HW4 SOLUTION

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Only problems with provided solutions will be graded. Solutions might be concise for some problems, but please be noticed that they don't reflect the wanted level of detailedness of your answer.

Armstrong.1.4
1pt for each for correct answer of the symmetries below. 2pts for general completeness.
flip along the axis determined by the midpoints of 13 and 24: $(rs)^{-1}s(rs) = sr^2srs$.
rotate $\frac{2\pi}{3}$ along the axis determined by the point 3: $(sr)^{-1}r(sr) = r^2srsr$.
rotate $\frac{4\pi}{3}$ along the axis determined by the point 3: $(sr)^{-1}r^2(sr) = r^2sr^2sr$.
Correct answers may take forms different from above.

Armstrong.1.5
Because rotate along the L-axis by angel $\frac{2\pi}{3}$ three times we get the original position back, $r^3 = e$. Because flip twice along the M-axis we get the original position back, $s^2 = e$.
Clearly $r^3 = e$ implies $r^{-1} = rr$ and $s^2 = e$ implies $s = s^{-1}$. Also

$$(rs)^{-1} = srr \iff srrrs = e \iff ses = e \iff s^2 = e,$$
$$(sr)^{-1} = rrs \iff e = rrssr \iff e = rrer \iff e = r^3,$$

which is what we want.

Problem.A
(a) For (i), consider

$$f(e_G) = f(e_G * e_G) = f(e_G) * f(e_G),$$

multiplying $f(e_G)^{-1}$ to both sides, we obtain $f(e_G) = e_H$.
For (ii) just check

$$f(a^{-1}) * f(a) = f(a^{-1} * a) = f(e_G) = e_H,$$
$$f(a) * f(a^{-1}) = f(a * a^{-1}) = f(e_G) = e_H,$$

and conclude by the definition (or uniqueness) of inverse.

Problem.C
(a) R: rotate counterclockwisely by angel $\frac{2\pi}{N}$. S: reflection with respect to the y-axis.
(b) Take an arbitrary $z = x + yi \in \mathbb{C}$.
$R^N(z) = \underbrace{R \circ R \cdots \circ R}_{n-times}(z) = \zeta^N z = z$ b/c $\zeta^N = 1$ by trigonometry. $S^2(z) = \bar{\bar{z}} =$

$\overline{x-yi} = x + yi = z$ by definition of conjugation.
To show $R \circ S = S \circ R^{N-1}$, we compute

$$R \circ S(z) = \zeta \bar{z}$$

and

$$S \circ R^{N-1}(z) = \overline{\zeta^{N-1} z} = \overline{\zeta^{N-1}} \bar{z}.$$

Again by trigonometry, $\zeta^{N-1} = \cos(\frac{2\pi(N-1)}{N}) + \sin(\frac{2\pi(N-1)}{N})i$ and hence

$$\begin{aligned}\overline{\zeta^{N-1}} &= \cos(\frac{2\pi(N-1)}{N}) - \sin(\frac{2\pi(N-1)}{N})i \\ &= \cos(-\frac{2\pi(N-1)}{N}) + \sin(-\frac{2\pi(N-1)}{N})i = \cos(\frac{2\pi}{N} - 2\pi) + \sin(\frac{2\pi}{N} - 2\pi)i \\ &= \cos(\frac{2\pi}{N}) + \sin(\frac{2\pi}{N})i = \zeta.\end{aligned}$$

Hence $R \circ S(z) = S \circ R^{N-1}(z)$ as wanted.
(c) By theorem in lecture notes 15, we know if R, S satisfies the relation proved in (b), we can list the elements of the group generated by R, S

$$\langle R, S\rangle = \{R^0, R^1, \dots, R^{N-1}\} \cup \{SR^0, SR^1, \dots, SR^{N-1}\}.$$

Similarly, if we denote the rotation and reflection in the dihedral group D_N as r, s respectively, we have

$$D_N = \{r^0, r^1, \dots, r^{N-1}\} \cup \{sr^0, sr^1, \dots, sr^{N-1}\}.$$

We define a map as follows

$$f : \langle R, S\rangle \to D_N, \quad R^i \mapsto r^i, SR^i \mapsto sr^i.$$

Clearly f is bijective, so it suffices to check f is a homomorphism. Take arbitrary $a, b \in \langle R, S\rangle$, WTS $f(ab) = f(a)f(b)$. There are four cases.
Case I: $a = R^i$ and $b = R^j$ for some i, j. Then $f(R^iR^j) = f(R^{i+j}) = r^{i+j} = r^ir^j = f(R^i)f(R^j)$ as wanted.
Case II: $a = SR^i$ and $b = R^j$ for some i, j. Then $f(SR^iR^j) = f(SR^{i+j}) = sr^{i+j} = sr^ir^j = f(SR^i)f(R^j)$ as wanted.
Case III: $a = R^i$ and $b = SR^j$ for some i, j. Then $R^iSR^j = SR^{j-i}$ by the rule $RS = SR^{-1}$. Similarly $r^isr^j = sr^{j-i}$ as $rs = sr^{-1}$. So $f(R^iSR^j) = f(SR^{j-i}) = sr^{j-i} = r^isr^j = f(R^i)f(SR^j)$ as wanted.
Case IV: $a = SR^i$ and $b = SR^j$ for some $i.j$. Similarly to the previous case we have $SR^iSR^j = R^{j-i}$ and $sr^isr^j = r^{j-i}$. So $f(SR^iSR^j) = f(R^{j-i}) = r^{j-i} = sr^isr^j = f(SR^i)f(SR^j)$ as wanted.

Problem.D
(a) $[a, b] = e$ means $aba^{-1}b{-}1 = e$, and we multiply ba to the right of both side, we see this identity is equivalent to $ab = ba$.
(b) Take arbitrary $a, b \in D_N$, we want to compute $[a, b]$. There are four cases.
Case I: $a = R^i$ and $b = R^j$ for some i, j. Then $[R^i, R^j] = e = R^0$ because R^i and R^j commute.
Case II: $a = SR^i$ and $b = R^j$ for some i, j. Then

$$\begin{aligned}[SR^i, R^j] &= SR^iR^j(SR^i)^{-1}(R^j)^{-1} = SR^iR^jR^{-i}SR^{-j} \\ &= SR^jSR^{-j} = SSR^{-j}R^{-j}, \text{ following from } RS = SR^{-1}\end{aligned}$$

$$= R^{-2j}$$

as wanted.
Case III: $a = R^i$ and $b = SR^j$ for some i, j. Similarly we can compute $[R^i, SR^j] = R^{2i}$.
Case IV: $a = SR^i$ and $b = SR^j$ for some i, j. Similarly we can compute $[SR^i, SR^j] = R^{2(j-i)}$.
So $[a, b]$ can always be written as R^{2k} for some $k \in \mathbb{Z}$. Conversely, $\forall k \in Z$, by our computation in Case III, $R^{2k} = [R^k, S]$.
(c) We know the dihedral group consists of $D_N = \{R^0, R^1, \ldots, R^{N-1}\} \cup \{SR^0, SR^1, \ldots, SR^{N-1}\}$, we wish to find the number of distinct elements $a \in D_N$ s.t. $a = R^{2k}$ for some $k \in \mathbb{Z}$.
If N is odd: Clearly $SR^i \neq R^{2k}$ for any choice of k. Consider every R^i with $i = 0, \ldots, N-1$. If i is even then $i = 2k$ for some k. If i is odd, then $R^i = R^{N+i}$ where $N + i$ is even. So every such R^i is in $\langle R^2 \rangle$. In particular, $|\langle R^2 \rangle| = N$.
If N is even: Clearly $SR^i \neq R^{2k}$ for any choice of k. Consider every R^i with $i = 0, \ldots, N-1$. $R^i = R^{2k}$ is equivalent to $N | i - 2k$. Since N is even, so $i - 2k$ is even hence i is even. So we only have those elements $\{R^i; 0 \leq i \leq N-1, 2|i\}$. In particular $|\langle R^2 \rangle| = \frac{N}{2}$.