Strong local-global compatibility in the p-adic Langlands program for $U(2)$

Przemyslaw Chojecki   Claus Sorensen

ABSTRACT – For certain mod p Galois representations $\bar{\rho}$, arising from modular forms on definite unitary groups in two variables, we express the $\bar{\rho}$-part of completed cohomology $\hat{H}^0_{\bar{\rho}}$ (away from $\Sigma = \Sigma_p \cup \Sigma_0$) as a tensor product $\Pi_p \otimes \Pi_{\Sigma_0}$. Here Π_p is attached to the universal deformation ρ^{univ} via the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, and Π_{Σ_0} is given by the local Langlands correspondence in families, of Emerton and Helm. [1] [2]

1. Introduction

We continue the study from [CS15] of the completed cohomology $\hat{H}^0$ of definite unitary groups in two variables; or more precisely, the piece associated with a modular mod p Galois representation $\bar{\rho}$, which is irreducible at p. The goal of this paper is to explicitly describe $\hat{H}^0_{\bar{\rho}}$ as a tensor product of representations arising from p-adic local Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$ and local Langlands in families (defined away from p) respectively.

Our unitary groups are defined relative to a fixed CM-extension $\mathcal{K}/F$. Thus $U = U(D, \star)_{/F}$, where D is a quaternion $\mathcal{K}$-algebra with involution $\star$. We will always assume $G = \mathrm{Res}_{F/\mathbb{Q}}(U)$ is compact at infinity, and split at p. Here $p > 2$ is a fixed prime, which splits in $\mathcal{K}$. Thus $G(\mathbb{R})$ is a product of copies of $U(2)$, and $G(\mathbb{Q}_p)$ is a product of copies of $\mathrm{GL}_2(\mathbb{Q}_p)$. Let Σ_p denote the set of places v of F above p. Choose a finite set of banal places Σ_0, disjoint from Σ_p, and

Przemyslaw Chojecki, Mathematical Institute University of Oxford Andrew Wiles Building Radcliffe Observatory Quarter Woodstock Road Oxford OX2 6GG England

E-mail: chojecki@maths.ox.ac.uk

Claus Sorensen, Department of Mathematics, UCSD 9500 Gilman Dr. 0112 La Jolla, CA 92093-0112 USA

E-mail: csorensen@ucsd.edu

let $\Sigma = \Sigma_0 \cup \Sigma_p$. We assume U splits at all these places, and for each $v \in \Sigma$ we once and for all choose a divisor $\tilde{v}|v$ in $\mathcal{K}$. Via this selection, we identify $U(F_v) \simeq \mathrm{GL}_2(\mathcal{K}_{\tilde{v}})$ (well-defined up to conjugacy).

Throughout, we fix a large enough coefficient field; a finite extension $E/\mathbb{Q}_p$, with integers $\mathcal{O} = \mathcal{O}_E$, uniformizer $\varpi = \varpi_E$, and residue field $k = k_E$. We start with an absolutely irreducible mod p Galois representation,

$$\bar{\rho} : \Gamma_\mathcal{K} = \mathrm{Gal}(\bar{\mathcal{K}}/\mathcal{K}) \longrightarrow \mathrm{GL}_2(k),$$

which we assume to be modular. That is, $\bar{\rho} \simeq \bar{\rho}_\pi$, for some automorphic representation π of $G(\mathbb{A})$ of weight V and level $K = \prod_v K_v$. Throughout, we impose the following conditions on $\bar{\rho}$.

- $\bar{\rho}(\Gamma_{F(\zeta_p)})$ is big (in a formal sense, cf. [CHT08]).

- $\bar{\rho}|_{\Gamma_{\mathcal{K}_{\tilde{v}}}}$ is irreducible, for all $v|p$.

- $K = K_{max} \subset G(\mathbb{A}_f)$ is maximal, and hyperspecial outside Σ.

- The weights of V are in the range $[0, p-2)$ at each infinite place.

All but the second one can be traced back to the hypotheses in [CHT08], which are needed to obtain modularity lifting theorems via the usual Taylor-Wiles patching machinery.

Let $\mathfrak{m} = \mathfrak{m}_{\bar{\rho}}$ be the associated maximal ideal in the Hecke algebra. Here we are a bit imprecise about which Hecke algebra we use; we pull-back $\mathfrak{m}$ via surjections of Hecke algebras without mention. Usually we view $\mathfrak{m}$ as an ideal of $\mathbb{T}_\Sigma$, the quotient of the polynomial algebra $\mathbb{T}_\Sigma^{abs}$, in variables $T_w^{(1)}$ and $T_w^{(2)}$, which acts faithfully on the completed cohomology,

$$\hat{H}^0(K^\Sigma)_\mathcal{O} = \varinjlim_{K_{\Sigma_0}} \hat{H}^0(K_{\Sigma_0} K^\Sigma)_\mathcal{O}, \quad \hat{H}^0(K^p)_\mathcal{O} = (\varinjlim_{K_p} H^0(K_p K^p)_\mathcal{O})^\wedge.$$

(These spaces, and their topologies, will be discussed in more detail in section 2.1 below.) We let $\rho_\mathfrak{m} : \Gamma_\mathcal{K} \to \mathrm{GL}_2(\mathbb{T}_{\Sigma,\mathfrak{m}})$ be the universal modular deformation of $\bar{\rho}$ over the (big) localized Hecke algebra $\mathbb{T}_{\Sigma,\mathfrak{m}}$, which acts faithfully on $\hat{H}^0(K^\Sigma)_{\mathcal{O},\mathfrak{m}}$. This is our main object of interest. Our goal in this paper is to factor the $U(F_\Sigma)$-action as a tensor product of a $G(\mathbb{Q}_p)$-representation Π_p, and a $U(F_{\Sigma_0})$-representation Π_{Σ_0} (both over $\mathbb{T}_{\Sigma,\mathfrak{m}}$).

At each place $v|p$, consider the restriction $\bar{\rho}|_{\Gamma_{\mathcal{K}_{\tilde{v}}}}$, which is irreducible by assumption – and in particular in the image of Colmez's Montreal functor. (Recall that $\mathcal{K}_{\tilde{v}} = \mathbb{Q}_p$, so this makes sense.) Thus $\bar{\rho}|_{\Gamma_{\mathcal{K}_{\tilde{v}}}} \leftrightarrow \bar{\pi}_{\tilde{v}}$ for some k-representation $\bar{\pi}_{\tilde{v}}$ of $\mathrm{GL}_2(\mathcal{K}_{\tilde{v}})$. We refer to section 3.4 in [CS15] for a review of the salient facts. The correspondence pairs deformations, so $\rho_{\mathfrak{m},\tilde{v}}$ corresponds to some deformation $\Pi_{\mathfrak{m},\tilde{v}}$ of $\bar{\pi}_{\tilde{v}}$ over $\mathbb{T}_{\Sigma,\mathfrak{m}}$. We let $\Pi_p = \hat{\otimes}_{v|p}\Pi_{\mathfrak{m},\tilde{v}}$, viewed as a representation of $G(\mathbb{Q}_p)$ over $\mathbb{T}_{\Sigma,\mathfrak{m}}$. Here $\hat{\otimes}$ denotes the ϖ-adically completed tensor product.

At each place $v \in \Sigma_0$, the restriction $\rho_{\mathfrak{m},\tilde{v}} : \Gamma_{\mathcal{K}_{\tilde{v}}} \to \mathrm{GL}_2(\mathbb{T}_{\Sigma,\mathfrak{m}})$ yields a corresponding representation $\Pi_{\mathfrak{m},\tilde{v}}$ of $\mathrm{GL}_2(\mathcal{K}_{\tilde{v}})$ over $\mathbb{T}_{\Sigma,\mathfrak{m}}$ via local Langlands correspondence in families – as developed by Emerton and Helm in [EH14] and [He13] (see section 2.3 for a quick summary of its defining properties). The normalization here is that $\Pi_{\mathfrak{m},\tilde{v}}$ is *cotorsion*-free (as opposed to torsion-free), and thus its smooth *dual* is the representation which interpolates the local Langlands correspondence, in the natural sense. We let $\Pi_{\Sigma_0} = (\otimes_{v \in \Sigma_0} \Pi_{\mathfrak{m},\tilde{v}})_{ctf}$ be the largest cotorsion-free submodule of the tensor product (cf. Proposition 6.4.2 in [EH14]). As always, via the selection of divisors $\{\tilde{v}\}_{v \in \Sigma_0}$, we view Π_{Σ_0} as a representation of $U(F_{\Sigma_0})$.

The main result we present in this note is the following.

Theorem 1. *With notation and assumptions as above, there is a $G(\mathbb{Q}_p) \times U(F_{\Sigma_0})$-equivariant isomorphism of $\mathbb{T}_{\Sigma,\mathfrak{m}}$-modules,*

$$\Pi_p \overset{\wedge}{\otimes}_{\mathbb{T}_{\Sigma,\mathfrak{m}}} \Pi_{\Sigma_0} \overset{\sim}{\longrightarrow} \widehat{H}^0(K^\Sigma)_{\mathcal{O},\mathfrak{m}}.$$

(Here $\overset{\wedge}{\otimes}_{\mathbb{T}_{\Sigma,\mathfrak{m}}}$ denotes Emerton's completed tensor product; see section 3 below.)

This result is formally similar to the (main) local-global compatibility conjecture 6.1.6 in [Em11] for $\hat{H}^1$ of the tower of modular curves which Emerton proves for p-distinguished $\bar{\rho}$ in Theorem 6.2.13 of loc. cit. (i.e., for $\bar{\rho}$ such that $\bar{\rho}|_{\Gamma_{\mathbb{Q}_p}}^{ss}$ is not of the form $\chi \oplus \chi$).

In our setup, the arithmetic manifolds are finite sets, and we only have cohomology in degree zero. Furthermore, $\hat{H}^0$ is much simpler. It can be realized as continuous functions on a compact set, and there is no Galois-action. That being said, our approach relies *heavily* on that of Emerton, and for the most part our proof is an almost formal application of his general results on coadmissible modules in his Appendix C. However, there seems to be one step in [Em11] which does not straightforwardly adapt to the unitary setting. Namely how Ihara's lemma ensures that a certain candidate representation for $(\Pi_{\Sigma_0}/\varpi\Pi_{\Sigma_0})[\mathfrak{m}]$ is indeed *essentially AIG* (a key notion from [EH14]). In the companion paper [Sor14], this was taken care of – even for $U(n)$, where Ihara's lemma is still a big open problem for $n > 2$, although significant progress has been made very recently by Pascal Boyer, cf. [Boy15]. This led to the writing of this note.

Note that in the special case where $\Sigma_0 = \varnothing$, Theorem 1 realizes Π_p globally as $\widehat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}}$, with K^p being a product of hyperspecials away from p. This suggests that the elusive p-adic local Langlands correspondence for n-dimensional ρ should be closely related to the analogous piece of completed cohomology for $U(n)$ (at least when $\bar{\rho}$ satisfies our criteria).

2. Notation and recollections

2.1 – *Definite unitary groups and their Hecke algebras*

We briefly recall the setting from our paper [CS15].

The unitary groups we work with are defined relative to a CM-extension $\mathcal{K}/F$. More precisely, we let D be a quaternion algebra over $\mathcal{K}$, endowed with an F-linear anti-involution $\star$ of the second kind ($\star|_{\mathcal{K}} = c$). This pair defines a unitary group $U = U(D, \star)_{/F}$, an inner form of $\mathrm{GL}(2)$ over $\mathcal{K}$. Indeed, $U \times_F \mathcal{K} \simeq D^\times$. We consider the restriction of scalars, $G = \mathrm{Res}_{F/\mathbb{Q}}(U)$. We always assume $G(\mathbb{R})$ is *compact* and that D splits at all places of $\mathcal{K}$ which lie above a fixed prime number p, which splits in $\mathcal{K}$.

For each place $v|p$ of F, we choose a place $\tilde{v}|v$ of $\mathcal{K}$ above it (note that $\mathcal{K}_{\tilde{v}} = \mathbb{Q}_p$ canonically). Using this selection of places $\{\tilde{v}\}_{\Sigma_*}$ (where $* = 0$ or p), we have identifications

$$G(\mathbb{R}) \xrightarrow{\sim} U(2)^{\mathrm{Hom}(F, \mathbb{R})}, \quad G(\mathbb{Q}_p) \xrightarrow{\sim} \prod_{v|p} \mathrm{GL}_2(\mathcal{K}_{\tilde{v}}).$$

Of course, $\mathcal{K}_{\tilde{v}} = \mathbb{Q}_p$, but we wish to incorporate $\tilde{v}$ in our notation to emphasize how our identification depends on this choice.

Throughout, we fix a finite set Σ_0 of finite places of F, none of which lie above p. We assume each $v \in \Sigma_0$ splits in $\mathcal{K}$, and that D splits at every place above such a v. Let $\Sigma = \Sigma_0 \cup \Sigma_p$, where $\Sigma_p = \{v|p\}$ consists of all places above p. We enlarge our selection $\{\tilde{v}\}$, and choose a divisor $\tilde{v}$ of each $v \in \Sigma$.

For any compact open subgroup $K \subset G(\mathbb{A}_f)$ we consider the arithmetic manifold

$$Y(K) = G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K$$

which is just a finite set. For any commutative ring A we denote by $H^0(K)_A$ the set of functions $Y(K) \to A$. For each tame level $K^p \subset G(\mathbb{A}_f^p)$, one defines (following Emerton, cf. [Em06]) the *completed* cohomology, with coefficients in some p-adic ring of integers $\mathcal{O} = \mathcal{O}_E$, having uniformizer ϖ,

$$\hat{H}^0(K^p)_{\mathcal{O}} := \varprojlim_s \varinjlim_{K_p} H^0(K_p K^p)_{\mathcal{O}/\varpi^s \mathcal{O}},$$

where the direct limit runs over compact open subgroups $K_p \subset G(\mathbb{Q}_p)$, and $s > 0$. Unwinding the definitions one easily identifies $\hat{H}^0(K^p)_{\mathcal{O}}$ with the module of all continuous functions $f : G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K^p \to \mathcal{O}$. Since the domain $\varprojlim_{K_p} Y(K_p K^p)$ is compact (indeed profinite) the supremum norm $\|f\|$ is finite. Thus $\hat{H}^0(K^p)_{\mathcal{O}}$ is the unit ball in the p-adic Banach space $\hat{H}^0(K^p)_E$ of all continuous E-valued functions f, on which $U(F_{\Sigma_p}) \simeq \prod_{v|p} \mathrm{GL}_2(\mathcal{K}_{\tilde{v}})$ acts unitarily. Next, we factor $K^p = K_{\Sigma_0} K^\Sigma$ and let K_{Σ_0} shrink to the identity. We define

$$\hat{H}^0(K^\Sigma)_{\mathcal{O}} := \hat{H}^0_{\mathcal{O}, \Sigma} = \varinjlim_{K_{\Sigma_0}} \hat{H}^0(K_{\Sigma_0} K^\Sigma)_{\mathcal{O}},$$

which by definition carries a smooth $U(F_{\Sigma_0})$-action. Here we endow $\hat{H}^0(K^\Sigma)_{\mathcal{O}}$ with the direct limit topology. Alternatively, as above, $\hat{H}^0(K^\Sigma)_{\mathcal{O}}$ can be thought of as the continuous functions $f : G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K^\Sigma \to \mathcal{O}$ which are smooth at the places in Σ_0. This guarantees that the supremum norm $\|f\|$ is finite, and the direct limit topology on $\hat{H}^0(K^\Sigma)_{\mathcal{O}}$ coincides with the norm topology.

In what follows, we will always take K^Σ to be a product of hyperspecial maximal compact subgroups $K_v = U(\mathcal{O}_v)$, for $v \notin \Sigma$. Note that there are *two* conjugacy classes of hyperspecials when $U_{/F_v}$ is a p-adic unitary group in an even number of variables, cf. [Min11].

We denote by $\mathbb{T}_\Sigma^{abs}$ the polynomial $\mathcal{O}$-algebra in the variables $T_w^{(1)}$ and $(T_w^{(2)})^{\pm 1}$, one for each place w of $\mathcal{K}$ lying above $v = w|_F \notin \Sigma$, which splits in $\mathcal{K}$. (Here the superscript *abs* stands for abstract as a reminder that this is a Hecke algebra of infinite type.) Once and for all, we choose an isomorphism $\iota_w : U(F_v) \xrightarrow{\sim} \mathrm{GL}_2(\mathcal{K}_w)$ in such a way that it identifies $U(\mathcal{O}_v) \simeq \mathrm{GL}_2(\mathcal{O}_{\mathcal{K}_w})$.

The algebra $\mathbb{T}_\Sigma^{abs}$ acts naturally on the module $\hat{H}^0_{\mathcal{O},\Sigma}$. Explicitly, $T_w^{(j)}$ acts via the usual double coset operator

$$\iota_w^{-1}\left[\mathrm{GL}_2(\mathcal{O}_{\mathcal{K}_w}) \begin{pmatrix} 1_{2-j} & \\ & \varpi_w 1_j \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{\mathcal{K}_w}) \right],$$

where ϖ_w is a choice of uniformizer in $\mathcal{O}_{\mathcal{K}_w}$. We let $\mathbb{T}_\Sigma$ denote the quotient of $\mathbb{T}_\Sigma^{abs}$ which acts faithfully on $\hat{H}^0_{\mathcal{O},\Sigma}$, and we may view $T_w^{(j)}$ as an operator on $\hat{H}^0_{\mathcal{O},\Sigma}$ or a variable in $\mathbb{T}_\Sigma^{abs}$, interchangeably. For a compact open subgroup K_{Σ_0} as above we let $\mathbb{T}_\Sigma(K_{\Sigma_0})$ be the quotient of $\mathbb{T}_\Sigma$ cut out by the submodule of K_{Σ_0}-invariants, $\hat{H}^0(K_{\Sigma_0} K^\Sigma)_{\mathcal{O}}$.

For each maximal ideal $\mathfrak{m} \subset \mathbb{T}_\Sigma^{abs}$, say with residue field $k = \mathbb{T}_\Sigma^{abs}/\mathfrak{m}$, which is a finite extension of $k_E \supset \mathbb{F}_p$, we may consider the localization $\hat{H}^0(K^\Sigma)_{\mathcal{O},\mathfrak{m}}$, which will be our main object of study. Those maximal ideals we will eventually look at arise from a modular Galois representation, $\bar{\rho} : \Gamma_\mathcal{K} \to \mathrm{GL}_2(k_E)$, which is unramified outside Σ, as follows: $\mathfrak{m} = \mathfrak{m}_{\bar{\rho}} \subset \mathbb{T}_\Sigma^{abs}$ is generated by $\varpi = \varpi_E$ and all the elements

$$((-1)^j \mathrm{Norm}(w)^{j(j-1)/2} T_w^{(j)} - a_w^{(j)})_{j,w}$$

where $j \in \{1, 2\}$, and w ranges over places of $\mathcal{K}$ lying above split $v \notin \Sigma$. Here the $a_w^{(j)} \in \mathcal{O}$ are lifts of $\bar{a}_w^{(j)}$, where $X^2 + \bar{a}_w^{(1)} X + \bar{a}_w^{(2)}$ is the characteristic polynomial of $\bar{\rho}(\mathrm{Frob}_w)$. We will occasionally write $\bar{\rho} = \bar{\rho}_\mathfrak{m}$.

2.2 – *Essentially AIG representations*

Let $L/\mathbb{Q}_\ell$ be a finite extension, for some prime $\ell \neq p$. Eventually this L will be one of the completions $F_v \simeq \mathcal{K}_{\tilde{v}}$, for $v \in \Sigma_0$. We say (after 3.2.1 in [EH14]) that a representation V of $\mathrm{GL}_2(L)$, with coefficients in E or k_E, is *essentially AIG* (which stands for absolutely irreducible and generic) if the conditions below are

fulfilled. We stress that *generic* is equivalent to *infinite-dimensional* here, as we are working with GL_2 and not higher rank general linear groups.

(1) The $GL_2(L)$-socle $soc(V)$ is absolutely irreducible and generic.
(2) The quotient $V/soc(V)$ contains no generic Jordan-Holder factors.
(3) The representation V is the sum (or equivalently, the union) of its finite length submodules.

Observation 1. *If V is essentially AIG, then so is any non-zero subrepresentation $U \subset V$. Moreover, $soc(U) = soc(V)$. We will use this simple observation multiple times later on (for example in the proof of Corollary 1 below).*

We will use the term *essentially AIG* also for representations of $\prod_i GL_2(L_i)$, for a finite collection of fields $L_i/\mathbb{Q}_{\ell_i}$ (possibly of varying residue characteristics ℓ_i), by which we mean a tensor product $\otimes_i V_i$ of essentially AIG representations V_i of each $GL_2(L_i)$. The context below will be that of representations of $U(F_{\Sigma_0})$, which we always identify with $\prod_{v \in \Sigma_0} GL_2(\mathcal{K}_{\tilde{v}})$ via the selection of places $\{\tilde{v}\}_{v \in \Sigma}$.

2.3 – The local Langlands correspondence in families

One of the main theorems of [EH14] is a characterization of the local Langlands correspondence for $GL(n)$ in families. The proof that it actually does exist (at least in the banal case, where p is prime to the pro-order of the groups in question) is given in the recent work [He13]. Local Langlands in families puts the classical local Langlands correspondence in a family over a reduced complete Noetherian local $\mathcal{O}$-algebra A. Emerton and Helm give a short list of desiderata, which we verify for a candidate-representation occuring naturally in the course of the proof of our main result. For convenience, we recall the criteria here.

Theorem 2. *[The local Langlands correspondence in families] Let A be a reduced complete Noetherian local $\mathcal{O}$-algebra. Let Σ_0 be the set of places of F from above. Suppose that for each $v \in \Sigma_0$ we are given a representation $\rho_{\tilde{v}} : \Gamma_{\mathcal{K}_{\tilde{v}}} \to GL_2(A)$. Then there is at most one (up to isomorphism) coadmissible [3] smooth A-representation $\Pi_{\Sigma_0} = \pi(\{\rho_{\tilde{v}}\}_{v \in \Sigma_0})$ of $U(F_{\Sigma_0}) \simeq \prod_{v \in \Sigma_0} GL_2(\mathcal{K}_{\tilde{v}})$ such that:*

(1) Π_{Σ_0} is cotorsion free over A (that is, the smooth dual is torsion free – which in turn means nonzero elements can only be annihilated by zero-divisors).

(2) $(\Pi_{\Sigma_0}/\varpi\Pi_{\Sigma_0})[\mathfrak{m}]$ is an essentially AIG representation (over $k = A/\mathfrak{m}$).

(3) There is a Zariski dense subset of closed points $\mathcal{S} \subset SpecA[1/p]$, such that for each point $\mathfrak{p} \in \mathcal{S}$, there exists an isomorphism:

$$(\Pi_{\Sigma_0} \otimes_{\mathcal{O}} E)[\mathfrak{p}] \simeq \otimes_{v \in \Sigma_0} \pi(\rho_{\tilde{v}} \otimes_A \kappa(\mathfrak{p}))$$

where $\kappa(\mathfrak{p})$ is the fraction field of $A/\mathfrak{p}$, and $\pi(\rho_{\tilde{v}} \otimes_A \kappa(\mathfrak{p}))$ is the representation of $GL_2(\mathcal{K}_{\tilde{v}})$ associated to $\rho_{\tilde{v}}(\mathfrak{p}) := \rho_{\tilde{v}} \otimes_A \kappa(\mathfrak{p})$ via the generic local Langlands correspondence (of Breuil-Schneider [BS07]).

---

[3]i.e. its smooth A-dual is admissible.

PROOF. This is the content of Proposition 6.3.14 in [EH14], except that we have dualized everything – using Proposition C.5 and Lemma C.35 in [Em11] □

In this article we specialize the theorem to the following situation: We take $A = \mathbb{T}_{\Sigma,\mathfrak{m}}$, for some maximal ideal $\mathfrak{m} = \mathfrak{m}_{\bar{\rho}} \subset \mathbb{T}_\Sigma$, and the Galois input is $\rho_{\tilde{v}} = \rho_{\mathfrak{m},\tilde{v}} = \rho_{\mathfrak{m}}|_{\Gamma_{\mathcal{K}_{\tilde{v}}}}$, where $\rho_{\mathfrak{m}} : \Gamma_{\mathcal{K}} \to \mathrm{GL}_2(\mathbb{T}_{\Sigma,\mathfrak{m}})$ is the universal modular Σ-deformation of $\bar{\rho}$ over the big Hecke algebra $\mathbb{T}_{\Sigma,\mathfrak{m}}$. We write Π_{Σ_0} for the resulting representation. For any $\mathfrak{p} \in \mathrm{Spec}\,\mathbb{T}_{\Sigma,\mathfrak{m}}$ we will write

$$\Pi_{\Sigma_0}(\mathfrak{p}) := \kappa(\mathfrak{p}) \otimes_{\mathbb{T}_{\Sigma,\mathfrak{m}}} \Pi_{\Sigma_0}.$$

Similarly, for $v \in \Sigma_p$, let Π_v be the $\mathbb{T}_{\Sigma,\mathfrak{m}}$-representation of $\mathrm{GL}_2(\mathcal{K}_{\tilde{v}})$ associated with $\rho_{\mathfrak{m}}|_{\Gamma_{\mathcal{K}_{\tilde{v}}}}$ under the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ (of Berger, Breuil, Colmez, Emerton, Kisin, Paskunas and others). Keep in mind that $\mathcal{K}_{\tilde{v}} = \mathbb{Q}_p$, so this is well-defined. Let $\Pi_p = \otimes_{v|p}\Pi_v$ – viewed as a $\mathbb{T}_{\Sigma,\mathfrak{m}}$-representation of $G(\mathbb{Q}_p) \simeq \prod_{v|p} \mathrm{GL}_2(\mathcal{K}_{\tilde{v}})$. As above, we may specialize,

$$\Pi_p(\mathfrak{p}) := \kappa(\mathfrak{p}) \otimes_{\mathbb{T}_{\Sigma,\mathfrak{m}}} \Pi_p.$$

When $\mathfrak{p} = \mathfrak{m}$, we will write $\bar{\pi}_p$ instead of $\Pi_p(\mathfrak{m}) = \Pi_p/\mathfrak{m}\Pi_p$ (which is a smooth $G(\mathbb{Q}_p)$-representation over k).

3. The strategy of the proof of our main result

The overall argument is to relate Π_{Σ_0} to the module

$$X := \mathrm{Hom}_{\mathbb{T}_{\Sigma,\mathfrak{m}}[G(\mathbb{Q}_p)]}(\Pi_p, \widehat{H}^0_{\mathcal{O},\Sigma,\mathfrak{m}}),$$

of $\mathbb{T}_{\Sigma,\mathfrak{m}}[G(\mathbb{Q}_p)]$-linear continuous homomorphisms $\Pi_p \to \widehat{H}^0_{\mathcal{O},\Sigma,\mathfrak{m}}$. Here Π_p is given the $\mathfrak{m}$-adic topology, and $\widehat{H}^0_{\mathcal{O},\Sigma,\mathfrak{m}}$ is given the induced topology from $\widehat{H}^0_{\mathcal{O},\Sigma}$ (as described in section 2.1). Note that X is naturally a $\mathbb{T}_{\Sigma,\mathfrak{m}}$-module, with a smooth action of $U(F_{\Sigma_0})$, and there is a natural evaluation map $ev_X : \Pi_p \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\Sigma,\mathfrak{m}}} X \to \widehat{H}^0_{\mathcal{O},\Sigma,\mathfrak{m}}$, which is equivariant for the $U(F_\Sigma)$-action. The curly-wedge tensor product $\overset{\curlywedge}{\otimes}_{\mathbb{T}_{\Sigma,\mathfrak{m}}}$ is Emerton's notation (see Definition C.43 in [Em11]); it is the direct limit of ϖ-adically completed tensor products,

$$\Pi_p \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\Sigma,\mathfrak{m}}} X := \varinjlim_{K_{\Sigma_0}} \Pi_p \hat{\otimes}_{\mathbb{T}_{\Sigma,\mathfrak{m}}} X^{K_{\Sigma_0}},$$

over compact open subgroups $K_{\Sigma_0} \subset U(F_{\Sigma_0})$.

For every coadmissible $\mathbb{T}_{\Sigma,\mathfrak{m}}[U(F_{\Sigma_0})]$-submodule $Y \subset X$, we let ev_Y denote the restriction of ev_X,

$$ev_Y : \Pi_p \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\Sigma,\mathfrak{m}}} Y \hookrightarrow \Pi_p \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\Sigma,\mathfrak{m}}} X \longrightarrow \widehat{H}^0_{\mathcal{O},\Sigma,\mathfrak{m}}.$$

(The fact that the initial map is an embedding is Lemma C.48 in [Em11].) In what follows, we use the map $ev_{Y,E}$ into $\widehat{H}^0_{E,\Sigma,\mathfrak{m}}$, obtained from ev_Y by tensoring $(-)\otimes_{\mathcal{O}} E$. For each prime $\mathfrak{p} \subset \mathbb{T}_{\Sigma,\mathfrak{m}}[1/p]$, we take the $\mathfrak{p}$-torsion and look at the induced map

$$ev_{Y,E}[\mathfrak{p}] : \Pi_p(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} (Y \otimes_{\mathcal{O}} E)[\mathfrak{p}] \longrightarrow \widehat{H}^0_{E,\Sigma,\mathfrak{m}}[\mathfrak{p}].$$

On the other hand, we may first reduce ev_Y modulo ϖ, and then take the $\mathfrak{m}$-torsion. By Lemma C.45 in [Em11] this gives rise to a map

$$ev_{Y,k}[\mathfrak{m}] : \bar{\pi}_p \otimes_k (Y/\varpi Y)[\mathfrak{m}] \longrightarrow H^0_{k,\Sigma,\mathfrak{m}}[\mathfrak{m}],$$

where $H^0_{k,\Sigma}$ is the space of all k-valued K^Σ-invariant functions on $G(\mathbb{Q})\backslash G(\mathbb{A}_f)$.

In view of these definitions, our main result can be restated as saying there exists a coadmissible $\mathbb{T}_{\Sigma,\mathfrak{m}}[U(F_{\Sigma_0})]$-submodule $Y \subset X$ such that ev_Y is an isomorphism and $Y \simeq \Pi_{\Sigma_0}$. To carry this out we need two results whose proofs are given in the following sections, which form the technical core of the argument. We recall that $X_{ctf} \subset X$ denotes the maximal cotorsion free coadmissible submodule of X.

Proposition 1. *Let $Y \subset X$ be a coadmissible $\mathbb{T}_{\Sigma,\mathfrak{m}}[U(F_{\Sigma_0})]$-module. Then the following two conditions are equivalent:*

(1) The map ev_Y is an isomorphism.

(2) Y is a faithful $\mathbb{T}_{\Sigma,\mathfrak{m}}$-module, and the map $ev_{Y,k}[\mathfrak{m}]$ is injective.

Moreover, if Y satisfies these conditions, then $Y = X_{ctf}$.

We will combine this result with:

Proposition 2. *If $ev_{X_{ctf}}$ is an isomorphism, then $X_{ctf} \simeq \Pi_{\Sigma_0}$.*

Before proving our main Theorem 1, granting the above Propositions, let us mention one elementary lemma of Emerton's (Lemma 6.4.15, [Em11]). We recall its proof for completeness.

Lemma 1. *Suppose that $\bar{\pi}_1$ and $\bar{\pi}_2$ are smooth representations of G over k, U is a vector space over k, and $f : \bar{\pi}_1 \otimes_k U \to \bar{\pi}_2$ is a G-equivariant k-linear map (the G-action on the source is defined via its action on $\bar{\pi}_1$). Assume that $\bar{\pi}_1$ is admissible and the G-socle $soc(\bar{\pi}_1)$ of $\bar{\pi}_1$ is multiplicity free. If for every non-zero element $u \in U$, the map $\bar{\pi}_1 \simeq \bar{\pi}_1 \otimes_k (k \cdot u) \to \bar{\pi}_2$ induced by f is an embedding, then f itself is an embedding.*

PROOF. Write $soc(\bar{\pi}_1) = \oplus_{i=1}^s \bar{\pi}_{1,i}$ where $\bar{\pi}_{1,i}$ are pair-wise non-isomorphic irreducible admissible smooth G-representations. We have the isomorphism

$$\bigoplus_{i=1}^s \bar{\pi}_{1,i} \otimes_k U \simeq soc(\bar{\pi}_1) \otimes_k U \simeq soc(\bar{\pi}_1 \otimes_k U)$$

If f has a non-zero kernel, then this kernel has a non-zero socle, hence it has a non-zero intersection with $\bar{\pi}_{1,i} \otimes_k U$ for at least one i. Thus it contains $\bar{\pi}_{1,i} \otimes_k (k \cdot u)$ for some non-zero $u \in U$. But this means that the map $\bar{\pi}_1 \otimes_k (k \cdot u) \to \bar{\pi}_2$ is not injective, which contradicts our assumption. $\qquad \square$

PROOF OF THEOREM 1. By Proposition 2 it is enough to show that $ev_{X_{ctf}}$ is an isomorphism. By Proposition 1, this is equivalent to $ev_{X_{ctf},k}[\mathfrak{m}]$ being injective, and X_{ctf} being faithful over $\mathbb{T}_{\Sigma,\mathfrak{m}}$. First, in regards to the faithfulness: We will show later that $ev_{X,E}$ is certainly onto (this is part of Proposition 4 below). In particular, X is faithful. Now Proposition C.40 in [Em11] tells us that so is X_{ctf}. What remains is the injectivity of the evaluation map

$$\bar{\pi}_p \otimes_k (X_{ctf}/\varpi X_{ctf})[\mathfrak{m}] \longrightarrow H^0_{k,\Sigma,\mathfrak{m}}[\mathfrak{m}].$$

By Lemma 1 it suffices to show, for every nonzero $u \in (X_{ctf}/\varpi X_{ctf})[\mathfrak{m}]$, that the induced map $\bar{\pi}_p \to H^0_{k,\Sigma,\mathfrak{m}}[\mathfrak{m}]$ is injective. However, each such $G(\mathbb{Q}_p)$-equivariant map is either trivial or injective by our standing hypothesis on $\bar{\rho}$. Namely, by known properties of mod p local Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$, we know that $\bar{\pi}_p$ is irreducible since each restriction $\bar{\rho}|_{\Gamma_{\mathcal{K}_{\tilde{v}}}}$ is assumed to be irreducible for all $v|p$. $\quad \square$

4. Coadmissible Hecke modules

4.1 – Allowable points

Following standard terminology in the subject, we define *allowable points* to be those prime ideals $\mathfrak{p} \in \mathrm{Spec}\,\mathbb{T}_{\Sigma,\mathfrak{m}}[1/p]$ for which the specialization $\rho_{\mathfrak{m}}(\mathfrak{p})$ is irreducible and crystalline above p, with distinct Hodge-Tate weights.

We record the following lemma for later use.

Lemma 2. *If $\mathfrak{p} \in \mathrm{Spec}\,\mathbb{T}_{\Sigma,\mathfrak{m}}[1/p]$ is an allowable point, then there is a $U(F_{\Sigma_0})$-equivariant $\kappa(\mathfrak{p})$-linear isomorphism*

$$(X \otimes_{\mathcal{O}} E)[\mathfrak{p}] \simeq \otimes_{v \in \Sigma_0} \pi(\rho_{\mathfrak{m},\tilde{v}}(\mathfrak{p})) =: \pi_{\Sigma_0}(\rho_{\mathfrak{m}}(\mathfrak{p})).$$

Furthermore, the evaluation map induces an isomorphism

$$ev_{X,E}[\mathfrak{p}] : \Pi_p(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} (X \otimes_{\mathcal{O}} E)[\mathfrak{p}] \xrightarrow{\sim} (\widehat{H}^0_{E,\Sigma,\mathfrak{m}}[\mathfrak{p}]_{loc.alg.})^- =: M(\mathfrak{p})_E.$$

($M(\mathfrak{p})_E$ is the closure of the space of locally algebraic vectors in $\widehat{H}^0_{E,\Sigma,\mathfrak{m}}[\mathfrak{p}]$.)

PROOF. Since $\rho_{\mathfrak{m},\tilde{v}}(\mathfrak{p})$ is irreducible crystalline and regular, a fundamental result of Berger and Breuil ([BB10]) tells us that $\Pi_v(\mathfrak{p})$ is the universal unitary completion of the locally algebraic representation $BS(\rho_{\mathfrak{m},\tilde{v}}(\mathfrak{p}))$ associated with $\rho_{\mathfrak{m},\tilde{v}}(\mathfrak{p})$ by Breuil and Schneider. A fortiori, $\Pi_p(\mathfrak{p}) = \hat{\otimes}_{v|p}\Pi_v(\mathfrak{p})$ is the universal unitary completion of $BS(\mathfrak{p}) := \otimes_{v|p} BS(\rho_{\mathfrak{m},\tilde{v}}(\mathfrak{p}))$, which is a locally algebraic representation

of $G(\mathbb{Q}_p)$ over $\kappa(\mathfrak{p})$, which we factor as $\xi(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \pi(\mathfrak{p})$. Here the algebraic part $\xi(\mathfrak{p})$ records the Hodge-Tate weights, and the smooth part $\pi(\mathfrak{p})$ corresponds to the Weil-Deligne representations $WD(\rho_{\mathfrak{m},\tilde{v}}(\mathfrak{p}))$ under the classical local Langlands correspondence.

The locally algebraic vectors in $\widehat{H}^0_{E,\Sigma,\mathfrak{m}}[\mathfrak{p}]$ have a description in terms of automorphic representations π of G (see either Proposition 3.2.4 in [Em06], or Corollary 3.2 in [CS15]),

$$(1) \qquad \widehat{H}^0_{E,\Sigma,\mathfrak{m}}[\mathfrak{p}]_{loc.alg.} = \bigoplus_{\pi:\pi_\infty=\xi(\mathfrak{p})} BS(\mathfrak{p}) \otimes \pi_{\Sigma_0} \otimes (\pi_f^\Sigma)^{K^\Sigma}[\mathfrak{p}].$$

The sum ranges over automorphic π with $\pi_\infty = \xi(\mathfrak{p})$, which are unramified outside Σ, such that $\mathbb{T}_{\Sigma,\mathfrak{m}}$ acts on the line $(\pi_f^\Sigma)^{K^\Sigma}$ through $\mathbb{T}_{\Sigma,\mathfrak{m}}/\mathfrak{p}$. We have used multiplicity one for G (see [Mok15] and [KMSW] for instance), and local-global compatibility at p (due to Caraiani and others, cf. [BGGT] and [Car14]).

In fact there is a unique $\pi = \otimes_v \pi_v$ contributing to the right-hand side of (1). Indeed, by [Rog90] (Theorem 11.5.1, part (b), on p. 166) any such π admits a base change $\Pi := BC_{\mathcal{K}/F}(\pi)$ to $\mathrm{GL}_2(\mathbb{A}_{\mathcal{K}})$. Note that Π is a cuspidal automorphic representation (as opposed to just isobaric) since $\bar{\rho}_\pi \simeq \bar{\rho}$ is irreducible. Moreover, the local base change $\Pi_w = BC_{w|v}(\pi_v)$ is unramified for all $v \notin \Sigma$ as K_v is hyperspecial. When v splits in $\mathcal{K}$, and $w|v$ is a place dividing it, the spherical Hecke algebra acts on $\Pi_w^{\mathrm{GL}_2(\mathcal{O}_{\mathcal{K}_w})}$ via the eigensystem

$$\mathcal{H}(\mathrm{GL}_2(\mathcal{K}_w), \mathrm{GL}_2(\mathcal{O}_{\mathcal{K}_w})) \hookrightarrow \mathbb{T}^{abs}_\Sigma \twoheadrightarrow \mathbb{T}_\Sigma \to \mathbb{T}_{\Sigma,\mathfrak{m}}/\mathfrak{p} \simeq \mathcal{O},$$

which shows that Π_w is completely determined by $\mathfrak{p}$ for such v. At an inert place $v = w \notin \Sigma$, the actions of the spherical Hecke algebras on $\Pi_w^{\mathrm{GL}_2(\mathcal{O}_{\mathcal{K}_w})}$ and $\pi_v^{K_v}$ correspond to each other via the base change morphism

$$b_{w|v} : \mathcal{H}(\mathrm{GL}_2(\mathcal{K}_w), \mathrm{GL}_2(\mathcal{O}_{\mathcal{K}_w})) \longrightarrow \mathcal{H}(U(F_v), K_v).$$

(This is the morphism $b(\xi)$ from (2.7) on p. 11 of [Min11], with $\xi = BC$ as given in (4.1) on p. 16 of op. cit.) By Corollary 4.2 of [Min11] the morphism $b_{w|v}$ is surjective, which is to say the local base change map is injective: $BC_{w|v}(\pi_v) \simeq BC_{w|v}(\pi'_v) \Rightarrow \pi_v \simeq \pi'_v$. By Corollary B (of the main Theorem) in [Ram15] a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$ is determined by its components Π_w for all but finitely many degree one primes w. In particular $\Pi = BC_{\mathcal{K}/F}(\pi)$ is determined by $\mathfrak{p}$. By the above remark on injectivity at the inert places, π_v is then determined for all $v \notin \Sigma$. Finally, all places $v \in \Sigma$ split in $\mathcal{K}$ by assumption, so π_v is determined here as well.

As observed in the previous paragraph, the base change $BC_{\mathcal{K}/F}(\pi)$ is cuspidal hence (globally) generic. Therefore the factors of π_{Σ_0} are generic (so local Langlands and generic local Langlands coincide at the places in Σ_0). By local-global compatibility at the places in Σ_0, and the fact that π_v is unramified for $v \notin \Sigma$, the closure of $\widehat{H}^0_{E,\Sigma,\mathfrak{m}}[\mathfrak{p}]_{loc.alg.}$ in $\widehat{H}^0_{E,\Sigma,\mathfrak{m}}$ factors as

$$M(\mathfrak{p})_E := \overline{\widehat{H}^0_{E,\Sigma,\mathfrak{m}}[\mathfrak{p}]_{loc.alg.}} \simeq \Pi_p(\mathfrak{p}) \otimes_E \pi_{\Sigma_0}(\rho_\mathfrak{m}(\mathfrak{p})),$$

again using the fact that $\Pi_p(\mathfrak{p})$ is the completion of the irreducible (by Theorem 1, part 2, p. 126 in Prasad's Appendix to [ST01]) locally algebraic representation $BS(\mathfrak{p})$. Furthermore,

$$
\begin{aligned}
\mathrm{Hom}_{E[G(\mathbb{Q}_p)]}(\Pi_p(\mathfrak{p}), \widehat{H}^0_{E,\Sigma,\mathfrak{m}}[\mathfrak{p}]) &\simeq \mathrm{Hom}_{E[G(\mathbb{Q}_p)]}(BS(\mathfrak{p}), \widehat{H}^0_{E,\Sigma,\mathfrak{m}}[\mathfrak{p}]) \\
&\simeq \mathrm{Hom}_{E[G(\mathbb{Q}_p)]}(BS(\mathfrak{p}), \widehat{H}^0_{E,\Sigma,\mathfrak{m}}[\mathfrak{p}]_{loc.alg.}) \\
&\simeq \mathrm{Hom}_{E[G(\mathbb{Q}_p)]}(BS(\mathfrak{p}), M(\mathfrak{p})_E) \\
&\simeq \mathrm{Hom}_{E[G(\mathbb{Q}_p)]}(\Pi_p(\mathfrak{p}), M(\mathfrak{p})_E).
\end{aligned}
$$

This string of isomorphisms, together with the natural identification

$$
(X \otimes_{\mathcal{O}} E)[\mathfrak{p}] \simeq \mathrm{Hom}_{E[G(\mathbb{Q}_p)]}(\Pi_p(\mathfrak{p}), \widehat{H}^0_{E,\Sigma,\mathfrak{m}}[\mathfrak{p}]),
$$

implies

$$
(X \otimes_{\mathcal{O}} E)[\mathfrak{p}] \simeq \mathrm{Hom}_{E[G(\mathbb{Q}_p)]}(\Pi_p(\mathfrak{p}), M(\mathfrak{p})_E),
$$

from which both statements of the Lemma are easily deduced. $\qquad\square$

Of course, the way this will be used below, is by showing that the allowable points $\mathcal{S}_{al}$ are Zariski dense – so that we may take $\mathcal{S} = \mathcal{S}_{al}$ in part (3) of Theorem 2 which characterizes Π_{Σ_0}. The previous Lemma identifies $(X \otimes_{\mathcal{O}} E)[\mathfrak{p}]$ with $\Pi_{\Sigma_0} \otimes_{\mathbb{T}_{\Sigma,\mathfrak{m}}} \kappa(\mathfrak{p})$, for $\mathfrak{p} \in \mathcal{S}_{al}$. We will eventually use this in the proof of Proposition 2.

4.2 – *Proof of Proposition* 1 *following Emerton*

Recall that a submodule Y of a $\mathcal{O}$-torsion free module X is *saturated* if X/Y is $\mathcal{O}$-torsion free (see Definition C.6 in [Em11]). The proof of the following Proposition is based on that of Proposition 6.4.2, p. 82, in [Em11]. Here, for convenience, we will walk the reader through the relevant facts from Emerton's comprehensive Appendix C on coadmissible modules.

Proposition 3. *If* $Y \subset X$ *is a saturated coadmissible* $\mathbb{T}_{\Sigma,\mathfrak{m}}[U(F_{\Sigma_0})]$-*submodule, then the following conditions are equivalent:*

(a) *Y is a faithful $\mathbb{T}_{\Sigma,\mathfrak{m}}$-module.*

(b) *For each level $K_{\Sigma_0} \subset U(F_{\Sigma_0})$, the submodule of invariants $Y^{K_{\Sigma_0}}$ is a faithful $\mathbb{T}_{\Sigma}(K_{\Sigma_0})_{\mathfrak{m}}$-module.*

(c) *For each classical allowable closed point $\mathfrak{p} \in Spec\,\mathbb{T}_{\Sigma,\mathfrak{m}}[1/p]$, the inclusion $(Y \otimes_{\mathcal{O}} E)[\mathfrak{p}] \subset (X \otimes_{\mathcal{O}} E)[\mathfrak{p}]$ is an equality.*

(d) *For each level $K_{\Sigma_0} \subset U(\Sigma_0)$, and for each classical allowable closed point $\mathfrak{p} \in Spec\,\mathbb{T}_{\Sigma,\mathfrak{m}}[1/p]$, the inclusion $(Y^{K_{\Sigma_0}} \otimes_{\mathcal{O}} E)[\mathfrak{p}] \subset (X^{K_{\Sigma_0}} \otimes_{\mathcal{O}} E)[\mathfrak{p}]$ is an equality.*

(e) *$X_{ctf} \subset Y$.*

PROOF. As a $\mathbb{T}_{\Sigma,\mathfrak{m}}[U(F_{\Sigma_0})]$-module, X is cofinitely generated and coadmissible (as explained in [CS15], subsection 3.9). Hence, by Corollary C.34 from [Em11], since Y is saturated in X, we infer that Y is cofinitely generated over $\mathbb{T}_{\Sigma,\mathfrak{m}}$.

Observe from the outset that clearly (c) $\Leftrightarrow$ (d), and (b) $\Rightarrow$ (a).

(a) $\Leftrightarrow$ (c): Lemma 2 above shows that for any allowable point $\mathfrak{p} \subset \mathbb{T}_{\Sigma,\mathfrak{m}}[1/p]$ the $U(F_{\Sigma_0})$-representation $(X \otimes_{\mathcal{O}} E)[\mathfrak{p}]$ is in particular irreducible. Hence, for any allowable $\mathfrak{p}$, we have $(Y \otimes_{\mathcal{O}} E)[\mathfrak{p}] \neq 0$ if and only if $(Y \otimes_{\mathcal{O}} E)[\mathfrak{p}] = (X \otimes_{\mathcal{O}} E)[\mathfrak{p}]$. We know from our previous work [CS15] that the allowable points are Zariski dense in $\mathrm{Spec}\,\mathbb{T}_{\Sigma,\mathfrak{m}}[1/p]$ (see Corollary 3.11 of [CS15]), and since $\mathbb{T}_{\Sigma,\mathfrak{m}}$ is reduced, we conclude from Proposition C.36 of [Em11], that (a) and (c) are equivalent.

(d) $\Rightarrow$ (b): By the irreducibility remarks just made, $(Y^{K_{\Sigma_0}} \otimes_{\mathcal{O}} E)[\mathfrak{p}] \neq 0$ for each allowable point $\mathfrak{p} \in \mathrm{Spec}\,\mathbb{T}_{\Sigma}(K_{\Sigma_0})_{\mathfrak{m}}$. Referring to Proposition C.22 in [Em11], $\mathfrak{p}$ therefore lies in the cosupport of $Y^{K_{\Sigma_0}}$ which by the preceding remarks in loc. cit. means $\mathfrak{p}$ contains the annihilator ideal $\mathrm{Ann}_{\mathbb{T}_{\Sigma}(K_{\Sigma_0})_{\mathfrak{m}}}(Y^{K_{\Sigma_0}})$. By the aforementioned density, $\cap \mathfrak{p} = 0$. Consequently, the annihilator ideal is trivial.

Altogether this shows the first four conditions are equivalent. We end by showing they are all equivalent to (e). Suppose we can prove that X_{ctf} can be characterized as the unique saturated, coadmissible $\mathbb{T}_{\Sigma,\mathfrak{m}}[U(F_{\Sigma_0})]$-submodule of X which is both faithful and cotorsion free over $\mathbb{T}_{\Sigma,\mathfrak{m}}$. Note that, by Proposition C.40 of [Em11], Y_{ctf} is faithful if Y is. As a result, admitting the posited characterization of X_{ctf}, we find that (a) implies the equality $Y_{ctf} = X_{ctf}$, and in particular condition (e) that $X_{ctf} \subset Y$. In conclusion (a) $\Rightarrow$ (e). For the converse, observe that the claim implies X_{ctf} satisfies (a), the faithfulness. Thus, if $X_{ctf} \subset Y$, then Y automatically satisfies (a) as well. To summarize, (a) and (e) are equivalent.

We are left with proving the characterization of X_{ctf}. As X trivially satisfies (c) it also satisfies (a), and by Proposition C.40 of [Em11], we infer that X_{ctf} also satisfies (a). Hence X_{ctf} *is* a faithful $\mathbb{T}_{\Sigma,\mathfrak{m}}$-cotorsion free $\mathbb{T}_{\Sigma,\mathfrak{m}}[U(F_{\Sigma_0})]$-submodule of X. Let Y be any saturated, coadmissible $\mathbb{T}_{\Sigma,\mathfrak{m}}[U(F_{\Sigma_0})]$-submodule of X which is faithful and cotorsion free over $\mathbb{T}_{\Sigma,\mathfrak{m}}$. Obviously $Y \subset X_{ctf}$. Moreover, since we know (a) implies (c), we get an equality $(Y \otimes_{\mathcal{O}} E)[\mathfrak{p}] = (X_{ctf} \otimes_{\mathcal{O}} E)[\mathfrak{p}]$ for allowable points $\mathfrak{p}$ (which we already observed are Zariski dense in $\mathrm{Spec}\,\mathbb{T}_{\Sigma,\mathfrak{m}}$). Thus Proposition C.41 of [Em11] implies that $Y = X_{ctf}$, as desired. $\square$

This is one of the key technical ingredients which goes into the proof of Proposition 1. The other is the following variant of "Zariski density of crystalline points".

Proposition 4. *Let $Y \subset X$ be a saturated coadmissible $\mathbb{T}_{\Sigma,\mathfrak{m}}[U(F_{\Sigma_0})]$-submodule, which satisfies the equivalent conditions (a)–(e) of the previous Proposition 3. Then $ev_{Y,E}$ is surjective. Equivalently, for each sufficiently small level K_{Σ_0}, the map*

$$ev_{Y,E}(K_{\Sigma_0}) : \Pi_p \hat{\otimes}_{\mathbb{T}_{\Sigma,\mathfrak{m}}} (Y^{K_{\Sigma_0}} \otimes_{\mathcal{O}} E) \longrightarrow \hat{H}^0(K_{\Sigma_0} K^{\Sigma})_{E,\mathfrak{m}}$$

is surjective.

PROOF. The last equivalence of the Proposition follows by taking K_{Σ_0}-invariants or, conversely, passing to the limit over all levels K_{Σ_0}, using the very definition of the curly wedge tensor product $\overset{\wedge}{\otimes}_{\mathbb{T}_{\Sigma,\mathfrak{m}}}$ (cf. section 3).

We first prove that the image of $ev_{Y,E}$ contains $\oplus_{\mathfrak{p}\in\mathcal{C}}\widehat{H}^0_{E,\Sigma,\mathfrak{m}}[\mathfrak{p}]_{loc.alg.}$, where $\mathcal{C}$ is the set of classical allowable points $\mathfrak{p}\in\mathrm{Spec}\,\mathbb{T}_{\Sigma,\mathfrak{m}}[1/p]$. Proposition 3 part (c) gives us the equality $(Y\otimes_{\mathcal{O}}E)[\mathfrak{p}]=(X\otimes_{\mathcal{O}}E)[\mathfrak{p}]$. Therefore,

$$\mathrm{im}(ev_{Y,E}[\mathfrak{p}])=\mathrm{im}(ev_{X,E}[\mathfrak{p}])=M(\mathfrak{p})_E\supset\widehat{H}^0_{E,\Sigma,\mathfrak{m}}[\mathfrak{p}]_{loc.alg.},$$

by the second half of Lemma 2. Letting $\mathfrak{p}$ vary, we deduce our first claim that $\mathrm{im}(ev_{Y,E})$ contains $\oplus_{\mathfrak{p}\in\mathcal{C}}\widehat{H}^0_{E,\Sigma,\mathfrak{m}}[\mathfrak{p}]_{loc.alg.}$. However, the latter space is dense in $\widehat{H}^0_{E,\Sigma,\mathfrak{m}}$ by Zariski density of crystalline points (Proposition 3.10 of [CS15]). Thus $\mathrm{im}(ev_{Y,E})$ is dense in $\widehat{H}^0_{E,\Sigma,\mathfrak{m}}$. However, $\mathrm{im}(ev_{Y,E})$ is in fact closed, as we now explain.

Using Lemma 3.1.16 on p. 17 in [Em11], we first note that $\Pi_p\widehat{\otimes}_{\mathbb{T}_{\Sigma,\mathfrak{m}}}Y^{K_{\Sigma_0}}$ is a ϖ-adically admissible representation of $G(\mathbb{Q}_p)$ over $\mathbb{T}_{\Sigma,\mathfrak{m}}$ (acting through the first factor of the tensor product). Admissibility of $\widehat{H}^0(K_{\Sigma_0}K^{\Sigma})_{\mathcal{O},\mathfrak{m}}$ is immediate (by finiteness of the class number of G). We may then apply Proposition 3.1.3 of [Em11] to the $G(\mathbb{Q}_p)$-map $ev_Y(K_{\Sigma_0})$ between admissible representations. It states precisely that the induced map on Banach E-spaces, $ev_{Y,E}(K_{\Sigma_0})$, has *closed* image. We conclude that each $ev_{Y,E}(K_{\Sigma_0})$ must in fact be onto. $\qquad\square$

Having established these preliminary results, we can finally proceed to the proof of Proposition 1. We remark that the proof is very similar to that of Emerton's Theorem 6.4.9, p. 85, in [Em11].

PROOF OF PROPOSITION 1. Suppose first that ev_Y is an isomorphism. Since $\widehat{H}^0_{\mathcal{O},\Sigma,\mathfrak{m}}$ is a faithful $\mathbb{T}_{\Sigma,\mathfrak{m}}$-module, we see that Y must be a faithful $\mathbb{T}_{\Sigma,\mathfrak{m}}$-module. Reducing the map ev_Y modulo ϖ, and taking $\mathfrak{m}$-torsion, we find that $ev_{Y,k}[\mathfrak{m}]$ is at least injective (it may not be onto, a priori). This shows that $(1)\Rightarrow(2)$.

$(2)\Rightarrow(1)$: Conversely, if $ev_{Y,k}[\mathfrak{m}]$ is injective, then by Lemma C.46 of [Em11], we see that ev_Y is injective, with saturated image. Lemma C.52 of [Em11] then implies that Y must be saturated in X. If Y is furthermore faithful as a $\mathbb{T}_{\Sigma,\mathfrak{m}}$-module, then Proposition 4 shows that $ev_{Y,E}$ is surjective. It follows that ev_Y is in fact an isomorphism (again because $\mathrm{im}(ev_Y)$ is saturated in $\widehat{H}^0_{\mathcal{O},\Sigma,\mathfrak{m}}$).

Finally, assume ev_Y satisfies conditions (1) and (2). We must show $Y=X_{ctf}$. As we have already noted in the previous paragraph, it follows that Y is saturated in X, and so (e) of Proposition 3 shows one inclusion, $X_{ctf}\subset Y$. Since X_{ctf} is saturated in X, and therefore also in Y, we conclude that the induced map

$$(X_{ctf}/\varpi X_{ctf})[\mathfrak{m}]\longrightarrow(Y/\varpi Y)[\mathfrak{m}]$$

is an embedding (even before taking $\mathfrak{m}$-torsion), and thus that $ev_{X_{ctf},k}[\mathfrak{m}]$ is injective, since $ev_{Y,k}[\mathfrak{m}]$ is. The equivalence $(1)\Leftrightarrow(2)$ applied to X_{ctf} then tells us

that $ev_{X_{ctf}}$ is an isomorphism. In other words, the inclusion $X_{ctf} \subset Y$ induces an isomorphism

$$\Pi_p \overset{\wedge}{\otimes}_{\mathbb{T}_{\Sigma,\mathfrak{m}}} X_{ctf} \overset{\sim}{\longrightarrow} \Pi_p \overset{\wedge}{\otimes}_{\mathbb{T}_{\Sigma,\mathfrak{m}}} Y.$$

It follows from Lemma C.51 of [Em11] that the inclusion $X_{ctf} \subset Y$ is an equality as required. □

4.3 – *Ihara's lemma and the proof of Proposition* 2

Let $W^{gl}(\bar{\rho})$ be the set of global Serre weights of $\bar{\rho}$. By definition, those are the finite-dimensional irreducible $k[G(\mathbb{Z}_p)]$-representations V for which

$$\mathrm{Hom}_{k[G(\mathbb{Z}_p)]}(V, H^0_{k,\Sigma,\mathfrak{m}}[\mathfrak{m}]) \neq 0.$$

It is easy to see that this space can be identified with $S_V(G(\mathbb{Z}_p)K^\Sigma, k)_{\mathfrak{m}}[\mathfrak{m}]$, where $S_V(G(\mathbb{Z}_p)K^\Sigma, k)$ is the space of mod p algebraic modular forms of weight V, and level $G(\mathbb{Z}_p)K^\Sigma$. More concretely,

$$S_V(G(\mathbb{Z}_p)K^\Sigma, k) = \varinjlim_{K_{\Sigma_0}} S_V(G(\mathbb{Z}_p)K_{\Sigma_0}K^\Sigma, k),$$

where $S_V(G(\mathbb{Z}_p)K_{\Sigma_0}K^\Sigma, k)$ is the (finite-dimensional) space of functions

$$f : G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_{\Sigma_0}K^\Sigma \to V^\vee, \quad f(gu) = u^{-1}f(g), \quad u \in G(\mathbb{Z}_p).$$

The Buzzard-Diamond-Jarvis conjecture (proved by Gee et al. in the case of unitary groups in two variables, cf. [GLS14]) gives an explicit description of $W^{gl}(\bar{\rho})$ in terms of the restrictions to inertia, $\{\bar{\rho}|_{I_{\mathcal{K}_{\tilde{v}}}}\}_{v|p}$. Recall from the introduction that $\bar{\rho}$ is assumed to occur at *maximal* level K_{Σ_0}, with weight V in the *Fontaine-Laffaille* range (which means the highest weights of V lie in $[0, p-2)$). Once and for all, choose such a $V \in W^{gl}(\bar{\rho})$. One of the goals of [Sor14] was to prove the following:

Proposition 5. $S_V(G(\mathbb{Z}_p)K^\Sigma, k)_{\mathfrak{m}}[\mathfrak{m}]$ *is essentially AIG.*

PROOF. Up to notational inconsistency this is Theorem 4 in section 5.5 of [Sor14], which in fact gives the analogous result for $U(n)$ contingent on the conjectural Ihara lemma. The latter is known (and almost a trivial consequence of the strong approximation theorem) for $n = 2$. □

Corollary 1. $Hom_{k[G(\mathbb{Q}_p)]}(\bar{\pi}_p, H^0_{k,\Sigma,\mathfrak{m}}[\mathfrak{m}])$ *is essentially AIG.*

PROOF. Since it is known that $\mathrm{soc}_{k[G(\mathbb{Z}_p)]}(\bar{\pi}_p) = \oplus_{V \in W^{gl}(\bar{\rho})}V$ (see [GLS12, Theorem A]), we have an embedding $V \hookrightarrow \bar{\pi}_p|_{G(\mathbb{Z}_p)}$ whose image generates $\bar{\pi}_p$ over $G(\mathbb{Q}_p)$ because $\bar{\pi}_p \leftrightarrow \{\bar{\rho}|_{\Gamma_{\mathcal{K}_{\tilde{v}}}}\}_{v|p}$ is irreducible – by known properties of mod p local Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$. We deduce that the restriction map

$$\mathrm{Hom}_{k[G(\mathbb{Q}_p)]}(\bar{\pi}_p, H^0_{k,\Sigma,\mathfrak{m}}[\mathfrak{m}]) \longrightarrow \mathrm{Hom}_{k[G(\mathbb{Z}_p)]}(V, H^0_{k,\Sigma,\mathfrak{m}}[\mathfrak{m}])$$

is injective, and the result follows from Proposition 5 and Observation 1. □

We need one more preliminary result in order to show X_{ctf} interpolates the local Langlands correspondence. It is analogous to Lemma 6.4.5, p. 83, in [Em11], except that we only need allowable *prime* ideals.

Lemma 3. *The closure (in the sense of definition C.28 of* [Em11]*) of*

$$\cup_{\mathfrak{p} \ allowable} X[\mathfrak{p}]$$

in X coincides with X_{ctf}.

PROOF. We start with a general remark, which will be used for both inclusions. Let $Y \subset X$ be an arbitrary saturated, coadmissible $\mathbb{T}_{\Sigma,\mathfrak{m}}[U(F_{\Sigma_0})]$-submodule, which is a faithful $\mathbb{T}_{\Sigma,\mathfrak{m}}$-module. By Proposition 3 part (c) we know that $Y[\mathfrak{p}] = X[\mathfrak{p}]$ for all allowable $\mathfrak{p}$. (A priori, we only know this after tensoring with E, but Y is saturated – see also C.41 in [Em11].) In particular, any such Y contains the union $\cup_{\mathfrak{p} \ allowable} X[\mathfrak{p}]$, and therefore its closure W since Y is closed. By C.31 in loc. cit., this is equivalent to Y being coadmissible.

$\subset$: Take $Y = X_{ctf}$ in the previous paragraph.

$\supset$: To show that $W \supset X_{ctf}$, first observe that W is saturated (C.30), coadmissible (C.31), and *faithful*: If $t \in \mathbb{T}_{\Sigma,\mathfrak{m}}$ acts trivially in W, it acts trivially on each $X[\mathfrak{p}]$. In other words $t \in \mathrm{Ann}_{\mathbb{T}_{\Sigma,\mathfrak{m}}} X[\mathfrak{p}] = \mathfrak{p}$. By Zariski density $\cap \mathfrak{p} = 0$ so that $t = 0$. From the first paragraph of the proof we conclude that W is the *minimal* saturated coadmissible faithful submodule of X. Since $W_{ctf} \subset W$ has all these three properties, $W = W_{ctf}$ must be cotorsion-free. As we have already seen in the proof of Proposition 3 part (e) this implies the desired equality $W = X_{ctf}$. $\square$

PROOF OF PROPOSITION 2. Let $\mathcal{S}$ denote the set of closed points $\mathfrak{p} \in \mathrm{Spec}\,\mathbb{T}_{\Sigma,\mathfrak{m}}$, which are allowable. As noted above, $\mathcal{S}$ is Zariski dense. We want to show that $X_{ctf}, \mathcal{S}$ and $A = \mathbb{T}_{\Sigma,\mathfrak{m}}$ satisfy the three conditions determining the local Langlands correspondence in families (see section 2.3).

(1) X_{ctf} is cotorsion-free over $\mathbb{T}_{\Sigma,\mathfrak{m}}$ by definition.

(2) $(X_{ctf}/\varpi X_{ctf})[\mathfrak{m}]$ is essentially AIG, since (by saturation) it sits inside

$$(X/\varpi X)[\mathfrak{m}] \simeq \mathrm{Hom}_{k[G(\mathbb{Q}_p)]}(\bar{\pi}_p, H^0_{k,\Sigma,\mathfrak{m}}[\mathfrak{m}]),$$

which was just shown to be essentially AIG.

(3) By Lemma 2, for all $\mathfrak{p} \in \mathcal{S}$ we have

$$(X \otimes_{\mathcal{O}} E)[\mathfrak{p}] \simeq \otimes_{v \in \Sigma_0} \pi(\rho_{\mathfrak{m},\tilde{v}}(\mathfrak{p})).$$

On the left-hand side, Lemma 3 allows us to replace X by X_{ctf}.

This proves Proposition 2, and altogether finishes the proof of our main Theorem 1. $\square$

REFERENCES

[BB10]   L. Berger and C. Breuil, *Sur quelques représentations potentiellement cristallines de $GL_2(\mathbb{Q}_p)$*. Astérisque No. 330 (2010), 155–211.

[BGGT]   T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor, *Local-global compatibility for l=p, II.* Ann. Sci. Éc. Norm. Supér. (4) 47 (2014), no. 1, 165–179.

[Boy15]   Pascal Boyer, *Lowering the level and Ihara's lemma for some unitary groups.* Preprint, arXiv:1511.00144.

[BS07]   C. Breuil and P. Schneider, *First steps towards p-adic Langlands functoriality.* J. Reine Angew. Math. 610, 2007, 149-180.

[Car14]   A. Caraiani, *Monodromy and local-global compatibility for l=p.* Algebra Number Theory 8 (2014), no. 7, 1597–1646.

[CHT08]   L. Clozel, M. Harris and R. Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations.* With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vigneras. Publ. Math. Inst. Hautes Etudes Sci. No. 108 (2008), 1181.

[CS15]   P. Chojecki and C. Sorensen, *Weak local-global compatibility in the p-adic Langlands program for $U(2)$.* Accepted for publication in Rendiconti del Seminario Matematico della Università di Padova.

[Em06]   M. Emerton, *On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms.* Invent. Math. 164 (2006), no. 1, 1–84.

[Em11]   M. Emerton, *Local-global compatibility in the p-adic Langlands programme for $GL_{2/\mathbb{Q}}$.* Preprint (2011).

[EH14]   M. Emerton and D. Helm, *The local Langlands correspondence for $GL(n)$ in families.* Ann. Sci. E.N.S. (4) 47 (2014), no. 4, 655-722.

[GLS12]   T. Gee, T. Liu, and D. Savitt, *Crystalline extensions and the weight part of Serre's conjecture.* Algebra Number Theory 6 (2012), no. 7, 15371559.

[GLS14]   T. Gee, T. Liu, and D. Savitt, *The Buzzard-Diamond-Jarvis conjecture for unitary groups.* J. Amer. Math. Soc. 27 (2014), no. 2, 389–435.

[He13]   D. Helm, *Whittaker models and the integral Bernstein center for $GL_n(F)$.* Preprint (2013).

[KMSW]   T. Kaletha, A. Minguez, S. W. Shin, P.-J. White, *Endoscopic Classification of Representations: Inner Forms of Unitary Groups.* Preprint, arXiv:1409.3731

[Min11]   A. Mínguez, *Unramified representations of unitary groups.* On the stabilization of the trace formula, 389–410, Stab. Trace Formula Shimura Var. Arith. Appl., 1, Int. Press, Somerville, MA, 2011.

[Mok15]   C. P. Mok, *Endoscopic classification of representations of quasi-split unitary groups.* Mem. Amer. Math. Soc. 235 (2015), no. 1108, vi+248 pp.

[Ram15]   D. Ramakrishnan, *A mild Tchebotarev theorem for GL(n).* J. Number Theory 146 (2015), 519–533.

[Rog90]   J. Rogawski, *Automorphic representations of unitary groups in three variables.* Annals of Mathematics Studies, 123. Princeton University Press, Princeton, NJ, 1990. xii+259 pp.

[Sor14] C. Sorensen, *The local Langlands correspondence in families and Ihara's lemma for U(n)*. Preprint 2014, arXiv:1406.1830.

[ST01] P. Schneider and J. Teitelbaum, $U(g)$-*finite locally analytic representations*. With an appendix by Dipendra Prasad. Represent. Theory 5 (2001), 111128.