Math 103A, Fall 2019

Modern Algebra I, HW 3

Due Friday October 25th at 11:30AM in Ji Zeng's box outside B402A.

From Armstrong's Groups and Symmetry:

- Exercises (Chapter 4, pages 18–19):

  4.3, 4.6, 4.8, 4.9

Problem A. Let $(G, *)$ be a cyclic group of size 15. Choose a generator $a \in G$.

(a) List all elements of $\langle a^3 \rangle$ and $\langle a^5 \rangle$.

(b) Which of the following elements are generators for G?

$$e,\ a,\ a^2,\ a^3,\ a^4,\ a^5,\ a^6,\ a^7,\ a^8,\ a^9,\ a^{10},\ a^{11},\ a^{12},\ a^{13},\ a^{14}.$$

(Circle those elements x for which $G = \langle x \rangle$ holds.)

(c) Find the order of a^{2019}.

Problem B. $(\mathbb{Z}_{12}, +)$ is the additive group of all residue classes modulo 12.

(a) Explain why [5] is a generator for $\mathbb{Z}_{12}$.

(b) Find <u>all</u> integers x in the range $0 \leq x < 12$ for which $[x]$ generates $\mathbb{Z}_{12}$.

(c) List all elements of $\langle [3] \rangle$ and $\langle [4] \rangle$.

Problem C. Recall that $(\mathbb{Z}_{13}^{\times}, \bullet)$ denotes the multiplicative group of invertible residue classes modulo 13.

(a) Check that [2] is a generator for $\mathbb{Z}_{13}^{\times}$. Conclude that $\mathbb{Z}_{13}^{\times}$ is cyclic.

(b) Find <u>all</u> integers x in the range $0 < x < 13$ for which $[x]$ generates $\mathbb{Z}_{13}^{\times}$.

(c) List all elements of $\langle [3] \rangle$ and $\langle [4] \rangle$.

Problem D. Let $(G, *)$ and $(H, \star)$ be two groups. Recall that the direct product $G \times H$ is the set of all pairs (g, h) with $g \in G$ and $h \in H$ arbitrary. Define a composition law $\bullet$ on $G \times H$ by working componentwise:

$$(g, h) \bullet (g', h') =\!= (g * g', h \star h').$$

(a) Verify in detail that $G \times H$ with $\bullet$ is a group. What is its neutral element?

(b) Suppose $g \in G$ and $h \in H$ both have finite order. Prove the formula

$$\operatorname{ord}(g, h) = \operatorname{LCM}\big(\operatorname{ord}(g), \operatorname{ord}(h)\big).$$

(c) Use the observation in (b) to deduce the following: If G and H are finite cyclic groups then so is $G \times H$ — **provided** $\operatorname{GCD}\big(|G|, |H|\big) = 1$.

(d) Can you give a different proof of (c) using the Chinese remainder theorem?