Math 103A, Fall 2019

Modern Algebra I, HW 5

Due Friday November 8th at 11:30AM in Ji Zeng's box outside B402A.

From Armstrong's Groups and Symmetry:

- Exercises (Chapter 1, page 5):

  1.2, 1.3, 1.4, 1.5

Problem A. Let $(G, *)$ and $(H, \star)$ be two groups. A function $f : G \longrightarrow H$ is said to be a homomorphism if the following equality in H holds for any two elements $a, b \in G$:

$$f(a * b) == f(a) \star f(b).$$

If f is also bijective we say that f is an isomorphism.

(a) Let $f : G \longrightarrow H$ be a homomorphism. Show that f has the two properties below.

  (i) $f(e_G) = e_H$ ("preserves neutral elements");

  (ii) $f(a^{-1}) = f(a)^{-1}$ ("preserves inverses").

(b) Suppose $f : G \longrightarrow H$ is an isomorphism. Prove that the inverse function $f^{-1} : H \longrightarrow G$ is (automatically) a homomorphism. That is, check that

$$f^{-1}(x \star y) == f^{-1}(x) * f^{-1}(y) \qquad \forall x, y \in H.$$

(`Hint`: Use that f is injective.)

Notation and terminology: When a function f is an isomorphism we often add a tilde above the arrow as in $f : G \xrightarrow{\sim} H$. We say the two groups G and H are isomorphic and write $G \simeq H$ if there is <u>some</u> isomorphism between them (in which case there may be many).

Problem B. Define a function $f : \mathbb{C}^\times \longrightarrow \mathrm{GL}_2(\mathbb{R})$ by the following formula

$$f(a + ib) = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right).$$

(a) Check that f is a homomorphism. Is f injective? Is f surjective?

(b) Verify that f takes the complex unit circle $\mathbb{C}^1$ into the group $\mathrm{SO}_2(\mathbb{R})$ of rotation matrices $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$. Prove that the resulting map

$$f : \mathbb{C}^1 \longrightarrow \mathrm{SO}_2(\mathbb{R})$$

is an isomorphism. List all matrices of the form $f(z)$ for some $z \in U_3$.

Problem C. Fix an integer $N > 0$ and let $\zeta = \cos(\frac{2\pi}{N}) + i\sin(\frac{2\pi}{N})$. We introduce the two linear transformations $R, S : \mathbb{C} \longrightarrow \mathbb{C}$ defined as follows. For $z \in \mathbb{C}$,

$$R(z) =\!= \zeta z \qquad\qquad S(z) =\!= \bar{z} \ \ (\text{"complex conjugation"}).$$

(a) Explain the geometric effect of R, S in the complex plane. What do they do – how do they transform the point z in geometric terms?

(b) Check the relations $R^N = \mathrm{Id}$ and $S^2 = \mathrm{Id}$. Then show that

$$\boxed{R \circ S =\!= S \circ R^{N-1}}$$

(c) Deduce from part (b) that the group generated by R and S is isomorphic to the dihedral group D_N.

Problem D. Let $(G, *)$ be a group. The <u>commutator</u> of $a, b \in G$ is defined as

$$[a, b] =\!= a * b * a^{-1} * b^{-1}.$$

(a) Show that $[a, b] = e$ if and only if a, b commute ($a * b = b * a$).

(b) For the dihedral group D_N, show that every commutator is an **even** power of R and vice versa. (R=rotation by $\frac{2\pi}{N}$ in the counterclockwise direction.)

(c) By part (b) the set of commutators in D_N coincides with the subgroup $\langle R^2 \rangle$. What is its size? (`Hint`: Divide into two cases according to whether N is even or odd.)