Problem A. Let \((G, \circ)\) and \((H, \bullet)\) be two groups. We equip the set of pairs
\[
G \times H = \{(g, h) : g \in G, \; h \in H\}
\]
with the composition law \((g, h) \ast (g', h') = (g \circ g', h \bullet h')\). This is known as the direct product of the two groups.

(a) Show that \((G \times H, \ast)\) is a group. Furthermore, verify that it is abelian if and only if both \((G, \circ)\) and \((H, \bullet)\) are abelian.

(b) Note that \(\{\pm 1\}\) is a group under multiplication. Write down the composition table for the direct product
\[
\{\pm 1\} \times \{\pm 1\} = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}.
\]
This is called the Klein four-group and is usually denoted by \(V_4\).

Problem B. Let \(G\) be a set with exactly three elements.

(a) In how many ways can one define a composition law on \(G\)? (That is, how many functions \(G \times G \rightarrow G\) are there?)

(b) Suppose \((G, \circ)\) is a group, and \(G = \{e, a, b\}\) where \(e\) is the neutral element.
Give the composition table for \((G, \circ)\). Which elements \(x\) satisfy \(x^3 = e\)?

Problem C. Which of the following sets with composition laws are groups?

\[\text{Hint: Verify (and use) that } a \text{ and } n \text{ are relatively prime precisely when } n - a \text{ and } n \text{ are.}\]

\[\text{See Example 2.1.11 on page 57 for the definition of } O_2(\mathbb{R}).\]
(a) $\mathbb{Z}$ with addition;
(b) $\mathbb{Z}/\mathbb{N}$ with multiplication;
(c) $\mathbb{N} = \{0, 1, 2, \ldots\}$ with addition;
(d) $\mathbb{R}_{>0} = (0, \infty)$ with multiplication;
(e) $\mathbb{Q}^\times = \{x \in \mathbb{Q} : x \neq 0\}$ with $a \circ b = ab^{-1}$;
(f) $\text{Sym}_n(\mathbb{R}) = \{\text{symmetric } n \times n\text{-matrices}\}$ with matrix addition$^3$;
(g) $\mathbb{Z}$ with $a \bullet b = \max\{a, b\}$;
(h) $\text{SL}_n(\mathbb{R}) = \{A : \det(A) = 1\}$ with matrix multiplication$^4$;
(i) $\{\alpha \in S_n : \alpha(1) = 1\}$ with composition (of permutations);
(j) $\mathbb{R}$ with subtraction: $(a, b) \mapsto a - b$.

**Problem D.** Let $f : X \to Y$ and $g : Y \to Z$ be two functions, and let $g \circ f$ be their composite. I.e., $(g \circ f)(x) = g(f(x))$ for all $x \in X$.

(a) Show that if both $f$ and $g$ are injective so is $g \circ f$.

(b) Show that if both $f$ and $g$ are surjective so is $g \circ f$.

(c) Show that if both $f$ and $g$ are bijective so is $g \circ f$. Furthermore, verify that the inverse functions are related by the formula

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$ 

$^3$Recall that an $n \times n$-matrix $A$ is said to be symmetric if it equals its transpose $A^T$.

$^4$Recall from linear algebra that $\det(AB) = \det(A) \det(B)$ for $n \times n$-matrices $A$ and $B$. 