Math 103A, Fall 2019

Modern Algebra I, HW 6

Due Friday November 15th at 11:30AM in Ji Zeng's box outside B402A.

From Armstrong's Groups and Symmetry:

- Exercises (Chapter 6, pages 30–31):

  6.2, 6.3, 6.6, 6.8

Problem A. Let $\alpha \in S_7$ be the permutation given by $\alpha = (5137)(1352)$.

(a) Express α in array form. That is, fill in the blank boxes below.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \square & \square & \square & \square & \square & \square & \square \end{pmatrix}$$

(b) Is α a cycle? If not, find its decomposition into disjoint cycles.

(c) Compute $\mathrm{ord}(\alpha)$ and $\mathrm{sign}(\alpha)$. Does α belong to A_7?

Problem B. Let $\alpha \in S_9$ be the permutation given by its array form:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 9 & 1 & 7 & 6 & 2 & 4 & 3 & 5 \end{pmatrix}.$$

(a) Write α as a composition of disjoint cycles. Then write α as a composition of transpositions.

(b) Is α even or odd? What is the order of α?

(c) Express α^{2019} and α^{-2019} in array form.

Problem C. Consider the subset $V \subseteq A_4$ consisting of the identity e and the three permutations

$$\alpha = (12)(34) \qquad \beta = (13)(24) \qquad \gamma = (14)(23).$$

(a) Verify that V is a subgroup of A_4, and write down the composition table for V. Deduce that V is isomorphic to Klein's four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) Explain why V contains precisely the elements of A_4 of order at most two.

(c) Deduce from part (b) that $\boxed{\delta V \delta^{-1} = V}$ for all $\delta \in S_4$. (This means V is a "normal" subgroup of S_4.)

Problem D.

(a) Let (ab) be an arbitrary transposition in S_n. For any permutation $\alpha \in S_n$, prove the formula:

$$\alpha(ab)\alpha^{-1} = (\alpha(a)\alpha(b)).$$

(Check that the left-hand side interchanges $\alpha(a)$ and $\alpha(\beta)$.)

(b) Suppose some $\alpha \in S_n$ <u>commutes</u> with every other element of S_n, such as (ab). Use part (a) to conclude that $\alpha = e$, assuming that $n > 2$.

(c) Now suppose an $\alpha \in A_n$ commutes with every element of A_n. Is it true or false that α must be the identity e? Assume that $n > 3$. (`Hint`: Extend the formula in part (a) to all 3-cycles (abc).)