} \text{Hom}_{U_v}(\rho_v^{\vee}, \pi_v) \right) \otimes \left(\bigotimes_{v \notin S(B)} \pi_v^{U_v} \right),$$

where π runs over automorphic representations of $G(\mathbb{A}_{F^+})$ with $\pi_{\infty} \simeq \xi_a^{\vee}$.

3.3. Hecke algebras and Galois representations

Let $T \supset S_\ell \cup S(B)$ be a finite set of split places of F^+ . Suppose $U_v = G(\mathcal{O}_{F_v^+})$ for all split places $v \notin T$. For each of the two divisors $w|v$, there are Hecke operators $T_w^{(j)}$ on modular forms, where $j = 1, \dots, n$, defined as the coset operators

$$T_w^{(j)} = [\mathrm{GL}_n(\mathcal{O}_{F_w})\zeta_{w,j}\mathrm{GL}_n(\mathcal{O}_{F_w})], \quad \zeta_{w,j} = \begin{pmatrix} \varpi_w \cdot I_j & \\ & I_{n-j} \end{pmatrix}.$$

(See p. 99 of [CHT] for more details; they act via the fixed isomorphism i_w .) We let

$$\mathbb{T}_{a,\{\rho_v\},\emptyset}^T(U) = \mathcal{O}[T_w^{(1)}, \dots, T_w^{(n)\pm 1}]_{w|v \notin T} \subset \mathrm{End}_{\mathcal{O}}(S_{a,\{\rho_v\},\emptyset}(U, \mathcal{O}))$$

denote the \mathcal{O} -subalgebra generated by the $T_w^{(j)}$ (along with $T_w^{(n)-1}$) where w ranges over places of F such that $v = w|_{F^+} \notin T$ splits in F . Thus $\mathbb{T}_{a,\{\rho_v\},\emptyset}^T(U)$ is a reduced commutative algebra, finite free over \mathcal{O} .

For each maximal ideal $\mathfrak{m} \subset \mathbb{T}_{a,\{\rho_v\},\emptyset}^T(U)$ (whose residue field is a finite extension of k) there is an associated continuous semisimple Galois representation

$$\bar{r}_{\mathfrak{m}} : \Gamma_F = \mathrm{Gal}(\bar{F}/F) \rightarrow \mathrm{GL}_n(\mathbb{T}_{a,\{\rho_v\},\emptyset}^T(U)/\mathfrak{m}),$$

as in 3.4.2 of [CHT]. Suppose $v = ww^c \notin T$; then $\bar{r}_{\mathfrak{m}}$ is unramified at w , and the characteristic polynomial of $\bar{r}_{\mathfrak{m}}(\mathrm{Frob}_w)$ is the GL_n -Hecke polynomial. We call \mathfrak{m} *non-Eisenstein* when $\bar{r}_{\mathfrak{m}}$ is absolutely irreducible – in which case it has a natural continuous lifting $r_{\mathfrak{m}}$ to the localization $\mathbb{T}_{a,\{\rho_v\},\emptyset}^T(U)_{\mathfrak{m}}$, as described in 3.4.4 of [CHT]. (We may have to enlarge K to ensure \mathfrak{m} has residue field k .)

4. Modularity lifting in the non-minimal case

The proof of $R = \mathbb{T}$ mimics Wiles’s proof for $\mathrm{GL}(2)$: It comes down to verifying a certain numerical criterion (comparing the size of the tangent space with that of a congruence module). In the minimal case, where no ramification is allowed, this can be done by a clever patching argument (Taylor–Wiles systems). The general case is then reduced to the minimal case by controlling level-raising congruences on the Hecke side, which at the end reduces to Ihara’s lemma.

4.1. Ihara’s lemma for unitary groups

Throughout, we will admit the following conjecture. (“Ihara’s lemma”.)

Conjecture 2 (Clozel, Harris, Taylor). *Let $U \subset G(\mathbb{A}_{F^+}^\infty)$ be a sufficiently small subgroup, and suppose*

- $v \in T - (S_\ell \cup S(B))$ is a place where $U_v = G(\mathcal{O}_{F_v^+})$,
- $\mathfrak{m} \subset \mathbb{T}_{a, \{\rho_x\}, \emptyset}^T(U)$ is a non-Eisenstein maximal ideal,
- $f \in S_{a, \{\rho_x\}, \emptyset}(U, \bar{k})[\mathfrak{m}]$ is an eigenform.

Then every irreducible $\bar{k}[G(F_v^+)]$ -submodule π ,

$$\pi \subset \langle G(F_v^+)f \rangle \subset S_{a, \{\rho_x\}, \emptyset}(U^v, \bar{k}),$$

is generic.

In fact, one can deduce it from the special case of trivial data ($a = 0$ and all $\rho_x = 1$), which is Conjecture I on p. 145 of [CHT]. Indeed, we may shrink U at the places $x \in S_\ell \cup S(B)$, and assume U_x is pro- ℓ . It therefore acts unipotently on $M_a \otimes \bar{k}$. Thus, up to semisimplification, $S_{a, \{\rho_x\}, \emptyset}(U^v, \bar{k})$ is just a direct sum of finitely many copies of $S_{0, \{1\}, \emptyset}(U^v, \bar{k})$. Compare with Lemma 5.3.2 in [CHT].

In [CHT] they propose a *stronger* conjecture, which they do not need however. The key difference with Conjecture 2 is that π is not assumed to sit in a cyclic submodule generated by a $G(\mathcal{O}_{F_v^+})$ -spherical eigenform f . We quote it below.

Conjecture 3 (Clozel, Harris, Taylor). *Let $U \subset G(\mathbb{A}_{F^+}^\infty)$ be a sufficiently small subgroup, and suppose*

- $v \in T - (S_\ell \cup S(B))$ is a place (split in F , but no restriction on U_v),
- $\mathfrak{m} \subset \mathbb{T}_{a, \{\rho_x\}, \emptyset}^T(U)$ is a non-Eisenstein maximal ideal.

Then every irreducible $\bar{k}[G(F_v^+)]$ -submodule $\pi \subset S_{a, \{\rho_x\}, \emptyset}(U^v, \bar{k})_{\mathfrak{m}}$ is generic.

This is Conjecture II on p. 146 in [CHT] (see also their Conjecture B in the introduction of [CHT]) where they take trivial data for simplicity, and without loss of generality (cf. their Lemma 5.3.2).

At the very end of this paper (section 5.5 below) we will appeal to Conjecture 3 at a point where Conjecture 2 is insufficient. It turns out a slightly weaker statement is enough for our purposes; namely that any $\pi \subset S_{a, \{\rho_x\}, \emptyset}(U^v, \bar{k})_{\mathfrak{m}}[\mathfrak{m}]$ is generic. We will apply this to a closely related group $G' = U(B^{op}, \star)$.

Lemma 5. *The conjectures above hold for $n = 2$.*

Proof. This is basically just the strong approximation theorem for G^{der} . If π is non-generic, it is one-dimensional, $\pi = \bar{k} \cdot f$ where f factors through the torus $\ker(N_{F/F^+})$. One easily deduces \mathfrak{m} must be Eisenstein. See 5.3.1 in [CHT]. \square

Conjectures 2 and 3 are still open for $n > 2$. The $R = \mathbb{T}$ theorems in [CHT] rely on 2 (in the non-minimal case). Taylor has since adapted a variant of the patching argument,

due to Kisin, which avoids Ihara’s lemma – but proves the a priori weaker result $R^{red} = \mathbb{T}$, cf. [Tay08]. As discussed in the introduction of [CHT], the stronger result $R = \mathbb{T}$ would still be of interest, and have potential applications to special value formulas for adjoint L -functions. One of the goals of this paper is to recast the “Ihara” conjectures in a more modern and robust framework; that of essentially AIG representations, which play a fundamental role in local Langlands in families, as studied by Emerton and Helm in [EH14].

4.2. $R = \mathbb{T}$ in the non-minimal case

Here we follow the discussion in section 5.4 of [CHT] rather closely. Recall that we assume F/F^+ is unramified everywhere, and that ℓ is unramified in F^+ .

We choose a finite set of auxiliary primes $S_a \neq \emptyset$, which all split in F . We assume S_a is disjoint from $S_\ell \cup S(B)$. Moreover, if $v \in S_a$ lies above the rational prime p , then $[F(\zeta_p) : F] > n$. The sole purpose of S_a is to ensure that the levels we work with below are sufficiently small.

Finally, S is a finite set of split places, disjoint from $S_\ell \cup S(B) \cup S_a$, consisting of **banal** places v . That is, $\ell \nmid \#\mathrm{GL}_n(k(v))$ for all $v \in S$. All these places taken together gives us

$$T = S \cup S_\ell \cup S(B) \cup S_a.$$

For each $v \in T$, we once and for all choose a place \tilde{v} of F dividing it, and let \tilde{T} be the collection of all these \tilde{v} .

Now, for each subset $\Sigma \subset S$, we introduce a level $U(\Sigma) \subset G(\mathbb{A}_{F^+}^\infty)$. (Here our notation differs slightly from [CHT], in that we use Σ instead of the somewhat cumbersome S_1 .) Its local factors are defined by

$$U(\Sigma)_v = \begin{cases} \text{hyperspecial,} & v \text{ non-split,} \\ G(\mathcal{O}_{F_v^+}), & v \notin \Sigma \cup S_a \text{ split,} \\ i_{\tilde{v}}^{-1}U_1(\tilde{v}^n), & v \in \Sigma, \\ i_{\tilde{v}}^{-1}U(\tilde{v}), & v \in S_a. \end{cases}$$

Here $U(\tilde{v})$ is the kernel of the reduction map to $\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}}/\varpi_{\tilde{v}})$, and $U_1(\tilde{v}^n)$ consists of matrices whose last row is $(0^{n-1}, 1) \bmod \tilde{v}^n$. (The exponent n is the same as in GL_n .) Let $U = U(\emptyset)$, which is maximal away from S_a .

We assume that the weight $a = (a_\tau)_{\tau:F \rightarrow K} \in \mathrm{Wt}_n$ is in the Fontaine–Laffaille range. That is, it satisfies the inequalities

$$\ell - n > a_{\tau,1} \geq \dots \geq a_{\tau,n} \geq 0$$

for all $\tau \in \tilde{\ell}$. (That is, those τ which define a place in $\tilde{\ell}$.) Furthermore, we recall the small technical hypothesis on the types $\{\rho_v\}_{v \in S(B)}$ mentioned in section 3.2.2 that

$\tilde{r}_{\bar{v}} \otimes_{\mathcal{O}} k$ should be absolutely irreducible, and not isomorphic to any cyclotomic twist $(\tilde{r}_{\bar{v}} \otimes_{\mathcal{O}} k)(i)$ for $i = 1, \dots, m_v$.

Now, let $\mathfrak{m} \subset \mathbb{T}_{a, \{\rho_x\}, \emptyset}^T(U)$ be a non-Eisenstein maximal ideal, with residue field k , and let $\bar{r}_{\mathfrak{m}}$ be its associated Galois representation. We will assume throughout that the (standard) “modularity lifting criteria” are fulfilled:

- $\bar{r}_{\mathfrak{m}}(\Gamma_{F^+(\zeta_\ell)})$ is big⁴ (in the sense of 2.5.1 on p. 55 of [CHT]).
- $\bar{r}_{\mathfrak{m}}$ is unramified at every $v \in S_a$, and $(\text{ad}\bar{r}_{\mathfrak{m}})(1)^{\Gamma_{F_{\bar{v}}}} = 0$.

For $\Sigma \subset S$, consider the deformation problem \mathcal{S}_{Σ} as in [CHT], where we only point out that one takes crystalline lifts at $v \in S_{\ell}$, and unramified lifts at $v \in S - \Sigma$. Let R_{Σ}^{univ} denote the corresponding universal deformation ring of $\bar{r}_{\mathfrak{m}}$.

Pull back \mathfrak{m} to a maximal ideal of $\mathbb{T}_{a, \{\rho_x\}, \emptyset}^T(U(\Sigma))$, and consider the localized module,

$$X_{\Sigma} = S_{a, \{\rho_x\}, \emptyset}(U(\Sigma), \mathcal{O})_{\mathfrak{m}, \mathfrak{n}},$$

where

$$\mathfrak{n} = (\lambda, U_{\bar{v}}^{(1)}, \dots, U_{\bar{v}}^{(n-1)})_{v \in \Sigma}.$$

The image of $\mathbb{T}_{a, \{\rho_x\}, \emptyset}^T(U(\Sigma))_{\mathfrak{m}}$ in $\text{End}(X_{\Sigma})$ is then a complete local Noetherian \mathcal{O} -algebra \mathbb{T}_{Σ} , which is reduced, and finite free over \mathcal{O} . By composition, $r_{\mathfrak{m}}$ gives a natural lifting of $\bar{r}_{\mathfrak{m}}$ to \mathbb{T}_{Σ} , of type \mathcal{S}_{Σ} , and consequently there is a natural surjective map,

$$R_{\Sigma}^{univ} \longrightarrow \mathbb{T}_{\Sigma}.$$

One of the main results of [CHT] shows this is an isomorphism if we admit the conjectural analogue of Ihara’s lemma recalled previously.

Theorem 3. *Assume Conjecture 2 is true. Then $R_{\Sigma}^{univ} \simeq \mathbb{T}_{\Sigma}$. In addition, X_{Σ} is a free module over \mathbb{T}_{Σ} . (Both hold unconditionally if $\Sigma = \emptyset$ or $n = 2$.)*

Proof. This is Theorem 5.4.1 on p. 155 in [CHT], except that the freeness is not mentioned explicitly. However, it follows directly from the proof combined with Diamond’s numerical freeness criterion (Theorem 2.4 in [Dia97]). \square

(We remark that the freeness of X_{\emptyset} is mentioned in Theorem 3.5.1 of [CHT], and the freeness of X_{Σ} is one of the hypotheses of their Lemma 5.3.3.)

The overall strategy of the proof follows that of Wiles. First one proves it in the minimal case, $\Sigma = \emptyset$, by patching together modules and algebras of auxiliary levels (the

⁴ This may be weakened to *adequate* by work of Thorne, cf. [Tho12].

Taylor–Wiles method). Secondly, one shows that R_Σ^{univ} and \mathbb{T}_Σ have the same growth rate as one enlarges Σ . To check this for the Hecke algebra, one needs to control the size of a congruence module by somehow measuring level-raising congruences, which is where [Conjecture 2](#) enters.

4.3. The rank of X_Σ is one

When $S_a = \emptyset$, one can make the exact same definitions as in the previous section, and get a \mathbb{T}_Σ -module X_Σ . The problem is that $U = U(\emptyset)$ is maximal everywhere, and hence not sufficiently small. However, freeness still holds.

Lemma 6. Admitting [Conjecture 2](#), X_Σ is free over \mathbb{T}_Σ ; even when $S_a = \emptyset$.

Proof. Pick a split place u , above a large enough rational prime p , which is banal for ℓ . That is, $\ell \nmid \#\mathrm{GL}_n(k(u))$. By large enough we mean that $[F(\zeta_p) : F] > n$ (recall the assumptions on the places we allow in S_a at the beginning of section 4.2).

Let $S_a = \{u\}$, and define corresponding X'_Σ and \mathbb{T}'_Σ as in the previous section, of level $U'(\Sigma)$ contained in $U(\Sigma)$ – they differ only at the place u , where $U'(\Sigma)_u \simeq U(\tilde{u})$, whereas $U(\Sigma)_u = G(\mathcal{O}_{F_u^+})$. By $R = \mathbb{T}$, we know that X'_Σ is free over \mathbb{T}'_Σ . Now, $X_\Sigma \hookrightarrow X'_\Sigma$, and we view both as \mathbb{T}'_Σ -modules via the restriction map $\mathbb{T}'_\Sigma \rightarrow \mathbb{T}_\Sigma$. The usual averaging-idempotent $e_{G(\mathcal{O}_{F_u^+})}$ (defined in section 2.1) is defined over \mathcal{O} , since u is banal for ℓ , and obviously commutes with the Hecke-action away from T . Consequently, $X_\Sigma \hookrightarrow X'_\Sigma$ admits a retraction, showing that X_Σ sits as a direct summand of X'_Σ , which is free. In other words, X_Σ is at least a projective \mathbb{T}'_Σ -module. Since \mathbb{T}'_Σ is a local ring, a general result of Kaplansky ([Kap58]) tells us X_Σ is in fact free over \mathbb{T}'_Σ . (In particular, $\mathbb{T}'_\Sigma \simeq \mathbb{T}_\Sigma$.) \square

From now on, we will assume $S_a = \emptyset$. Moreover, we will make the following multiplicity one hypothesis for the (non-quasi-split) unitary group G .

Hypothesis 1. Every automorphic representation π of $G(\mathbb{A}_{F^+})$ occurs with multiplicity $m_\pi = 1$.

We are optimistic that the trace formula experts have checked this, but have been unsuccessful in finding a precise reference. At the very least we need this for stable π – by which we mean its Galois representation r_π is irreducible.

Remark. For quasi-split $U(n)$, multiplicity one was proved for $n = 2$ and $n = 3$ by Rogawski. See Theorem 11.5.1(c) and Theorem 13.3.1 in [Rog90]. (For $n = 2$, see also Corollary 5 on p. 726 of [Fli82].) For general n , multiplicity one presumably follows from the much more general results of Mok (which mimics work of Arthur ([Art13]) for symplectic and orthogonal groups) – again, in the quasi-split case. See [Mok15]. For

general unitary groups G (that is, inner forms of $U(n)$, possibly non-quasi-split), $m_\pi = 1$ should follow from work of Kaletha, Shin, and White, cf. [KSW]. Mok has pointed out to me that it can also be deduced from his results in the quasi-split case, by transferring. However, for lack of an exact reference, it seems more prudent to keep $m_\pi = 1$ as a hypothesis. In Chapter 14 of [Rog90], Rogawski dealt with general G , in two or three variables, and established the analogue of the Jacquet–Langlands correspondence – from which one deduces multiplicity one. In summary, at least the multiplicity one hypothesis is fulfilled for $n = 2$ and $n = 3$.

For the next result we will need to impose a further condition on the types $\{\rho_v\}_{v \in S(B)}$ (in addition to those listed in section 4.2). Namely that $\rho_v(\chi) \not\cong \rho_v$ for all characters $\chi \neq 1$ of $G(F_v^+)/U_v \simeq B_{\bar{v}}^\times/\mathcal{O}_{B,\bar{v}}^\times \simeq \mathbb{Z}$ which are trivial on the center $F_{\bar{v}}^\times$.

Proposition 2. *Admitting Hypothesis 1, X_Σ is free of rank one over \mathbb{T}_Σ (assuming $S_a = \emptyset$).*

Proof. Let r be the rank of X_Σ . We compute it by tensoring with \bar{K} . Note,

$$X_\Sigma \otimes_{\mathcal{O}} \bar{K} \simeq \bigoplus_{\pi} (\otimes_{v \in S(B)} \text{Hom}_{U_v}(\rho_v^\vee, \pi_v)) \otimes (\otimes_{v \in S_\ell \cup S \setminus \Sigma} \pi_v^{U_v}) \otimes (\otimes_{v \notin T} \pi_v^{U_v})_{\mathfrak{m}} \otimes (\otimes_{v \in \Sigma} \pi_v^{U(\Sigma)_v})_{\mathfrak{n}},$$

where π runs over automorphic representations of $G(\mathbb{A}_{F^+})$ with $\pi_\infty \simeq \xi_a^\vee$. We have omitted m_π in the decomposition, making use of Hypothesis 1. This is a free module of rank r over

$$\mathbb{T}_\Sigma \otimes_{\mathcal{O}} \bar{K} \simeq \bar{K}^{\text{Hom}_{\mathcal{O}\text{-alg.}(\mathbb{T}_\Sigma, \bar{K})}}. \tag{1}$$

The tensor product $\otimes_{v \in S(B)}$ is one-dimensional over \bar{K} : The center $F_{\bar{v}}^\times$ acts on $\text{Hom}_{U_v}(\rho_v^\vee, \pi_v) \simeq (\rho_v \otimes \pi_v)^{U_v}$ via the central character $\chi_{\rho_v} \cdot \chi_{\pi_v}$ (which is necessarily unramified). Note that $F_{\bar{v}}^\times/\mathcal{O}_{F,\bar{v}}^\times$ sits as an index n subgroup of $G(F_v^+)/U_v \simeq B_{\bar{v}}^\times/\mathcal{O}_{B,\bar{v}}^\times$ (with cyclic quotient) since $\varpi_{F_{\bar{v}}} = \varpi_{B_{\bar{v}}}^n$, cf. [PR94] p. 29. We view $(\rho_v \otimes \pi_v)^{U_v}$ as a representation of $G(F_v^+)/U_v$. As such, it breaks up into a direct sum $\bigoplus_{\chi} \text{Hom}_{G(F_v^+)}(\rho_v(\chi)^\vee, \pi_v)$ with χ running over the n extensions of $\chi_{\rho_v} \cdot \chi_{\pi_v}$ to a character of $B_{\bar{v}}^\times$ trivial on $\mathcal{O}_{B,\bar{v}}^\times$. This is essentially Clifford theory, cf. [Cli37]. By the condition on ρ_v preceding Proposition 2, only one of the summands $\text{Hom}_{G(F_v^+)}(\rho_v(\chi)^\vee, \pi_v)$ is nonzero, and by Schur’s lemma it must be one-dimensional.

Secondly, the next factor $\otimes_{v \in S_\ell \cup S \setminus \Sigma}$ is one-dimensional since $U_v \simeq \text{GL}_n(\mathcal{O}_{F_{\bar{v}}})$. For the same reason, $\otimes_{v \notin T}$ is one-dimensional – U_v being hyperspecial for $v \notin T$. Finally, the \mathfrak{n} -localization of $\otimes_{v \in \Sigma}$ is one-dimensional – and this is the crux of the matter. We give the proof in the next three paragraphs.

For each π contributing to $X_\Sigma \otimes_{\mathcal{O}} \bar{K}$, we must show $\mathcal{L}_n(\pi_v) \simeq \bar{K}$ for all $v \in \Sigma$. Here we think of π_v as a representation of $\text{GL}_n(F_{\bar{v}})$ via our choice $i_{\bar{v}}$. By 3.3.4 in [CHT],

we know how to attach a Galois representation $r_\pi : \Gamma_F \rightarrow \mathrm{GL}_n(\mathcal{O}')$, where $\mathcal{O}' \supset \mathcal{O}$, which is now known to satisfy local–global compatibility at all finite primes (although we only need the result away from ℓ). Let $r_{\pi, \tilde{v}} = r_\pi|_{\Gamma_{F_{\tilde{v}}}}$ be shorthand notation. Via local Langlands, $r_{\pi, \tilde{v}} \leftrightarrow \pi_v$, and π_v is a *generic* representation: Indeed, $\bar{r}_\pi \simeq \bar{r}_\mathfrak{m}$, and the latter is irreducible (since \mathfrak{m} is assumed to be non-Eisenstein). A fortiori, r_π must be irreducible which implies the base change $BC_{F/F^+}(\pi)$ is cuspidal, and therefore (globally) generic. In particular π_v is generic. Thus local Langlands coincides with Breuil–Schneider’s *generic* local Langlands correspondence ([BS07]) in this case.

Now we use the (modified) mod p local Langlands correspondence $\bar{\pi}(\cdot)$ of Emerton and Helm, cf. Theorem 5.1.5 in [EH14], which builds on the semisimple correspondence of Vigneras given in [Vig01]. Since $r_{\pi, \tilde{v}}$ is a lift of $\bar{r}_{\mathfrak{m}, \tilde{v}} \otimes_k k'$, there is a unique (up to homothety) $\mathrm{GL}_n(F_{\tilde{v}})$ -invariant \mathcal{O}' -lattice $L \subset \pi_v$ such that $L \otimes k'$ is essentially AIG (as defined in the Introduction just before Theorem 1), and embeddable in $\bar{\pi}(\bar{r}_{\mathfrak{m}, \tilde{v}}) \otimes_k k'$. This is basically just paraphrasing part (2) of 5.1.5 in [EH14]. Since \mathcal{L}_n commutes with extension of scalars (Lemma 3), it suffices to show $\mathcal{L}_n(L) \simeq \mathcal{O}'$. Or in turn that $\mathcal{L}_n(L \otimes k') \simeq k'$. To do this, recall part (8) of 5.1.5 in [EH14], which says that all constituents of $\bar{\pi}(\bar{r}_{\mathfrak{m}, \tilde{v}})$ (and hence of $L \otimes k'$) have the same supercuspidal support. Namely,

$$|\cdot|^{\frac{n-1}{2}} \cdot \{\bar{\chi}_1, \dots, \bar{\chi}_n\}, \quad \text{if } \bar{r}_{\mathfrak{m}, \tilde{v}}^{ss} \simeq \begin{pmatrix} \bar{\chi}_1 & & \\ & \ddots & \\ & & \bar{\chi}_n \end{pmatrix}.$$

We remind ourselves that \mathfrak{m} comes from full level $U = U(\emptyset)$, so $\bar{r}_\mathfrak{m}$ is unramified away from ℓ ; in particular at \tilde{v} . Thus the $\bar{\chi}_i$ are unramified characters $\Gamma_{F_{\tilde{v}}} \rightarrow \bar{k}^\times$, which we tacitly view as unramified characters of $F_{\tilde{v}}^\times$ via the local Artin map. What is important to us, is that every constituent of $L \otimes k'$ is a constituent of an unramified principal series $\mathrm{Ind}_B^G(\bar{\chi}_1 \otimes \dots \otimes \bar{\chi}_n)$ (ignoring the twist $|\cdot|^{\frac{n-1}{2}}$ for clarity). We now invoke our main local result, Theorem 2: We showed that only the generic constituent of $\mathrm{Ind}_B^G(\bar{\chi}_1 \otimes \dots \otimes \bar{\chi}_n)$ has a nonzero \mathcal{L}_n , which was shown to be one-dimensional (parts (3) and (4) of Theorem 2). Therefore, since $L \otimes k'$ has a unique generic constituent (its socle), being essentially AIG, we deduce from the exactness of \mathcal{L}_n that $\mathcal{L}_n(L \otimes k')$ is one-dimensional, as we wished. Consequently, so is $\mathcal{L}_n(\pi_v)$.

With (1) in mind, it remains to show that a contributing π is completely determined by its Hecke eigensystem $\phi_\pi : \mathbb{T}_\Sigma \rightarrow \bar{K}$. Certainly π_v is determined for all $v \notin T$ split in F . Let $\Pi = BC_{F/F^+}(\pi)$ be the base change to $\mathrm{GL}_n(\mathbb{A}_F)$; whose existence is guaranteed by Proposition 3.3.2 of [CHT] for instance, which quotes results of Clozel and Labesse. Then Π_w is determined at all places w such that $w|_{F^+} \notin T$ splits. Since Π is isobaric, the main result (Theorem A) of [Ram15] shows Π is uniquely determined: If Π and Π' are isobaric automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, such that $\Pi_w \simeq \Pi'_w$ for all but finitely many degree-one places w of F (over F^+) then $\Pi \simeq \Pi'$. (Alternatively one could work with the Galois representation r_π from 3.3.4 in [CHT], and use Tchebotarev.) We still have to argue that π is determined by its base change Π . The potential problem is

what happens at the non-split places v of F^+ , where we must show the L -packets are singletons. However, at non-split v we take U_v to be a hyperspecial subgroup – so π_v is U_v -unramified. At least for unramified representations base change is injective: We refer the reader to Corollary 4.2 on p. 17 of [Min11]. We conclude that $r = 1$ as desired. \square

Remark. As pointed out by the referee, we do not really need the lattice L given by [EH14] (whose reduction is essentially AIG) for the previous proof; the semisimple correspondence of [Vig01] suffices. Indeed, for *any* lattice $L \subset \pi_v$, Vigneras shows its reduction has a unique generic constituent, and that all constituents have the same supercuspidal support (see Theorem 1.6 in [Vig01]).

5. End of the proof of the main result

5.1. More notation and finite generation

We keep the notation introduced in section 3. For $U^\Sigma := \prod_{v \notin \Sigma} U_v$, we consider

$$S_{a, \{\rho_v\}, \emptyset}(U^\Sigma, \mathcal{O}) := \varinjlim_{U_\Sigma} S_{a, \{\rho_v\}, \emptyset}(U_\Sigma U^\Sigma, \mathcal{O}),$$

consisting of smooth $M_{a, \{\rho_v\}, \emptyset}$ -valued functions f on $G(\mathbb{A}_{F^+}^\infty)$, with the usual invariance properties (as in 3.2). It carries a faithful action of the Hecke algebra,

$$\mathbb{T}_{a, \{\rho_v\}, \emptyset}^T(U^\Sigma) := \varprojlim_{U_\Sigma} \mathbb{T}_{a, \{\rho_v\}, \emptyset}^T(U_\Sigma U^\Sigma),$$

and we pull back \mathfrak{m} to a maximal ideal in here (non-Eisenstein, and with residue field k). Then we introduce the localized module

$$H = H_\Sigma := S_{a, \{\rho_v\}, \emptyset}(U^\Sigma, \mathcal{O})_{\mathfrak{m}}.$$

Thus $H \otimes_{\mathcal{O}} \bar{K}$ is an admissible representation of $G(F_\Sigma^+)$, which we always identify with $\prod_{v \in \Sigma} \text{GL}_n(F_{\bar{v}})$. This identification obviously depends on the choices $\tilde{\Sigma}$ and $\{i_{\bar{v}}\}_{v \in \Sigma}$. As a representation of $G(F_\Sigma^+)$, again admitting Hypothesis 1 ($m_\pi = 1$),

$$H \otimes_{\mathcal{O}} \bar{K} \simeq \bigoplus_{\pi} \pi_\Sigma, \tag{2}$$

where π runs over all automorphic representations of $G(\mathbb{A}_{F^+}^\infty)$, of infinity type $\pi_\infty \simeq \xi_a^\vee$, types ρ_v^\vee at $v \in S(B)$, with non-trivial U_v -invariants for $v \in S_\ell \cup S \setminus \Sigma$, U_v -unramified for $v \notin T$, and with $\bar{r}_\pi \simeq \bar{r}_\mathfrak{m}$ (cf. the decomposition of X_Σ in the proof of Proposition 2). Here $\pi_\Sigma := \otimes_{v \in \Sigma} \pi_v$.

Lemma 7. $H \otimes_{\mathcal{O}} \bar{K}$ has finite length as a $G(F_\Sigma^+)$ -representation. (Equivalently, $H \otimes_{\mathcal{O}} \bar{K}$ is finitely generated by Howe’s Theorem – see Theorem 4.1 on p. 37 in [BZ76].)

Proof. Since π_Σ is irreducible, we show there are only finitely many π contributing to $H \otimes_{\mathcal{O}} \bar{K}$ in the above decomposition: Any such π has a fixed weight a , and $\pi_v^{U_v} \neq 0$ for $v \in S_\ell \cup S \setminus \Sigma$, and for $v \notin T$. Moreover, $\pi_v^{U_v \cap \ker(\rho_v)} \neq 0$ for $v \in S(B)$. It remains to check that the condition $\bar{r}_\pi \simeq \bar{r}_m$ forces the conductor of π_v to be bounded for $v \in \Sigma$. This follows from an observation of Livne (and independently Carayol). Indeed, since r_π is a lift of \bar{r}_m , it follows from Proposition 1.1 on p. 135 of [Liv89] that $r_{\pi, \bar{v}}^{ss}$ and $\bar{r}_{\pi, \bar{v}}$ have the same Swan conductors ($\Sigma \cap S_\ell = \emptyset$), for $v \in \Sigma$. As a result, the Artin conductor of $r_{\pi, \bar{v}}$ is at most that of $\bar{r}_{\pi, \bar{v}} \simeq \bar{r}_{m, \bar{v}}$ plus n . We conclude from the fact that the local Langlands correspondence preserves ϵ -factors, that the conductor of π_v is at most n – since $\bar{r}_{m, \bar{v}}$ is unramified. \square

Remark. There is a perhaps more elementary way to deduce the conductor of π_v is at most n , which avoids [Liv89]. During the proof of Lemma 9 below we will show that $\mathcal{L}_n(\pi_v) \neq 0$ by a more direct argument. In particular $\pi_v^{U_1(\varpi_v^n)} \neq 0$.

5.2. Lattices and strong Brauer–Nesbitt

Let π be an automorphic representation appearing in the decomposition of $H \otimes_{\mathcal{O}} \bar{K}$, cf. (2). Fix a place $v \in \Sigma$, for now. Since $r_{\pi, \bar{v}}$ is a lift of $\bar{r}_{m, \bar{v}} \otimes_k \bar{k}$, there is an $\mathcal{O}_{\bar{K}}$ -lattice $L_{\pi_v} \subset \pi_v$ such that $L_{\pi_v} \otimes \bar{k}$ is essentially AIG, and embeddable in $\bar{\pi}(\bar{r}_{m, \bar{v}}) \otimes_k \bar{k}$. Moreover, such an L_{π_v} is unique up to homothety. This is part of Theorem 5.1.5 in [EH14], and was already used previously in the proof of Proposition 2 above. (The analogue holds over a finite extension $k' \supset k$, the residue field of a discrete valuation ring $\mathcal{O}' \supset \mathcal{O}$, and we may sometimes think of L_{π_v} as being defined over \mathcal{O}' – enlarged if necessary.) Let $L_{\pi_\Sigma} := \otimes_{v \in \Sigma} L_{\pi_v}$.

Lemma 8. $L = L_\Sigma := \oplus_\pi L_{\pi_\Sigma}$ is a finitely generated $G(F_\Sigma^+)$ -stable $\mathcal{O}_{\bar{K}}$ -lattice in $H \otimes_{\mathcal{O}} \bar{K}$. Furthermore, its reduction $L \otimes \bar{k}$ is of finite length, and $(L \otimes \bar{k})^{ss}$ is independent of the choice of (any) lattice $L \subset H \otimes_{\mathcal{O}} \bar{K}$. In particular,

$$(L \otimes \bar{k})^{ss} \simeq (H \otimes \bar{k})^{ss}. \tag{3}$$

Proof. This follows from the strong Brauer–Nesbitt principle of Vigneras (which is Theorem 1 in [Vig04]), which we briefly recall: Let G be a connected reductive p -adic group, and let V be a \mathbb{Q}_ℓ -representation of finite length, where $\ell \neq p$. Suppose V admits a G -stable free $\bar{\mathbb{Z}}_\ell$ -submodule $L \subset V$ such that $V \simeq L \otimes_{\bar{\mathbb{Z}}_\ell} \bar{\mathbb{Q}}_\ell$. Then (1) L is finitely generated over $\bar{\mathbb{Z}}_\ell[G]$; (2) $L \otimes_{\bar{\mathbb{Z}}_\ell} \bar{\mathbb{F}}_\ell$ has finite G -length; (3) $(L \otimes_{\bar{\mathbb{Z}}_\ell} \bar{\mathbb{F}}_\ell)^{ss}$ is independent of L . In the usual Brauer–Nesbitt principle, (1) is one of the hypotheses. The point of the strong version is to deduce it from the $\bar{\mathbb{Z}}_\ell$ -freeness of L , as done by Vigneras in Chapter I of [Vig04]. The strong principle applies to $H \otimes_{\mathcal{O}} \bar{K}$ since we verified in Lemma 7 that $H \otimes_{\mathcal{O}} \bar{K}$ is of finite length. \square

Note that the space on the right of (3),

$$H \otimes \bar{k} = S_{a, \{\rho_v\}, \emptyset}(U^\Sigma, \bar{k})_{\mathfrak{m}},$$

is a space of modular forms mod ℓ . We infer that any $G(F_\Sigma^+)$ -constituent hereof occurs in $L \otimes \bar{k}$ (with the same multiplicity), and therefore in $\otimes_{v \in \Sigma} (L_{\pi_v} \otimes \bar{k})$, for some π . The latter tensor product can be embedded in $\otimes_{v \in \Sigma} \bar{\pi}(\bar{r}_{\mathfrak{m}, \bar{v}})$, after extending scalars to \bar{k} . As we have used before (in the proof of Proposition 2), all constituents of $\bar{\pi}(\bar{r}_{\mathfrak{m}, \bar{v}})$ appear in the same unramified principal series $\text{Ind}_B^G(\bar{\chi}_1 \otimes \cdots \otimes \bar{\chi}_n)$. See 5.1.5 in [EH14]. The upshot is that any constituent of $H \otimes \bar{k}$ occurs in a tensor product of unramified principal series $\otimes_{v \in \Sigma} \text{Ind}_B^G(\bar{\chi}_{\bar{v}, 1} \otimes \cdots \otimes \bar{\chi}_{\bar{v}, n})$.

Below it will be useful to have alternative ways to think of \mathbb{T}_Σ . For instance, we will often think of H as a \mathbb{T}_Σ -module, via the following isomorphisms.

Lemma 9. $\mathbb{T}_{a, \{\rho_v\}, \emptyset}^T(U^\Sigma)_{\mathfrak{m}} \xrightarrow{\sim} \mathbb{T}_{a, \{\rho_v\}, \emptyset}^T(U(\Sigma))_{\mathfrak{m}} \xrightarrow{\sim} \mathbb{T}_\Sigma$.

Proof. Suppose $t \in \mathbb{T}_{a, \{\rho_v\}, \emptyset}^T(U^\Sigma)_{\mathfrak{m}}$ acts trivially on X_Σ . We must show t acts trivially on H , so that $t = 0$. It is enough to show t acts trivially on $H \otimes_{\mathcal{O}} \bar{K}$, which has the automorphic description (2). Given a π contributing to $H \otimes_{\mathcal{O}} \bar{K}$, with eigensystem $\phi_\pi : \mathbb{T}_{a, \{\rho_v\}, \emptyset}^T(U^\Sigma) \rightarrow \bar{K}$, we must show $\phi_\pi(t) = 0$. This follows from our hypothesis on t if we can verify π contributes to $X_\Sigma \otimes_{\mathcal{O}} \bar{K}$, as described in the beginning of the proof of Proposition 2. In other words, that $\mathcal{L}_n(\pi_v) \neq 0$ for all $v \in \Sigma$. It suffices to show $\mathcal{L}_n(L_{\pi_v}) \neq 0$. Or in turn that $\mathcal{L}_n(L_{\pi_v} \otimes \bar{k}) \neq 0$ – all because \mathcal{L}_n commutes with extension of scalars. However, as noted above, $L_{\pi_v} \otimes \bar{k}$ has a unique generic constituent, and all its constituents appear in the same unramified principal series $\text{Ind}_B^G(\bar{\chi}_1 \otimes \cdots \otimes \bar{\chi}_n)$. In particular, the generic constituent has a non-trivial \mathcal{L}_n , as shown in Theorem 2. We conclude by exactness of \mathcal{L}_n . \square

The remark after the proof of Proposition 2 applies here as well: We did not use the fact that the reduction of L_{π_v} is essentially AIG. The arguments above work for any lattice. We only used the fact that the reduction has a unique generic constituent, and that all constituents have the same supercuspidal support.

5.3. Generic constituents of $H/\mathfrak{m}H$

As a first approximation to our main result, we have:

Lemma 10. $H/\mathfrak{m}H$ has a unique generic $G(F_\Sigma^+)$ -constituent (occurring with multiplicity one).

Proof. First observe that $X_\Sigma = \mathcal{L}_{n, \Sigma}(H)$, where by $\mathcal{L}_{n, \Sigma}$ we mean the composition of all the functors $\mathcal{L}_{n, \bar{v}}$ for $v \in \Sigma$. It commutes with extension of scalars, so (viewing H as a \mathbb{T}_Σ -module)

$$\mathcal{L}_{n,\Sigma}(H/\mathfrak{m}H) = \mathcal{L}_{n,\Sigma}(H \otimes_{\mathbb{T}_\Sigma} k) = \mathcal{L}_{n,\Sigma}(H) \otimes_{\mathbb{T}_\Sigma} k = X_\Sigma \otimes_{\mathbb{T}_\Sigma} k = k,$$

where, in the last step, we have used that X_Σ is of rank one over \mathbb{T}_Σ , cf. Proposition 2. Since $\lambda = (\varpi_K) = \mathfrak{m} \cap \mathcal{O} \subset \mathfrak{m}$, there is a surjection $H/\lambda H \twoheadrightarrow H/\mathfrak{m}H$, from which we deduce that every constituent of $H/\mathfrak{m}H$ also occurs in $H/\lambda H = H \otimes_{\mathcal{O}} k$. From Lemma 8, and the pertaining discussion, we see that such constituents appear in $\otimes_{v \in \Sigma} \text{Ind}_B^G(\bar{\chi}_{\tilde{v},1} \otimes \cdots \otimes \bar{\chi}_{\tilde{v},n})$. The unique constituent of $H/\mathfrak{m}H$ with a nonzero $\mathcal{L}_{n,\Sigma}$ must therefore be the unique generic constituent of this (tensor product of) unramified principal series. \square

5.4. Duality

In Section 5.2 of [CHT], they define a Petersson pairing between automorphic forms on G and those on a related group $G' = U(B^{op}, \star)_{/F^+}$, isomorphic to G via the inversion map $I : G \xrightarrow{\sim} G'$. The choice of an order \mathcal{O}_B even yields an integral model of G' over \mathcal{O}_{F^+} , and I is defined over \mathcal{O}_{F^+} as well. We choose isomorphisms i_w^t as in the first few paragraphs on p. 143 in [CHT].

For each $a \in \text{Wt}_n$, there is a natural pairing $\langle \cdot, \cdot \rangle_{a_\tau}$ on W_{a_τ} , reflecting the fact that $\xi_{a_\tau}^\vee$ can be realized as $\xi_{a_\tau} \circ {}^t(\cdot)^{-1}$. We let $M'_{a_\tau} \subset W_{a_\tau}$ denote the \mathbb{Z} -lattice dual to M_{a_τ} . Correspondingly, there is an irreducible algebraic representation,

$$\xi'_a : G'(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \hookrightarrow \prod_{\tau \in \tilde{I}_\ell} \text{GL}_n(K) \longrightarrow \text{GL}(W_a) = \text{GL}(\otimes_{\tau \in \tilde{I}_\ell} W_{a_\tau}),$$

where we abuse notation and write W_{a_τ} instead of $W_{a_\tau} \otimes_{\mathbb{Q}} K$. By definition, the action of $G'(\mathcal{O}_{F^+} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)$ stabilizes the \mathcal{O} -lattice $M'_a = \otimes_{\tau \in \tilde{I}_\ell} M'_{a_\tau}$. Furthermore, there is a perfect pairing $\langle \cdot, \cdot \rangle_a : M_a \times M'_a \rightarrow \mathcal{O}$; contravariant in the sense that

$$\langle \xi_a(g)m, m' \rangle_a = \langle m, \xi'_a(I(g)^{-1})m' \rangle_a$$

for $g \in G(\mathcal{O}_{F^+} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)$. In particular, we identify $M'_a \simeq \text{Hom}_{\mathcal{O}}(M_a, \mathcal{O})$.

For each place $v = ww^c \in S(B)$, we introduce the finite free \mathcal{O} -module $M'_{\rho_v} = \text{Hom}_{\mathcal{O}}(M_{\rho_v}, \mathcal{O})$, equipped with a natural $G'(F_v^+)$ -action ρ'_v . Namely,

$$\langle \rho_v(g)m, m' \rangle = \langle m, \rho'_v(I(g)^{-1})m' \rangle$$

for $g \in G(F_v^+)$. We consolidate all these data in the following \mathcal{O} -module,

$$M'_{a, \{\rho_v\}, \emptyset} = M'_a \otimes_{\mathcal{O}} \left(\bigotimes_{v \in S(B)} M'_{\rho_v} \right),$$

which carries commuting actions of $G'(\mathcal{O}_{F^+} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)$ and $G'(F_{S(B)}^+)$. Then for each compact open subgroup $U' \subset G'(\mathbb{A}_{F^+}^\infty)$, and any \mathcal{O} -algebra A , we define $S'_{a, \{\rho_v\}, \emptyset}(U', A)$ to be the set of functions

$$f' : G'(F^+) \backslash G'(\mathbb{A}_{F^+}^\infty) \rightarrow M'_{a, \{\rho_v\}, \emptyset} \otimes_{\mathcal{O}} A, \quad f'(gv) = v_{S_\ell \cup S(B)}^{-1} f'(g).$$

When U is sufficiently small, and $U' = I(U)$, this is in duality with $S_{a,\{\rho_v\},\emptyset}(U, A)$ via the pairing defined at the bottom of p. 144 in [CHT] (taking $\eta = 1$),

$$\langle f, f' \rangle = \sum_{g \in G(F^+) \backslash G(\mathbb{A}_{F^+}^\infty) / U} \langle f(g), f'(I(g)) \rangle.$$

(We keep assuming $U_v \subset G(\mathcal{O}_{F_v^+})$ for $v \in S_\ell$.) This pairing has the following key adjointness property for the Hecke operators,

$$\langle [U\zeta U]f, f' \rangle = \langle f, [U'I(\zeta)^{-1}U']f' \rangle,$$

for any $\zeta \in G(\mathbb{A}_{F^+}^\infty)$ with $\zeta_\ell = 1$. As a special case, we see that $T_w^{(j)}$ is self-adjoint – in the natural sense. More precisely, the adjoint map, which sends $[U\zeta U]$ to $[U'I(\zeta)^{-1}U']$, defines an isomorphism of Hecke algebras,

$$\mathbb{T}_{a,\{\rho_v\},\emptyset}^T(U) \xrightarrow{\sim} \mathbb{T}_{a,\{\rho_v\},\emptyset}^T(U')', \quad T_w^{(j)} \mapsto T_w^{(j)}, \tag{4}$$

where the latter algebra is defined in the obvious way – sitting inside the endomorphisms of $S'_{a,\{\rho_v\},\emptyset}(U', A)$. In particular, without further comment, we will view maximal ideals $\mathfrak{m} \subset \mathbb{T}_{a,\{\rho_v\},\emptyset}^T(U)$ as maximal ideals of $\mathbb{T}_{a,\{\rho_v\},\emptyset}^T(U')'$, and vice versa.

Recall from commutative algebra that

$$\mathbb{T}_{a,\{\rho_v\},\emptyset}^T(U) \xrightarrow{\sim} \prod_{\mathfrak{m}} \mathbb{T}_{a,\{\rho_v\},\emptyset}^T(U)_{\mathfrak{m}},$$

where \mathfrak{m} runs over the finitely many maximal ideals (and similarly for the primed version). Correspondingly, the module $S_{a,\{\rho_v\},\emptyset}(U, \mathcal{O})$ decomposes as a direct sum $\bigoplus_{\mathfrak{m}} S_{a,\{\rho_v\},\emptyset}(U, \mathcal{O})_{\mathfrak{m}}$. Upon tensoring with \bar{K} , each $S_{a,\{\rho_v\},\emptyset}(U, \mathcal{O})_{\mathfrak{m}}$ breaks up into Hecke-eigenspaces, $\bigoplus_{\mathfrak{p} \subset \mathfrak{m}} S_{a,\{\rho_v\},\emptyset}(U, \bar{K})_{\mathfrak{p}}$. Given these observations, it is easy to see that \langle, \rangle restricts to a perfect pairing between \mathfrak{m} -localizations,

$$\langle, \rangle : S_{a,\{\rho_v\},\emptyset}(U, \mathcal{O})_{\mathfrak{m}} \times S'_{a,\{\rho_v\},\emptyset}(U', \mathcal{O})_{\mathfrak{m}} \longrightarrow \mathcal{O}.$$

Now consider levels $U = U_\Sigma U^\Sigma$, as in the definition of $H = H_\Sigma$. Our immediate goal is to extend the duality to infinite level, after shrinking $U_\Sigma \rightarrow \{1\}$. Since $\Sigma \subset S$ consists of banal places, the pro-order $|U_\Sigma|$ is invertible in \mathcal{O} , and we may modify the above pairing \langle, \rangle by suitable constants in order to make them compatible with change of level U_Σ . In the definition of H , we had a non-Eisenstein maximal ideal \mathfrak{m} occurring at full level $U(\emptyset)$, which we pulled back to each $\mathbb{T}_{a,\{\rho_v\},\emptyset}^T(U_\Sigma U^\Sigma)$. Then,

$$H = H_\Sigma = S_{a,\{\rho_v\},\emptyset}(U^\Sigma, \mathcal{O})_{\mathfrak{m}} = \varinjlim_{U_\Sigma} S_{a,\{\rho_v\},\emptyset}(U_\Sigma U^\Sigma, \mathcal{O})_{\mathfrak{m}}.$$

Similarly, let

$$H' = H'_\Sigma = S'_{a, \{\rho_v\}, \emptyset}(I(U^\Sigma), \mathcal{O})_{\mathfrak{m}} = \varinjlim_{U^\Sigma} S'_{a, \{\rho_v\}, \emptyset}(I(U_\Sigma U^\Sigma), \mathcal{O})_{\mathfrak{m}},$$

where we view \mathfrak{m} as a maximal ideal of the primed Hecke algebras, via the isomorphism (4) above.

The upshot of our discussion is the following duality.

Lemma 11. *There is a (smoothly) perfect pairing $\langle \cdot, \cdot \rangle : H \times H' \rightarrow \mathcal{O}$, relative to which each $T_w^{(j)}$ is self-adjoint, and such that $\langle gh, h' \rangle = \langle h, I(g)^{-1}h' \rangle$ for all $g \in G(F_\Sigma^+)$. In other words*

$$H' \simeq \text{Hom}_{\mathcal{O}}(H, \mathcal{O})^\infty \quad \text{and} \quad H \simeq \text{Hom}_{\mathcal{O}}(H', \mathcal{O})^\infty,$$

as smooth $G(F_\Sigma^+)$ -representations defined over \mathbb{T}_Σ . (Intertwining the contragredient action with the action via I on H' .)

The superscript ∞ means we take the smooth vectors in $\text{Hom}_{\mathcal{O}}(-, \mathcal{O})$ (viewed as a representation of $G(F_\Sigma^+)$).

5.5. A second application of Ihara’s lemma

We already used Ihara’s lemma (Conjecture 2) to get $R = \mathbb{T}$ in the non-minimal case, and hence the freeness of X_Σ over \mathbb{T}_Σ . In this subsection we will apply it – or rather the stronger version Conjecture 3 – in a different way, to see that the generic constituent of $H/\mathfrak{m}H$ must in fact be a quotient.

For the moment, we look at the full \mathcal{O} -linear dual $M = \text{Hom}_{\mathcal{O}}(H', \mathcal{O})$, with its natural $\mathbb{T}_\Sigma[G(F_\Sigma^+)]$ -module structure. In this notation $H \simeq M^\infty$.

Lemma 12. $M/\mathfrak{m}M \simeq \text{Hom}_k((H'/\varpi H')[\mathfrak{m}], k)$.

Proof. For each i there is an obvious reduction map (tensor with $\mathcal{O}/\varpi^i\mathcal{O}$)

$$M/\varpi^i M \hookrightarrow \text{Hom}_{\mathcal{O}}(H'/\varpi^i H', \mathcal{O}/\varpi^i \mathcal{O}). \tag{5}$$

On the other hand, at least M is ϖ -adically separated ($\bigcap_i \varpi^i M = 0$). That is, $M \hookrightarrow \varprojlim_i M/\varpi^i M$. Taking the inverse limit of the maps in (5) therefore yields an embedding

$$M \hookrightarrow \varprojlim_i M/\varpi^i M \hookrightarrow \varprojlim_i \text{Hom}_{\mathcal{O}}(H'/\varpi^i H', \mathcal{O}/\varpi^i \mathcal{O}).$$

The composite map is easily checked to be onto: Given a compatible sequence of maps $\phi_i : H'/\varpi^i H' \rightarrow \mathcal{O}/\varpi^i \mathcal{O}$, let $\phi = \varprojlim_i \phi_i : \varprojlim_i H'/\varpi^i H' \rightarrow \mathcal{O}$, which we inflate to a map $H' \rightarrow \mathcal{O}$. In particular M is ϖ -adically complete, and (5) is an isomorphism for all i . In particular, for $i = 1$ we find that $M/\varpi M \simeq \text{Hom}_k(H'/\varpi H', k)$. The largest quotient of

$M/\varpi M$ killed by \mathfrak{m} is $M/\mathfrak{m}M$. Its k -dual is the largest submodule of $H'/\varpi H'$ annihilated by \mathfrak{m} . That is, the \mathfrak{m} -torsion $(H'/\varpi H')[\mathfrak{m}]$. This proves the Lemma. \square

Lemma 13. *Assume Conjecture 3 holds. Then every irreducible $k[G(F_\Sigma^+)]$ -quotient of $H/\mathfrak{m}H$ is generic.*

Proof. First off, observe that

$$(H'/\varpi H')[\mathfrak{m}] \simeq S'_{a, \{\rho_v\}, \emptyset}(I(U^\Sigma), k)_{\mathfrak{m}}[\mathfrak{m}].$$

Let $\pi = \otimes_{v \in \Sigma} \pi_v$ be an absolutely irreducible $G'(F_\Sigma^+)$ -submodule hereof. Then, for every fixed $v \in \Sigma$, there is a $G'(F_v^+)$ -equivariant embedding,

$$\pi_v \hookrightarrow S'_{a, \{\rho_v\}, \emptyset}(U'^v, \bar{k})_{\mathfrak{m}}[\mathfrak{m}],$$

for some sufficiently small level $U'^v = U_{\Sigma'}^v I(U^\Sigma)$ away from v . However, it may not be the case that π_v sits inside a cyclic submodule $\langle G'(F_v^+)f \rangle$, for an eigenform f of full level $U'_v = G'(\mathcal{O}_{F_v^+})$ at v – in which case Conjecture 2 does not suffice. However, since we appeal to Conjecture 3, we can infer that π_v is generic. We conclude that π_Σ is a generic representation of $G'(F_\Sigma^+)$.

Finally, taking smooth vectors $(\cdot)^\infty$ commutes with extending scalars (by banality of the places in $\Sigma \subset S$), so by Lemma 12,

$$H/\mathfrak{m}H = M^\infty/\mathfrak{m}M^\infty = (M/\mathfrak{m}M)^\infty \simeq \text{Hom}_k((H'/\varpi H')[\mathfrak{m}], k)^\infty.$$

All the smooth representations involved here are admissible. We deduce that

$$(H'/\varpi H')[\mathfrak{m}] \simeq \text{Hom}_k(H/\mathfrak{m}H, k)^\infty.$$

Consequently, if π_Σ is an irreducible $k[G(F_\Sigma^+)]$ -quotient of $H/\mathfrak{m}H$, its smooth dual $\pi_\Sigma^\vee = \text{Hom}_k(\pi_\Sigma, k)^\infty$ becomes a submodule of $(H'/\varpi H')[\mathfrak{m}]$, and is therefore generic as just explained. Taking duals again, we see π_Σ itself must be generic. \square

This allows us to finish the proof of our main result:

Theorem 4. *Admit Conjecture 3. Then $H/\mathfrak{m}H$ has a unique irreducible quotient. This quotient is absolutely irreducible and generic. All other constituents of $H/\mathfrak{m}H$ are non-generic: In the terminology of [EH14], the smooth dual*

$$(H/\mathfrak{m}H)^\vee \simeq S'_{a, \{\rho_v\}, \emptyset}(U'^\Sigma, k)_{\mathfrak{m}}[\mathfrak{m}]$$

is essentially AIG (short for “absolutely irreducible and generic”).

Proof. Let π_Σ be an irreducible quotient of $H/\mathfrak{m}H$, which is necessarily generic by Lemma 13. Now, by Lemma 10, $H/\mathfrak{m}H$ has a unique generic constituent (counting multiplicity) – which must then be π_Σ . As a result, $H/\mathfrak{m}H$ has π_Σ as its unique irreducible quotient – which is generic – and all other constituents are non-generic. \square

For $n = 2$ this result is unconditional, and we expect it to play a key role in proving strong local–global compatibility results in the p -adic Langlands program for unitary groups in two variables, in the vein of [Eme11]. This is joint work in progress with Chojecki, cf. [CS2], in continuation of our *weak* local–global compatibility paper [CS1], in which we showed how the p -adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ appears in the completed cohomology of the tower of finite sets for G (assuming mod p irreducibility at p).

5.6. The interpolative property of H

We need to verify that H interpolates the (generic) local Langlands correspondence; or rather its dual $\tilde{\pi}(\cdot)$. (Property (2) in Theorem 6.2.1 in [EH14].) Our multiplicity one assumption (Hypothesis 1) remains in force.

Lemma 14. *If $\mathfrak{p} \subset \mathbb{T}_\Sigma$ is a minimal prime, with residue field $\kappa(\mathfrak{p})$, there is a $\kappa(\mathfrak{p})$ -linear $G(F_\Sigma^+)$ -equivariant isomorphism,*

$$\otimes_{v \in \Sigma} \tilde{\pi}(r_{\mathfrak{m}, \bar{v}} \otimes_{\mathbb{T}_\Sigma} \kappa(\mathfrak{p})) \xrightarrow{\sim} \kappa(\mathfrak{p}) \otimes_{\mathbb{T}_\Sigma} H.$$

(Recall that we always identify $G(F_\Sigma^+)$ with $\prod_{v \in \Sigma} \mathrm{GL}_n(F_{\bar{v}})$.)

Proof. For simplicity, let us assume $\mathcal{O} = \mathbb{T}_\Sigma/\mathfrak{p}$, so that $\kappa(\mathfrak{p}) = K$. Then,

$$\kappa(\mathfrak{p}) \otimes_{\mathbb{T}_\Sigma} H = \kappa(\mathfrak{p}) \otimes_{\mathbb{T}_\Sigma/\mathfrak{p}} (H/\mathfrak{p}H) = (H \otimes_{\mathcal{O}} K)/\mathfrak{p}(H \otimes_{\mathcal{O}} K).$$

Now keep in mind the automorphic description of $H \otimes_{\mathcal{O}} \bar{K}$ given in (2). It breaks up as $\oplus_\pi \pi_\Sigma$, for a certain family of automorphic representations π . Moreover, \mathbb{T}_Σ acts on π_Σ via the eigensystem $\phi_\pi : \mathbb{T}_\Sigma \rightarrow \bar{K}$. After possibly enlarging K ,

$$(H \otimes_{\mathcal{O}} K)/\mathfrak{p}(H \otimes_{\mathcal{O}} K) = \oplus_\pi \pi_\Sigma/\mathfrak{p}\pi_\Sigma.$$

Now, $\mathfrak{p}\pi_\Sigma = 0$ for $\mathfrak{p} = \ker(\phi_\pi)$, and it is easy to see that $\mathfrak{p}\pi_\Sigma = \pi_\Sigma$ for all other \mathfrak{p} . Therefore, the above sum reduces to just π_Σ , where $\pi = \pi(\mathfrak{p})$ now denotes *the* automorphic representation with $\ker(\phi_\pi) = \mathfrak{p}$. In other words, with

$$r_\pi \simeq r_{\mathfrak{m}} \otimes_{\mathbb{T}_\Sigma} \kappa(\mathfrak{p}).$$

It remains to note that $\tilde{\pi}(r_{\pi, \bar{v}}) = \pi_v$, at each $v \in \Sigma$, by local–global compatibility away from ℓ – again, under the identification $G(F_v^+) \simeq \mathrm{GL}_n(F_{\bar{v}})$. (See the proof of

Proposition 3.3.4 in [CHT] how to deduce this from the corresponding result for GL_n – due to Taylor, Yoshida, Shin, and others.) We should point out why π_v is generic. Indeed, r_π is irreducible (since $\bar{r}_\pi \simeq \bar{r}_m$ is; m being non-Eisenstein), so the base change of π to $GL_n(\mathbb{A}_F)$ must be cuspidal, and therefore (globally) generic. Thus, in our situation, there is no discrepancy between local Langlands and “generic” local Langlands. \square

5.7. H^{tf} and local Langlands in families

First recall that an A -module M is *torsionfree* if the map $m \mapsto am$ defines an injection $M \rightarrow M$, for any non-zero-divisor $a \in A$. Equivalently, in a fancier language, every associated prime of M is contained in an associated prime of A . When A is reduced, the latter are just the minimal primes. We let M_{tor} denote the subset of $m \in M$ for which $Ann_A(m)$ contains a non-zero-divisor. Note that M_{tor} is an A -submodule, because the non-zero-divisors form a multiplicative subset. We let $M^{tf} = M/M_{tor}$ be the maximal A -torsionfree quotient of M .

In the next subsection we show that in fact H is \mathbb{T}_Σ -torsionfree. Here we wish to point out that the interpolative property is somehow insensitive to torsion. Indeed, one can easily deduce the weaker statement that H^{tf} interpolates local Langlands (knowing H does).

Lemma 15. $H \otimes_{\mathbb{T}_\Sigma} \kappa(\mathfrak{p}) \xrightarrow{\sim} H^{tf} \otimes_{\mathbb{T}_\Sigma} \kappa(\mathfrak{p})$, for every minimal prime $\mathfrak{p} \subset \mathbb{T}_\Sigma$.

Proof. Let $\mathcal{I} \subset H \otimes_{\mathbb{T}_\Sigma} \kappa(\mathfrak{p})$ denote the image of $H_{tor} \otimes_{\mathbb{T}_\Sigma} \kappa(\mathfrak{p})$ under the natural map. Clearly $G(F_\Sigma^+)$ preserves H_{tor} , and therefore this \mathcal{I} is a $G(F_\Sigma^+)$ -invariant subspace. As we have just seen, as part of the proof of the previous Lemma, $H \otimes_{\mathbb{T}_\Sigma} \kappa(\mathfrak{p}) \simeq \pi_\Sigma$ is irreducible. Consequently, if $\mathcal{I} \neq 0$, it must be the whole space $H \otimes_{\mathbb{T}_\Sigma} \kappa(\mathfrak{p})$. Applying the (composite) functor $\mathcal{L}_{n,\Sigma}$ from Lemma 10,

$$\mathcal{L}_{n,\Sigma}(\mathcal{I}) = \mathcal{L}_{n,\Sigma}(H) \otimes_{\mathbb{T}_\Sigma} \kappa(\mathfrak{p}) = X_\Sigma \otimes_{\mathbb{T}_\Sigma} \kappa(\mathfrak{p}) = \kappa(\mathfrak{p}),$$

where, in the last step, we have used that $X_\Sigma \simeq \mathbb{T}_\Sigma$ (by Proposition 2, which relies on Ihara’s lemma). On the other hand, $\mathcal{L}_{n,\Sigma}(\mathcal{I})$ is a quotient of $\mathcal{L}_{n,\Sigma}(H_{tor}) \otimes_{\mathbb{T}_\Sigma} \kappa(\mathfrak{p})$, and

$$\mathcal{L}_{n,\Sigma}(H_{tor}) \subset \mathcal{L}_{n,\Sigma}(H)_{tor} = X_{\Sigma,tor} = 0,$$

again using $X_\Sigma \simeq \mathbb{T}_\Sigma$ in the last step. We deduce that $\mathcal{L}_{n,\Sigma}(\mathcal{I}) = 0$. This is a contradiction. We conclude that $\mathcal{I} = 0$. \square

Proof of Theorem 1 – up to torsion. H^{tf} is tautologically \mathbb{T}_Σ -torsionfree, and we have just shown (in the last two lemmas) that it interpolates the local Langlands correspondence. Moreover, there is a $G(F_\Sigma^+)$ -equivariant embedding,

$$(H^{tf}/\mathfrak{m}H^{tf})^\vee \hookrightarrow (H/\mathfrak{m}H)^\vee.$$

By our main result, [Theorem 4](#), $(H/\mathfrak{m}H)^\vee$ is essentially AIG. A fortiori, so is $(H^{tf}/\mathfrak{m}H^{tf})^\vee$. [Theorem 6.2.1](#) in [\[EH14\]](#) shows that there is at most one module with these three properties. Having proved its existence, we infer that

$$\tilde{\pi}(\{r_{\mathfrak{m},\bar{v}}\}_{v \in \Sigma}) \simeq H^{tf},$$

where we adapt the notation from [6.2.2](#) in [\[EH14\]](#). It remains to show $H = H^{tf}$. \square

5.8. H is torsion-free over \mathbb{T}_Σ

We finish off by showing that in fact H is \mathbb{T}_Σ -torsionfree.

Lemma 16. $H = H^{tf}$ (at least after enlarging the coefficient field K).

Proof. Suppose there is a non-zero element $h \in H_{tor}$. Thus $Ann_{\mathbb{T}_\Sigma}(h)$ contains a non-zero-divisor $t \in \mathbb{T}_\Sigma$, say. Recall the automorphic description of $H \otimes_{\mathcal{O}} \bar{K}$ from [\(2\)](#),

$$H \otimes_{\mathcal{O}} \bar{K} \simeq \bigoplus_{\pi} \pi_{\Sigma},$$

a finite direct sum (by [Lemma 7](#)). As explained at the end of the proof of [Proposition 2](#), each π is completely determined by its Hecke eigensystem, which we continue to denote $\phi_{\pi} : \mathbb{T}_\Sigma \rightarrow \bar{K}$. Since \mathbb{T}_Σ preserves H , in fact ϕ_{π} maps into $\mathcal{O}_{\bar{K}}$. We enlarge K , if necessary, to arrange for $\mathcal{O} = \mathcal{O}_K$ to accommodate the images of all the (finitely many) ϕ_{π} . Via the isomorphism,

$$\mathbb{T}_\Sigma \otimes_{\mathcal{O}} \bar{K} \simeq \bar{K}^{\text{Hom}_{\mathcal{O}\text{-alg.}}(\mathbb{T}_\Sigma, \bar{K})} = \bar{K}^{\text{Hom}_{\mathcal{O}\text{-alg.}}(\mathbb{T}_\Sigma, \mathcal{O})},$$

$t \otimes 1$ corresponds to the tuple $(\phi_{\pi}(t))_{\pi}$. Moreover, $h \otimes 1$ corresponds to some tuple, which we will label $(h_{\pi})_{\pi}$, for some vectors $h_{\pi} \in \pi_{\Sigma}$. Now, the assumption that $th = 0$ translates into $\phi_{\pi}(t)h_{\pi} = 0$ – for all π . Since h is nonzero, $h_{\pi} \neq 0$ for at least one π , which we fix for the rest of this proof. For this specific π , we deduce that $\phi_{\pi}(t) = 0$. Now let $e_{\pi} \in \mathbb{T}_\Sigma \otimes_{\mathcal{O}} \bar{K}$ be the element which corresponds to the “standard” vector $(0, \dots, 1, \dots, 0)$, which is 1 in the π -slot, and 0 elsewhere. Then obviously $e_{\pi}(t \otimes 1) = 0$ – which shows $t \otimes 1$ is a zero-divisor in the bigger ring $\mathbb{T}_\Sigma \otimes_{\mathcal{O}} \bar{K}$. However, it may not be the case that $e_{\pi} \in \mathbb{T}_\Sigma$. Nevertheless, we claim that $ae_{\pi} \in \mathbb{T}_\Sigma$ for some nonzero $a \in \mathcal{O}$. To see this, observe that the natural inclusion

$$i : \mathbb{T}_\Sigma \hookrightarrow \mathcal{O}^{\text{Hom}_{\mathcal{O}\text{-alg.}}(\mathbb{T}_\Sigma, \mathcal{O})} = \mathcal{O} \times \dots \times \mathcal{O}$$

becomes an isomorphism after $(-) \otimes_{\mathcal{O}} \bar{K}$. Thus $\text{cok}(i) \otimes_{\mathcal{O}} \bar{K} = 0$, which is to say $\text{cok}(i)$ is \mathcal{O} -torsion (since \bar{K} is flat over \mathcal{O}). e_{π} does lie in the target of i , and so some nonzero multiple ae_{π} lies in its image $\text{im}(i)$. In conclusion, we get an element in \mathbb{T}_Σ , which we

will continue to denote ae_π , such that $ae_\pi t = 0$. This shows t is a zerodivisor in \mathbb{T}_Σ , contrary to our assumption. \square

6. $H/\mathfrak{m}H$ and local Langlands mod ℓ

We end this paper with a few observations on how $H/\mathfrak{m}H$ relates to the mod ℓ local Langlands correspondence of Emerton–Helm (as defined in Chapter 5 of [EH14]). By our main result (Theorem 1 in the introduction) we know that Ihara’s lemma is *equivalent* to the isomorphism $\tilde{\pi}(\{r_{\mathfrak{m},\tilde{v}}\}_{v \in \Sigma}) \simeq H$. Reducing mod \mathfrak{m} ,

$$\tilde{\pi}(\{r_{\mathfrak{m},\tilde{v}}\}_{v \in \Sigma}) \otimes_{\mathbb{T}_\Sigma} k \simeq H/\mathfrak{m}H.$$

Now, Conjecture 6.2.9 on p. 52 in [EH14] predicts that there should be an equivariant *surjection*,

$$\otimes_{v \in \Sigma} \tilde{\tilde{\pi}}(\tilde{r}_{\mathfrak{m},\tilde{v}}) \twoheadrightarrow \tilde{\pi}(\{r_{\mathfrak{m},\tilde{v}}\}_{v \in \Sigma}) \otimes_{\mathbb{T}_\Sigma} k.$$

We prove this below – contingent on Ihara’s lemma. Conversely, dualizing such a surjection, $(H/\mathfrak{m}H)^\vee$ sits as a subrepresentation of $\otimes_{v \in \Sigma} \tilde{\pi}(\tilde{r}_{\mathfrak{m},\tilde{v}})$, which is essentially AIG. Therefore $(H/\mathfrak{m}H)^\vee$ is itself essentially AIG. Identifying it with a space of mod ℓ modular forms, using Petersson duality as in Theorem 4 above, one can *deduce* Ihara’s lemma. Thus we end up with following mod ℓ reformulation of Ihara:

Corollary 4. *Ihara’s lemma (Conjecture 1 in the introduction) for G is equivalent to the existence of an equivariant surjection,*

$$\otimes_{v \in \Sigma} \tilde{\tilde{\pi}}(\tilde{r}_{\mathfrak{m},\tilde{v}}) \twoheadrightarrow H/\mathfrak{m}H.$$

(Dually, $S'_{a,\{\rho_v\},\emptyset}(U'^\Sigma, k)_\mathfrak{m}[\mathfrak{m}] \hookrightarrow \otimes_{v \in \Sigma} \tilde{\pi}(\tilde{r}_{\mathfrak{m},\tilde{v}})$, with notation as in Theorem 4. In particular,

$$\otimes_{v \in \Sigma} \tilde{\pi}(\tilde{r}_{\mathfrak{m},\tilde{v}}) \hookrightarrow \text{env}(H/\mathfrak{m}H)^\vee,$$

where *env* denotes the essentially AIG envelope – introduced in Definition 3.2.6 on p. 21 in [EH14].)

Proof. Let us first assume $\Sigma = \{v\}$ is a singleton, and at the end reduce the general case to this special case. Our goal is to produce a surjection,

$$\tilde{\tilde{\pi}}(\tilde{r}_{\mathfrak{m},\tilde{v}}) \twoheadrightarrow \tilde{\pi}(r_{\mathfrak{m},\tilde{v}}) \otimes_{\mathbb{T}_\Sigma} k.$$

This would follow immediately if we knew $\tilde{\pi}(\cdot)$ has the strong property (2') in Remark 1.5.2 of [EH14] – but we don't. To make use of the weaker property (2) in

Theorem 1.5.1 of [EH14], we choose some minimal prime $\mathfrak{p} \subset \mathbb{T}_\Sigma$ and look at the deformation of $\bar{r}_{\mathfrak{m},\bar{v}}$ given by specializing $r_{\mathfrak{m},\bar{v}}$ at \mathfrak{p} ,

$$r_{\mathfrak{m},\bar{v}} \otimes_{\mathbb{T}_\Sigma} \mathbb{T}_\Sigma/\mathfrak{p} : \Gamma_{F_{\bar{v}}} \longrightarrow \mathrm{GL}_n(\mathbb{T}_\Sigma/\mathfrak{p}).$$

This is a representation over $\mathbb{T}_\Sigma/\mathfrak{p}$, which we may identify with just \mathcal{O} after possibly enlarging our coefficient field K/\mathbb{Q}_ℓ – as we did in the proof of Lemma 14 (except that we do not pass to the residue field $\kappa(\mathfrak{p})$). By property (2), alluded to earlier, we now *know* there is a surjection,

$$\tilde{\pi}(\bar{r}_{\mathfrak{m},\bar{v}}) \twoheadrightarrow \tilde{\pi}(r_{\mathfrak{m},\bar{v}} \otimes_{\mathbb{T}_\Sigma} \mathbb{T}_\Sigma/\mathfrak{p}) \otimes_{\mathbb{T}_\Sigma/\mathfrak{p}} k.$$

Moreover, by Lemma 6.3.12 on p. 60 in [EH14], we know that

$$\tilde{\pi}(r_{\mathfrak{m},\bar{v}} \otimes_{\mathbb{T}_\Sigma} \mathbb{T}_\Sigma/\mathfrak{p}) \simeq (\tilde{\pi}(r_{\mathfrak{m},\bar{v}}) \otimes_{\mathbb{T}_\Sigma} \mathbb{T}_\Sigma/\mathfrak{p})^{\mathbb{T}_\Sigma/\mathfrak{p}-tf},$$

the maximal $\mathbb{T}_\Sigma/\mathfrak{p}$ -torsionfree quotient. Once we show $\tilde{\pi}(r_{\mathfrak{m},\bar{v}}) \otimes_{\mathbb{T}_\Sigma} \mathbb{T}_\Sigma/\mathfrak{p}$ is already torsionfree, we are done. Indeed, we would then get a map from $\tilde{\pi}(\bar{r}_{\mathfrak{m},\bar{v}})$ onto

$$\tilde{\pi}(r_{\mathfrak{m},\bar{v}} \otimes_{\mathbb{T}_\Sigma} \mathbb{T}_\Sigma/\mathfrak{p}) \otimes_{\mathbb{T}_\Sigma/\mathfrak{p}} k \simeq \tilde{\pi}(r_{\mathfrak{m},\bar{v}}) \otimes_{\mathbb{T}_\Sigma} \mathbb{T}_\Sigma/\mathfrak{p} \otimes_{\mathbb{T}_\Sigma/\mathfrak{p}} k \simeq \tilde{\pi}(r_{\mathfrak{m},\bar{v}}) \otimes_{\mathbb{T}_\Sigma} k,$$

as desired. To show $H/\mathfrak{p}H$ is $\mathbb{T}_\Sigma/\mathfrak{p}$ -torsionfree, we tweak the proof of Lemma 16 a bit. Recall from the proof of Lemma 14 that

$$H/\mathfrak{p}H \otimes_{\mathcal{O}} \bar{K} = (H \otimes_{\mathcal{O}} \bar{K})/\mathfrak{p}(H \otimes_{\mathcal{O}} \bar{K}) = \pi_\Sigma,$$

where $\pi = \pi(\mathfrak{p})$ is the automorphic representation whose eigensystem ϕ_π has kernel \mathfrak{p} . Thus $t \in \mathbb{T}_\Sigma$ acts on $H/\mathfrak{p}H$ via multiplication by $\phi_\pi(t)$. If t annihilates some nonzero element of $H/\mathfrak{p}H$, we must have $\phi_\pi(t) = 0$ – which shows $H/\mathfrak{p}H$ is torsionfree, and concludes the proof of the Corollary in the singleton-case.

It remains to deal with the general case, where Σ is not necessarily a singleton. Taking the tensor product $\otimes_{v \in \Sigma}$ (over k) of all the surjections defined in the previous paragraph, we get a map

$$\otimes_{v \in \Sigma} \tilde{\pi}(\bar{r}_{\mathfrak{m},\bar{v}}) \twoheadrightarrow \otimes_{v \in \Sigma} (\tilde{\pi}(r_{\mathfrak{m},\bar{v}}) \otimes_{\mathbb{T}_\Sigma} k) \simeq (\otimes_{v \in \Sigma} \tilde{\pi}(r_{\mathfrak{m},\bar{v}})) \otimes_{\mathbb{T}_\Sigma} k,$$

(where the tensor product $\otimes_{v \in \Sigma} \tilde{\pi}(r_{\mathfrak{m},\bar{v}})$ obviously is over \mathbb{T}_Σ). Compose it with

$$\otimes_{v \in \Sigma} \tilde{\pi}(r_{\mathfrak{m},\bar{v}}) \twoheadrightarrow (\otimes_{v \in \Sigma} \tilde{\pi}(r_{\mathfrak{m},\bar{v}}))^{tf} \simeq \tilde{\pi}(\{r_{\mathfrak{m},\bar{v}}\}_{v \in \Sigma})$$

after tensoring with $(-) \otimes_{\mathbb{T}_\Sigma} k$. The last isomorphism above is the content of Proposition 6.2.4 on p. 51 in [EH14]. This proves the Corollary. \square

Acknowledgments

I wish to thank P. Chojecki for useful correspondence, which eventually led to the writing of this paper, and for his continued interest. I also wish to acknowledge the impact of many enjoyable conversations with D. Helm, F. Herzig, S.W. Shin, and L. Xiao on related topics. I thank M. Emerton for clarifications, and I am grateful to the anonymous referee for the thoughtful and detailed feedback provided.

References

- [Art13] J. Arthur, *The Endoscopic Classification of Representations. Orthogonal and Symplectic Groups*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013.
- [BZ76] I.N. Bernstein, A.V. Zelevinskii, Representations of the group $GL(n, F)$, where F is a local non-Archimedean field, *Uspekhi Mat. Nauk* 31 (3(189)) (1976) 5–70.
- [BZ77] I.N. Bernstein, A.V. Zelevinsky, Induced representations of reductive p-adic groups. I, *Ann. Sci. Éc. Norm. Supér.* (4) 10 (4) (1977) 441–472.
- [Boy15] P. Boyer, Diminution du niveau et lemme d’Ihara pour les groupes unitaires, Preprint, arXiv:1511.00144 (dated October 31, 2015).
- [BS07] C. Breuil, P. Schneider, First steps towards p-adic Langlands functoriality, *J. Reine Angew. Math.* 610 (2007) 149–180.
- [CS1] P. Chojecki, C. Sorensen, Weak local–global compatibility in the p-adic Langlands program for $U(2)$, *Rend. Semin. Mat. Univ. Padova* 137 (2017), in press.
- [CS2] P. Chojecki, C. Sorensen, Strong local–global compatibility in the p-adic Langlands program for $U(2)$, *Rend. Semin. Mat. Univ. Padova* 137 (2017), in press.
- [Cli37] A.H. Clifford, Representations induced in an invariant subgroup, *Ann. of Math.* (2) 38 (3) (1937) 533–550.
- [CHT] L. Clozel, M. Harris, R. Taylor, Automorphy for some l-adic lifts of automorphic mod l Galois representations, with Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vigneras, *Publ. Math. Inst. Hautes Études Sci.* 108 (2008) 1–181.
- [Dia97] F. Diamond, The Taylor–Wiles construction and multiplicity one, *Invent. Math.* 128 (2) (1997) 379–391.
- [Eme11] M. Emerton, Local–global compatibility in the p-adic Langlands programme for GL_2/\mathbb{Q} , Preprint, 2011.
- [EH14] M. Emerton, D. Helm, The local Langlands correspondence for GL_n in families, *Ann. Sci. Éc. Norm. Supér.* (4) 47 (4) (2014) 655–722.
- [Fli82] Y. Flicker, Stable and labile base change for $U(2)$, *Duke Math. J.* 49 (3) (1982) 691–729.
- [Hel13] D. Helm, Whittaker models and the integral Bernstein center for $GL_n(F)$, Preprint, 2013.
- [JS83] H. Jacquet, J. Shalika, The Whittaker models of induced representations, *Pacific J. Math.* 109 (1) (1983) 107–120.
- [KSW] T. Kaletha, S.W. Shin, P.-J. White, Endoscopic classification of representations: inner forms of unitary groups, Preprint, arXiv:1409.3731.
- [Kap58] I. Kaplansky, Projective modules, *Ann. of Math.* (2) 68 (1958) 372–377.
- [Liv89] R. Livne, On the conductors of mod l Galois representations coming from modular forms, *J. Number Theory* 31 (2) (1989) 133–141.
- [Man01] R. Mann, Local level-raising for $GL(n)$, Ph.D. Thesis, Harvard University, 2001.
- [Min11] A. Minguez, Unramified representations of unitary groups, in: *On the Stabilization of the Trace Formula*, in: *Stab. Trace Formula Shimura Var. Arith. Appl.*, vol. 1, Int. Press, Somerville, MA, 2011, pp. 389–410.
- [Mok15] C.P. Mok, Endoscopic classification of representations of quasi-split unitary groups, *Mem. Amer. Math. Soc.* 235 (1108) (2015).
- [PR94] V. Platonov, A. Rapinchuk, *Algebraic Groups and Number Theory*, translated from the 1991 Russian original by Rachel Rowen, Pure and Applied Mathematics, vol. 139, Academic Press, Inc., Boston, MA, 1994.

- [Ram15] D. Ramakrishnan, A mild Tchebotarev theorem for $GL(n)$, *J. Number Theory* 146 (2015) 519–533.
- [Rog90] J. Rogawski, *Automorphic Representations of Unitary Groups in Three Variables*, *Annals of Mathematics Studies*, vol. 123, Princeton University Press, Princeton, NJ, 1990.
- [Tay08] R. Taylor, Automorphy for some l -adic lifts of automorphic mod l Galois representations. II, *Publ. Math. Inst. Hautes Études Sci.* 108 (2008) 183–239.
- [Tho12] J. Thorne, On the automorphy of l -adic Galois representations with small residual image, with an appendix by Robert Guralnick, Florian Herzig, Richard Taylor and Thorne, *J. Inst. Math. Jussieu* 11 (4) (2012) 855–920.
- [Tho14] J. Thorne, Raising the level for GL_n , *Forum Math. Sigma* 2 (2014) e16, 35 pp.
- [Vig96] M.-F. Vignéras, Représentations l -modulaires d'un groupe réductif p -adique avec $l \neq p$, *Progress in Mathematics*, vol. 137, Birkhäuser Boston, Inc., Boston, MA, 1996.
- [Vig01] M.-F. Vignéras, Correspondance de Langlands semi-simple pour $GL(n, F)$ modulo $l \neq p$, *Invent. Math.* 144 (1) (2001) 177–223.
- [Vig04] M.-F. Vigneras, On highest Whittaker models and integral structures, in: *Contributions to Automorphic Forms, Geometry, and Number Theory*, Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 773–801.