Math 103A, Fall 2019

Modern Algebra I, HW 7

Due Friday November 22nd at 11:30AM in Ji Zeng's box outside B402A.

From Armstrong's Groups and Symmetry:

- Exercises (Chapter 6, pages 30–31):

  6.5, 6.11

- Exercises (Chapter 7, page 36):

  7.11, 7.12

Problem A. Let $\alpha \in S_9$ be the permutation $\alpha = (19462)(679)(735)$.

(a) Express α in array form. That is, fill in the blank boxes below.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \square & \square & \square & \square & \square & \square & \square & \square & \square \end{pmatrix}$$

(b) Write α as a composition of disjoint cycles. Then write α as a composition of transpositions.

(c) Compute $\operatorname{ord}(\alpha)$ and $\operatorname{sign}(\alpha)$. Does α belong to A_9?

Problem B. Fix an $n > 2$ and consider the alternating group A_n.

(a) Check that any 3-cycle (abc) can be decomposed as $(abc) = (ab)(bc)$.

(b) Let (ab) and (cd) be two disjoint transpositions in S_n. Show that

$$(ab)(cd) = (abc)(bcd).$$

(c) Deduce from parts (a) and (b) that A_n is generated by the set of all 3-cycles. (That is, every $\alpha \in A_n$ can be written as a composition of a finite number of 3-cycles.)

Problem C.

(a) Consider the additive group $(\mathbb{Z}_6, +)$ of all residue classes modulo 6, and let $H = \langle [2] \rangle$ be the subgroup of order 3.

  (i) Find the index $[\mathbb{Z}_6 : H]$.

  (ii) List all the elements of the two cosets $[2] + H$ and $[3] + H$.

(b) Consider the multiplicative group $(\mathbb{Z}_7^\times, \bullet)$ of all invertible residue classes modulo 7, and let $K = \langle [6] \rangle$ be the subgroup of order 2.

  (i) Find the index $[\mathbb{Z}_7^\times : K]$.

  (ii) List all the elements of each of the three cosets:

$$[1] \bullet K \qquad [2] \bullet K \qquad [3] \bullet K.$$

Problem D. Let $V \subset A_4$ be the subgroup $\{e, \alpha, \beta, \gamma\}$ introduced in Problem C on Homework 6.

(a) Compute the indices $[S_4 : V]$ and $[A_4 : V]$.

(b) List all the elements of the cosets $(123) \circ V$ and $(132) \circ V$. (Each is a subset consisting of four 3-cycles. Give each of the 3-cycles $(123) \circ \alpha = (abc)$ etc.)

(c) Prove or disprove that $(123) \circ V = V \circ (123)$.

(d) Find the cycle decomposition of all the elements of $(12) \circ V$.