Problem A. Let $a, b \in \mathbb{Z}$ be two integers. Recall that $a$ divides $b$ if one can write $b = qa$ for some $q \in \mathbb{Z}$. When that happens the notation we use is $a | b$. Prove the following two properties, for any three $a, b, c \in \mathbb{Z}$:

(a) $a | a$.

(b) $a | b \land b | c \implies a | c$.

Suppose $a | b$ and $b | a$. Can one deduce that $a = b$? What if $a, b$ are both nonzero?

Problem B. Given $a, b \in \mathbb{Z}$ with $a > 0$, there are uniquely determined $q, r \in \mathbb{Z}$ such that the following two conditions are satisfied simultaneously:

(i) $b = qa + r$,

(ii) $0 \leq r < a$.

(This is called “division with remainder”.)

(a) Let $a = 17$ and $b = 2019$. Find the corresponding pair $q, r$.

(b) In general, suppose $d$ is a common divisor of $a$ and $b$. That is $d | a$ and $d | b$. Deduce that $d | r$.

(c) Use the observation in part (b) to find the greatest common divisor of 143 and 221. Then write it in the form

$$\text{GCD}(143, 221) = 143x + 221y$$

for suitable integer coefficients $x, y \in \mathbb{Z}$. (“Euclid’s Algorithm”.)

Problem C. A prime number is an integer $p > 1$ whose only positive divisors are 1 and $p$.

(a) List all the prime numbers less than 20.
(b) Factor 60 as a product of prime numbers.

(c) Is there an \( n > 1 \) such that \( 4^n - 1 \) is a prime number?

**Problem D.** For any positive integer \( n \in \mathbb{N} \) and any real number \( x \neq 1 \) the following very useful formula holds:

\[
1 + x + x^2 + x^3 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}.
\]  

(1)

(Known as the ”Geometric Sum Formula“.)

(a) Prove the formula (1) above using mathematical induction on \( n \).

(b) Infer that 5 divides \( 6^n - 1 \) for all \( n \in \mathbb{N} \). (Hint: Take \( x = 6 \).)

**Problem E.** Consider the following two subsets of \( \mathbb{N} \).

\[ A = \{2, 3, 5, 7, 11, 13, 17, 19\}, \quad B = \{1, 3, 4, 11, 17, 18\}. \]

(a) Find their cardinalities \(|A|\) and \(|B|\).

(b) List the elements of their union \( A \cup B \) and intersection \( A \cap B \).

(c) How many subsets does \( B \) have?

(d) Which of the following statements are true?

(i) \( 1 \in A \)

(ii) \( A \subseteq B \)

(iii) \( B \subseteq A \)

(iv) \( x \in A \iff x \) is a prime number

(v) \( x \in B \implies x < 100 \)

**Problem F.** Let \( \text{GL}_N(\mathbb{R}) \) be the set of invertible \( N \times N \)-matrices \( A \) with all its entries in \( \mathbb{R} \). Verify in gory detail that \( \text{GL}_N(\mathbb{R}) \) constitutes a group under matrix multiplication. (Called the ”General Linear“ group.)

(a) Verify the relation \( \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) for all \( n \in \mathbb{Z} \) — possibly negative.

(b) For any integer \( n > 1 \) find a \( 2 \times 2 \)-matrix \( A \) such that \( A^n = I \) but none of the preceding powers \( A, A^2, \ldots, A^{n-1} \) equal \( I \). (Hint: Try a rotation matrix \( A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \) for a suitable angle \( \theta \).)

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\(^1\)This means there is an \( N \times N \)-matrix \( B \) such that \( AB = BA = I \) = the identity matrix.