From Lauritzen’s book:

- Exercises 2.11 (starting page 104): 3 (only the $a$–row), 4, 5, 16

**Problem A.** Let $(G, \cdot)$ be a group. For each element $a \in G$ we consider the function $f_a : G \to G$ defined by $f_a(x) = a \cdot x$ (i.e., left multiplication by $a$).

(a) Show that $f_a$ is bijective (and therefore a permutation of $G$).
(b) For any two elements $a, b \in G$ prove the identity

$$f_{a \cdot b} = f_a \circ f_b.$$  

(Here the $\circ$ on the right-hand side is composition of functions.)
(c) Deduce from (b) that $f_a$ has inverse function $f_a^{-1} = f_{a^{-1}}$.

**Problem B.** Introduce the following three permutations in $S_4$,

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

(a) Verify that the subset $V = \{e, \alpha, \beta, \gamma\}$ is in fact a subgroup of $S_4$, where $e$ denotes the neutral element.
(b) Give the composition table for $V$. (Does it remind you of another group?)

**Problem C.** Let $(G, \cdot)$ be a group, and let $H, H'$ be two subgroups.

(a) Give a detailed proof that the intersection $H \cap H'$ is then also a subgroup.
(b) Provide an example showing that the union $H \cup H'$ need not be a subgroup.
**Problem D.** Here we consider the (additive) group of integers $\mathbb{Z}$. Let $m, n$ be any two positive integers. Prove the following relations.

(a) $m\mathbb{Z} \cap n\mathbb{Z} = \text{LCM}(m, n)\mathbb{Z}$

(b) $m\mathbb{Z} + n\mathbb{Z} = \text{GCD}(m, n)\mathbb{Z}$.

(In (b) the left-hand side is the set of integers of the form $mx + ny$ with $x, y \in \mathbb{Z}$.)