Homework #3 Solutions:

#50

a, b group elements, \( |a| = 2, b = e \) s.t. \( ab = b^2 \). Find \( |b| \).

**Solu.** \( b^2 = e \) since if \( ab = e \) \( \Rightarrow \) \( b = e \) which it isn't.

Thus, \( b^4 = (aba)^2 = ababa \)

\[ = ab^2a \]

\[ = a(ab)a \]

\[ = b \]

\[ \Rightarrow b^3 = e \]

\[ \Rightarrow \quad |b| = 3 \]

#58

\( U(15) \) has six cyclic subgroups, list them.

**Solu.** \( \langle 1 \rangle, \langle 2 \rangle = \langle 8 \rangle, \langle 4 \rangle, \langle 7 \rangle = \langle 13 \rangle, \langle 11 \rangle, \langle 14 \rangle \)

#60

G a group with exactly eight elements of order 3. How many order 3 subgroups are in G?

**Solu.** Let \( H \) be of order 3, thus \( H \) is cyclic, say \( H = \langle x \rangle \)

So \( H = \{ 1, x, x^2 \} \), but \( x^2 = x^{-1} \) and \( |x| = 1 \times |x|^1 \), so \( H = \langle x^{-1} \rangle \). So, there are \( \frac{8}{3} = 4 \) subgroups of order 3.

#70

Let \( F \) be reflection, \( R \) rotation in \( D_n \). Find \( C(F) \) if \( n \) is even or odd, and \( C(F) \).

**Solu.** \( C(F) = \langle R \rangle \) since any element in \( D_n \) is of the form \( RF \) some \( k \) and \( R \in C(R) \) is clear. So, if \( R^k \in C(R) \) \( \Rightarrow RF \in C(R) \) \( \Rightarrow C(R) = D_n \) which is not true since \( FRF = R^{-1} \).

* For \( n \) even, use that \( R^k = R^{-k} \) to get \( R^k \in C(R) \).

* For \( n \) odd, show that \( R^k \in C(F) \) for any \( i \).
#80

G a finite group with more than one element. Show that G has an element of prime order.

Pf: \( |a| < \infty \) so let \( a \neq e \) be in \( G \). Consider \( < a > \leq G \).

1. Let \( m = |a| \), hence of \( < a > \). Write \( \mathbb{Z}_m = \prod_{p \mid m} p^{a(p)} \) (so \( a(p) = 0 \) for all but finitely many primes \( p \)).

2. Since \( m + 1 \) we know \( \exists p \) s.t. \( a(p) \geq 1 \), consider the subgroup \( < a > \leq < a > \). Then, \( |a^{a(p)}| = p \Rightarrow a^{a(p)} \in G \) is an element of order \( p \).

#9

Suppose \( < a >, < b >, < c > \) are cyclic groups of orders 6, 8, and 20 respectively. Find all generators of \( < a >, < b >, \) and \( < c > \).

- \( |< a >| = 6 \) so, using \( < a > = < a^{\text{gcd}(6,k)} > \), we consider the integers \( 1 \leq k \leq 6 \) s.t. \( \text{gcd}(6,k) = 1 \), namely \( k = 1, 5 \).

- \( |< b >| = 8 \) so find \( 1 \leq k \leq 8 \) s.t. \( \text{gcd}(8,k) = 1 \), namely \( k = 1, 3, 5, 7 \). Then, generators of \( < b > \) are \( b, b^3, b^5, b^7 \).

- \( |< c >| = 20 \) so find \( 1 \leq k \leq 20 \) s.t. \( \text{gcd}(20,k) = 1 \), namely \( k = 1, 3, 7, 9, 11, 13, 17, 19 \), so generators of \( < c > \) are \( c, c^3, c^7, c^9, c^9, c^{13}, c^{17}, c^{19} \).

#4

List elements of \( < 3 > \) and \( < 15 > \) in \( \mathbb{Z}_{18} \). Let \( |a| = 18 \), list elements of \( < a^3 > \) and \( < a^{15} > \).

Solu: \( < 3 > = \{ 3, 6, 9, 12, 15, 0 \} \) or more precisely \( \{ [3], [6], [9], [12], [15] \} \) let \( |a| = 18 \),

Similarly \( < 15 > = \{ [15], [12], [9], [6], [3], [0] \} \).
#8

\[ a \in G \text{ s.t. } |a| = 15. \text{ Find:} \]

1. \(|a^3|, |a^6|, |a^9|, |a^{12}| \)
2. \(|a^5|, |a^{10}| \)
3. \(|a^2|, |a^4|, |a^8|, |a^{14}| \)

\[ \text{Sol:} \]

Use the fact that if \(a \in G \) is of order \( n \), then \(|a^k| = \frac{n}{(n,k)} \).

So,

\[ |a^3| = \frac{15}{(3,15)} = 5, \quad |a^9| = \frac{15}{(9,15)} = 5 \]
\[ |a^6| = \frac{15}{(6,15)} = 5, \quad |a^{12}| = \frac{15}{(12,15)} = 5 \]
\[ |a^5| = \frac{15}{(5,15)} = 3, \quad |a^{10}| = \frac{15}{(10,15)} = 3 \]
\[ |a^2| = 15 = |a^4| = |a^8| = |a^{14}|. \]

#10

In \( \mathbb{Z}_{24} \), list all generators of the subgroup of order 8. Let \( G = \langle a \rangle \) s.t. \(|a| = 24 \), list all generators of the subgroup of order 8.

\[ \text{Sol:} \]

\( \mathbb{Z}_{24} : 6|24 \) so \( \exists! \) subgroup of order 8 in \( \mathbb{Z}_{24} \), namely the subgroup generated by \( \langle 3 \rangle \). The generators of \( \langle 3 \rangle \) are all of the form \( 3k \) where \((3,8) = 1\), namely \( 3, 9, 15, 21 \).

Note: There are \( \phi(8) = 2^3 - 2^2 = 4 \) such generators.

\( G = \langle a \rangle \) with \(|a| = 24 \). \( 8|24 \) so \( \exists! \) subgroup of order 8 in G namely \( \langle a^3 \rangle \). The generators of \( \langle a^3 \rangle \) are those of the form \( a^{3k} \) for \((3,8) = 1\), namely, \( a^3, a^9, a^{15}, a^{21} \).

#12

In \( \mathbb{Z} \), find all generators of \( \langle 3 \rangle \). If \(|a| = \infty \), find all generators of \( \langle a \rangle \).

\[ \text{Sol:} \]

If \( H \subseteq \mathbb{Z} \implies H = \langle k \rangle \) some \( k \in \mathbb{Z} \). So, since a generator \( x \) of \( \langle 3 \rangle \) would satisfy \( \langle 3 \rangle = \langle x \rangle \) we get \( x \in \langle 3 \rangle \) and \( 3 \in \langle x \rangle \).

But, \( x \in \langle 3 \rangle \implies 3|\text{ord } x \), and \( 3 \in \langle x \rangle \implies x|3 \), hence only choices for \( x \) are 3 or \(-3 \). Thus, only generators of \( \langle 3 \rangle \) are 3 and \(-3 \). Same for \( \langle a^3 \rangle \), only generators are \( a^3 \) and \( a^{-3} \).