Math 103A, Fall 2017

Modern Algebra, HW 5

Due Tuesday November 7th by 5PM in your TA's box/after discussion session

From Lauritzen's book:

- Exercises 2.11 (starting page 104): *3 (only the a–row), 4, 5, 16*

Problem A. Let $(G, \bullet)$ be a group. For each element $a \in G$ we consider the function $f_a : G \to G$ defined by $f_a(x) = a \bullet x$ (i.e., left multiplication by a).

(a) Show that f_a is bijective (and therefore a permutation of G).

(b) For any two elements $a, b \in G$ prove the identity

$$f_{a \bullet b} = f_a \circ f_b.$$

(Here the $\circ$ on the right-hand side is composition of functions.)

(c) Deduce from (b) that f_a has inverse function $f_a^{-1} = f_{a^{-1}}$.

Problem B. Introduce the following three permutations in S_4,

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

(a) Verify that the subset $V = \{e, \alpha, \beta, \gamma\}$ is in fact a **subgroup** of S_4, where e denotes the neutral element.

(b) Give the composition table for V. (Does it remind you of another group?)

Problem C. Let $(G, \bullet)$ be a group, and let H, H' be two subgroups.

(a) Give a detailed proof that the intersection $H \cap H'$ is then also a subgroup.

(b) Provide an example showing that the union $H \cup H'$ need **not** be a subgroup.

Problem D. Here we consider the (additive) group of integers $\mathbb{Z}$. Let m, n be any two positive integers. Prove the following relations.

(a) $m\mathbb{Z} \cap n\mathbb{Z} = \mathrm{LCM}(m, n)\mathbb{Z}$

(b) $m\mathbb{Z} + n\mathbb{Z} = \mathrm{GCD}(m, n)\mathbb{Z}$.

(In (b) the left-hand side is the set of integers of the form $mx+ny$ with $x, y \in \mathbb{Z}$.)