Math 103A, Fall 2017

Modern Algebra, HW 6

Due Tuesday November 14th by 5PM in your TA's box/after discussion session

From Lauritzen's book:

- Exercises 2.11 (starting page 104): *9, 10, 12, 20*

Problem A. Let $(G, \circ)$ and $(G', \bullet)$ be two groups. A function $f : G \to G'$ is said to be a **homomorphism** if it satisfies the relation

$$f(a \circ b) = f(a) \bullet f(b)$$

for any two elements $a, b \in G$. An **isomorphism** is a bijective homomorphism.

(a) Recall that $\mathbb{C}^\times = \mathbb{C} \backslash \{0\}$ is a group under multiplication. Show that

$$f : \mathbb{C}^\times \longrightarrow \mathrm{GL}_2(\mathbb{R}) \qquad a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

defines a homomorphism. Is it injective and/or surjective?

(b) We let $\mathbb{C}^1 = \{z \in \mathbb{C} : |z| = 1\}$ denote the complex unit circle, which is a subgroup of $\mathbb{C}^\times$. Verify that f restricts to an isomorphism

$$f : \mathbb{C}^1 \longrightarrow \mathrm{SO}_2(\mathbb{R}) \qquad a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Problem B. The complex exponential function $\exp : \mathbb{C} \to \mathbb{C}^\times$ is defined by

$$\exp(a + ib) = e^a \big(\cos b + i \sin b \big).$$

(a) Check that exp is a surjective homomorphism (cf. Problem A).

(b) Which $z \in \mathbb{C}$ satisfy the equation $\exp(z) = 1$? Plot them in the complex plane.

(c) Let $\Sigma = \{a + ib : 0 \leq b < 2\pi\}$. Draw Σ in the complex plane and show that the restriction $\exp : \Sigma \to \mathbb{C}^\times$ is bijective. Is Σ a subgroup of $\mathbb{C}$?

Problem C. Let $\zeta = \exp(2\pi i/6) = (1 + i\sqrt{3})/2$ and consider the group of 6th roots of unity (under multiplication)

$$U_6 = \{1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5\}.$$

(a) Plot the elements of U_6 in the complex plane.

(b) Give the composition table for U_6.

(c) Find the order[1] of each element of U_6. Identify ζ^{2017} on the above list.

(d) Prove that the function defined below is an isomorphism (cf. Problem A):

$$f : \mathbb{Z}/6\mathbb{Z} \longrightarrow U_6 \qquad [a] \mapsto \zeta^a$$

Problem D. Let $(G, \circ)$ be a group. The **center** of G is the subset of elements which commute with every other element. That is,

$$Z(G) = \{a \in G : a \circ b = b \circ a \ \ \forall b \in G\}.$$

(a) Verify that $Z(G)$ is a subgroup of G. Check that G is abelian precisely when $Z(G) = G$.

(b) Find the center of $\mathrm{GL}_n(\mathbb{R})$. (`Hint`: Suppose $AB = BA$ for all B. First take B to be a suitable diagonal matrix and conclude that A must be diagonal. Then test the condition with other elementary matrices B.)

(c) Can you think of a group G with $Z(G) = \{e\}$?

[1] Recall that the *order* of a root of unity $z \in \mathbb{C}^\times$ is the smallest $n \geq 1$ such that $z^n = 1$.