Problem A. Let \((G, \circ)\) and \((G', \bullet)\) be two groups. A function \(f : G \to G'\) is said to be a \textbf{homomorphism} if it satisfies the relation
\[
f(a \circ b) = f(a) \bullet f(b)
\]
for any two elements \(a, b \in G\). An \textbf{isomorphism} is a bijective homomorphism.

(a) Recall that \(\mathbb{C}^\times = \mathbb{C}\setminus\{0\}\) is a group under multiplication. Show that
\[
f : \mathbb{C}^\times \longrightarrow \text{GL}_2(\mathbb{R}) \quad a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}
\]
defines a homomorphism. Is it injective and/or surjective?

(b) We let \(\mathbb{C}^1 = \{z \in \mathbb{C} : |z| = 1\}\) denote the complex unit circle, which is a subgroup of \(\mathbb{C}^\times\). Verify that \(f\) restricts to an isomorphism
\[
f : \mathbb{C}^1 \longrightarrow \text{SO}_2(\mathbb{R}) \quad a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.
\]

Problem B. The complex exponential function \(\exp : \mathbb{C} \to \mathbb{C}^\times\) is defined by
\[
\exp(a + ib) = e^a (\cos b + i \sin b).
\]

(a) Check that \(\exp\) is a surjective homomorphism (cf. Problem A).

(b) Which \(z \in \mathbb{C}\) satisfy the equation \(\exp(z) = 1\)? Plot them in the complex plane.

(c) Let \(\Sigma = \{a + ib : 0 \leq b < 2\pi\}\). Draw \(\Sigma\) in the complex plane and show that the restriction \(\exp : \Sigma \to \mathbb{C}^\times\) is bijective. Is \(\Sigma\) a subgroup of \(\mathbb{C}\)?
**Problem C.** Let $\zeta = \exp(2\pi i/6) = (1 + i\sqrt{3})/2$ and consider the group of 6th roots of unity (under multiplication)

$$U_6 = \{1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5\}.$$  

(a) Plot the elements of $U_6$ in the complex plane.

(b) Give the composition table for $U_6$.

(c) Find the order\(^1\) of each element of $U_6$. Identify $\zeta^{2017}$ on the above list.

(d) Prove that the function defined below is an isomorphism (cf. Problem A):

$$f : \mathbb{Z}/6\mathbb{Z} \rightarrow U_6 \quad [a] \mapsto \zeta^a$$

**Problem D.** Let $(G, \circ)$ be a group. The **center** of $G$ is the subset of elements which commute with every other element. That is,

$$Z(G) = \{a \in G : a \circ b = b \circ a \ \forall b \in G\}.$$  

(a) Verify that $Z(G)$ is a subgroup of $G$. Check that $G$ is abelian precisely when $Z(G) = G$.

(b) Find the center of $\text{GL}_n(\mathbb{R})$. (Hint: Suppose $AB = BA$ for all $B$. First take $B$ to be a suitable diagonal matrix and conclude that $A$ must be diagonal. Then test the condition with other elementary matrices $B$.)

(c) Can you think of a group $G$ with $Z(G) = \{e\}$?

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\(^1\)Recall that the *order* of a root of unity $z \in \mathbb{C}^\times$ is the smallest $n \geq 1$ such that $z^n = 1$. 