From Armstrong’s Groups and Symmetry:

- Exercises (Chapter 6, pages 30–31):
  6.2, 6.3, 6.6, 6.8

**Problem A.** Let \( \alpha \in S_7 \) be the permutation given by \( \alpha = (5137)(1352) \).

(a) Express \( \alpha \) in array form. That is, fill in the blank boxes below.

\[
\alpha = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\square & \square & \square & \square & \square & \square & \square
\end{pmatrix}
\]

(b) Is \( \alpha \) a cycle? If not, find its decomposition into disjoint cycles.

(c) Compute \( \text{ord}(\alpha) \) and \( \text{sign}(\alpha) \). Does \( \alpha \) belong to \( A_7 \)?

**Problem B.** Let \( \alpha \in S_9 \) be the permutation given by its array form:

\[
\alpha = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
8 & 9 & 1 & 7 & 6 & 2 & 4 & 3 & 5
\end{pmatrix}
\]

(a) Write \( \alpha \) as a composition of disjoint cycles. Then write \( \alpha \) as a composition of transpositions.

(b) Is \( \alpha \) even or odd? What is the order of \( \alpha \)?

(c) Express \( \alpha^{2019} \) and \( \alpha^{-2019} \) in array form.

**Problem C.** Consider the subset \( V \subseteq A_4 \) consisting of the identity \( e \) and the three permutations

\[
\alpha = (12)(34) \quad \beta = (13)(24) \quad \gamma = (14)(23).
\]
(a) Verify that \( V \) is a subgroup of \( A_4 \), and write down the composition table for \( V \). Deduce that \( V \) is isomorphic to Klein’s four-group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \).

(b) Explain why \( V \) contains precisely the elements of \( A_4 \) of order at most two.

(c) Deduce from part (b) that \( \delta V \delta^{-1} = V \) for all \( \delta \in S_4 \). (This means \( V \) is a “normal” subgroup of \( S_4 \).)

**Problem D.**

(a) Let \((ab)\) be an arbitrary transposition in \( S_n \). For any permutation \( \alpha \in S_n \), prove the formula:
\[
\alpha(ab)\alpha^{-1} = (\alpha(a)\alpha(b)).
\]
(Check that the left-hand side interchanges \( \alpha(a) \) and \( \alpha(b) \)).

(b) Suppose some \( \alpha \in S_n \) commutes with every other element of \( S_n \), such as \((ab)\). Use part (a) to conclude that \( \alpha = e \), assuming that \( n > 2 \).

(c) Now suppose an \( \alpha \in A_n \) commutes with every element of \( A_n \). Is it true or false that \( \alpha \) must be the identity \( e \)? Assume that \( n > 3 \). (Hint: Extend the formula in part (a) to all 3-cycles \((abc)\).)