Math 103A, Modern Algebra I, Final

Friday, December 13th, 2019, 8–11am, APM B402A

- *Your Name*:
- *ID Number*:
- *Section*:

B01 (5:00 PM)   B02 (6:00 PM)

Problem #	Points (out of 10)
1	
2	
3	
4	
5	
6	
7	
8	
9	
Total (out of 90):	

Problem 1. Let $(G, *)$ be a cyclic group of size 10. Choose a generator $a \in G$.

(a) Find the <u>order</u> of each of its elements:

$$e \qquad a \qquad a^2 \qquad a^3 \qquad a^4 \qquad a^5 \qquad a^6 \qquad a^7 \qquad a^8 \qquad a^9$$

Circle those x above for which $G = \langle x \rangle$ holds.

(b) List the elements of the two non-trivial subgroups $\langle a^2 \rangle$ and $\langle a^5 \rangle$.

(c) Find all the elements of the cosets $a * \langle a^2 \rangle$ and $a * \langle a^5 \rangle$.

Problem 2. Recall that $(\mathbb{Z}_{11}^{\times}, \bullet)$ denotes the multiplicative group of all the invertible residue classes modulo 11.

(a) Find $|\mathbb{Z}_{11}^{\times}|$ and check that the residue class $[2]$ generates $\mathbb{Z}_{11}^{\times}$.

(b) Give the order of each of the elements:

$$[1] \quad [2] \quad [3] \quad [4] \quad [5] \quad [6] \quad [7] \quad [8] \quad [9] \quad [10]$$

Circle those $[x]$ above for which $\mathbb{Z}_{11}^{\times} = \langle [x] \rangle$ holds.

(c) List all the elements of the cosets $[2] \bullet \langle [4] \rangle$ and $[3] \bullet \langle [4] \rangle$.

Problem 3. Recall that $(\mathbb{Z}_{15}, +)$ denotes the additive group of all residue classes modulo 15.

(a) Give all integers x in the range $0 \leq x < 15$ such that $\mathbb{Z}_{15} = \langle [x] \rangle$.

(b) Write down all elements of the two non-trivial subgroups of $\mathbb{Z}_{15}$.

(c) Explain why the quotient group $\mathbb{Z}_{15}/\langle [5] \rangle$ is isomorphic to $\mathbb{Z}_5$.

Problem 4. Consider the additive group $(\mathbb{Z}, +)$ of all integers. Recall that $N\mathbb{Z}$ denotes the subgroup of $\mathbb{Z}$ consisting of all integer multiples of N.

(a) Find the positive integer M such that $65\mathbb{Z} \cap 91\mathbb{Z} == M\mathbb{Z}$.

(b) Find the positive integer N such that $65\mathbb{Z} + 91\mathbb{Z} == N\mathbb{Z}$, and express N as a linear combination $65x + 91y$ for suitable integers $x, y \in \mathbb{Z}$.

(c) Let $f : 65\mathbb{Z} \longrightarrow \mathbb{Z}_{91}$ be the homomorphism sending an $a \in 65\mathbb{Z}$ to its residue class $[a]$ modulo 91. Calculate the following two quantities:

   (i) The cardinality of $\mathrm{im}(f)$.

   (ii) The index of $\ker(f)$ in $65\mathbb{Z}$.

Problem 5. Let $\alpha \in S_9$ be the permutation $\alpha = (1234)(25)(617)(389)$.

(a) Express α in array form. That is, fill in the blank boxes below.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \square & \square & \square & \square & \square & \square & \square & \square & \square \end{pmatrix}$$

(b) Is α a cycle? If not, find its decomposition into disjoint cycles.

(c) Compute $\text{ord}(\alpha)$ and $\text{sign}(\alpha)$. Does α belong to A_9?

Problem 6. For each of the five statements below indicate whether it is true or false. Justify your answers.

(a) The group of rotational symmetries of a tetrahedron is isomorphic to S_3.

(b) The group of rotational symmetries of a cube is isomorphic to S_4.

(c) The direct product $G \times G$ is **not** cyclic for any non-trivial group G.

(d) All subgroups of an abelian group are normal.

(e) If G is any group, and $H \subset G$ is a normal subgroup, the following holds:

$$G \text{ cyclic} \iff H \text{ and } G/H \text{ are cyclic.}$$

Problem 7. Consider the alternating group A_5.

(a) Compute its cardinality $|A_5|$ and its index in S_5.

(b) Prove or disprove the existence of a non-trivial homomorphism

$$f : A_5 \longrightarrow \{\pm 1\}.$$

(c) Let $H \subset A_5$ be the subgroup generated by the 3-cycle (135).

  (i) Find the index $[A_5 : H]$.

  (ii) List all elements of the coset $(12345) \circ H$. (Express all permutations as a composition of disjoint cycles.) Is $(12345) \circ H = H \circ (12345)$?

Problem 8. Let $(G, *)$ be a group. The commutator of $a, b \in G$ is the element

$$[a, b] == a * b * a^{-1} * b^{-1}.$$

Let $H \subset G$ be the subset consisting of all finite products[1] of commutators.

(a) Show that $[a, b]^{-1} = [b, a]$. Deduce that H is a subgroup of G.

(b) Verify the formula below for all $g, a, b \in G$:

$$g * [a, b] * g^{-1} == [g * a * g^{-1}, g * b * g^{-1}].$$

Deduce that H is a **normal** subgroup of G.

(c) Prove that the quotient group G/H is abelian.

[1]I.e., all expressions $[a_1, b_1] * \cdots * [a_N, b_N]$ for varying N and $a_i, b_i \in G$. This includes e.

Problem 9. Consider the dihedral group D_5 of all symmetries of a pentagon centered at the origin. Recall that D_5 is generated by elements r, s satisfying:

$$\mathrm{ord}(r) = 5 \qquad \mathrm{ord}(s) = 2 \qquad rs = sr^{-1}$$

(a) Write down its cardinality $|D_5|$. Is D_5 an abelian group?

(b) What is the order of the element rs?

(c) Prove the following two statements:

  (i) The **only** two elements of D_5 commuting with s are e and s.

  (ii) The **only** elements of D_5 commuting with r are the powers of r.

Extra I.

Extra II.

Extra III.