Math 103A, Fall 2019

Modern Algebra I, HW 4

Due Friday November 1st at 11:30AM in Ji Zeng's box outside B402A.

From Armstrong's Groups and Symmetry:

- Exercises (Chapter 5, pages 24–25):

  5.2, 5.5, 5.10, 5.12

Problem A. Recall that $\mathbb{Z}_{11}^{\times}$ is the multiplicative group of all invertible residue classes modulo 11.

(a) Check that [2] is a generator for $\mathbb{Z}_{11}^{\times}$. Conclude that $\mathbb{Z}_{11}^{\times}$ is a cyclic group.

(b) Find the order of each element of $\mathbb{Z}_{11}^{\times}$.

(c) The group $\mathbb{Z}_{11}^{\times}$ has exactly two non-trivial subgroups. List their elements.

Problem B. Let U_{12} be the multiplicative group of 12th roots of unity. That is, all solutions $z \in \mathbb{C}$ to the equation $z^{12} = 1$. We know $U_{12} = \langle \zeta \rangle$ where

$$\zeta == \cos(\frac{2\pi}{12}) + i \sin(\frac{2\pi}{12}).$$

(a) Find the order of the elements of U_{12}. Circle those ζ^n for which $U_{12} = \langle \zeta^n \rangle$:

$1 \quad \zeta \quad \zeta^2 \quad \zeta^3 \quad \zeta^4 \quad \zeta^5 \quad \zeta^6 \quad \zeta^7 \quad \zeta^8 \quad \zeta^9 \quad \zeta^{10} \quad \zeta^{11}$

(b) The group U_{12} has exactly four non-trivial subgroups. List the elements of each subgroup, and plot them as points in the complex plane $\mathbb{C}$.

(c) Describe all the inclusions ($\subset$) among the subgroups you found in (b).

Problem C. Consider the additive group $\mathbb{Z}_{18}$ of all residue classes modulo 18.

(a) List all $x \in \mathbb{Z}$ in the range $0 \leq x < 18$ for which $\mathbb{Z}_{18} = \langle [x] \rangle$.

(b) The group $\mathbb{Z}_{18}$ has exactly four non-trivial subgroups. List the elements of each subgroup, and find all inclusions among the subgroups.

(c) Which of the groups below are cyclic? Circle the cyclic ones (and explain).

$$\mathbb{Z}_{18} \times \mathbb{Z}_{18} \qquad \mathbb{Z}_{18} \times \mathbb{Z}_{19} \qquad \mathbb{Z}_{18} \times U_{12} \qquad \mathbb{Z}_{18} \times \mathbb{Z}_{11}^{\times}.$$

(Recall the definition of direct products in Problem D on Homework 3.)

Problem D. Let $\mathbb{Z}$ be the additive group of all integers. Recall that $N\mathbb{Z}$ is the subgroup consisting of all integer multiple of N. Given two $M, N \in \mathbb{Z}$ consider

$$M\mathbb{Z} + N\mathbb{Z} == \{Mx + Ny : x, y \in \mathbb{Z}\}.$$

(a) Explain why $M\mathbb{Z} + N\mathbb{Z}$ is a subgroup of $\mathbb{Z}$, and prove that

$$M\mathbb{Z} + N\mathbb{Z} == \mathrm{GCD}(M, N)\mathbb{Z}.$$

(b) Check that $M\mathbb{Z} \cap N\mathbb{Z}$ is a subgroup of $\mathbb{Z}$, and show that

$$M\mathbb{Z} \cap N\mathbb{Z} == \mathrm{LCM}(M, N)\mathbb{Z}.$$

(c) Give an example showing that $M\mathbb{Z} \cup N\mathbb{Z}$ is <u>not</u> always a subgroup.