Math 103A, Final

Tuesday, December 16th, 2014, 11:30–2:29 pm, WLH 2111

• Your Name:
Problem 1. Let $G$ be a group with precisely 60 elements, $|G| = 60$.

(a) Does $G$ have a subgroup with precisely 7 elements?

(b) How many elements of $G$ have order 8?

(c) If $a \in G$ has order 30, what’s the order of $a^{42}$?
Problem 2.

(a) List all the invertible residue classes in $\mathbb{Z}_{10}$. (That is, elements of $U(10)$.)

(b) Write down all the cyclic subgroups of $U(10)$. Any non-cyclic subgroups?

(c) Is $U(10) \simeq U(5)$?
Problem 3.

(a) Decompose the permutation
\[ \alpha = (3941)(967)(421)(53) \in S_9 \]
into disjoint cycles; then compute \( \text{ord}(\alpha) \) and \( \text{sign}(\alpha) \).

(b) Introduce the permutation
\[ \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 7 & 9 & 2 & 8 & 3 & 5 & 1 & 6 \end{bmatrix} \in S_9. \]
Write \( \beta^3 = \beta \circ \beta \circ \beta \) in the same "array" form.

(c) Is \( \beta^{2014} \) an even permutation?
Problem 4. Let $D_9$ be the dihedral group of symmetries of a regular 9-gon, and let $H \leq D_9$ be the subgroup consisting of all 9 rotations.

(a) What’s the index $[D_9 : H]$?

(b) Is $H$ a normal subgroup of $D_9$?

(c) Let $s \in D_9$ be a reflection. Is the coset $sH$ a subgroup of $D_9$?

(d) Find $|sH|$, the size of $sH$. 
Problem 5. Which of the following functions are group homomorphisms? Explain why or why not.

(a) $\alpha : \mathbb{C}^\times \rightarrow \mathbb{C}^\times$, $\alpha(z) = z^7$. 
   (Here $\mathbb{C}^\times = \mathbb{C} - \{0\}$ is a group under multiplication.)

(b) $\beta : \mathbb{Z} \rightarrow \mathbb{Z}$, $\beta(n) = n^2$. 
   (Here $\mathbb{Z}$ is a group under addition.)

(c) $\gamma : \mathbb{R} \rightarrow \text{GL}_2(\mathbb{R})$, $\gamma(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$.

(d) $\delta : A_n \rightarrow \{\pm 1\}$, $\delta(\sigma) = \begin{cases} -1 & \text{if } \sigma \text{ is a 3-cycle} \\ +1 & \text{otherwise} \end{cases}$
Problem 6. Let $\phi : \mathbb{Z}_{25} \to \mathbb{Z}_{15}$ be a homomorphism such that $\phi([3]) = [9]$.

(a) Compute $\phi([1])$. (Hint: Try to find an $a \in \mathbb{Z}$ such that $1 \equiv 3a \pmod{25}$.)

(b) Underline the elements of $\mathbb{Z}_{15}$ which belong to the image $\text{im}(\phi)$:

\[ [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. \]

(c) Analogously, write down all the elements of $\mathbb{Z}_{25}$ which belong to $\ker(\phi)$. 

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Problem 7. Let $G$ be the set of complex numbers $z$ such that $z^n = 1$ for some positive integer $n$ (which depends on $z$). Symbolically, we have $G = \bigcup_{n=1}^{\infty} \mu_n$.

(a) Verify that $G$ contains $i = \sqrt{-1}$. What’s the ”corresponding” $n$?

(b) Check that $G$ is a subgroup of $\mathbb{C}^\times$. Is it finite?

(c) Obtain an isomorphism $\mathbb{Q}/\mathbb{Z} \rightarrow G$. (Here $\mathbb{Q}/\mathbb{Z}$ is a quotient group.)

(d) Is there a finite subset $S \subset \mathbb{Q}/\mathbb{Z}$ which generates all of $\mathbb{Q}/\mathbb{Z}$?

\footnote{Recall that $\mu_n = \{z \in \mathbb{C} : z^n = 1\}$.}
Problem 8.

(a) Show that $S_n$ is isomorphic to a subgroup of $A_{n+2}$.

(Hint: Consider $\alpha \circ (n+1,n+2)$, for odd $\alpha \in S_n$.)

(b) Is every finite group $G$ isomorphic to a subgroup of an alternating group? Explain why, or give a counterexample.