HW7 SOLUTION

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Only problems with provided solutions will be graded. Solutions might be concise for some problems, but please be noticed that they don't reflect the wanted level of detailedness of your answer.

Armstrong.6.5
For $\alpha = (143)$, because α is even, $\alpha P(x_1, x_2, x_3, x_4) = P(x_1, x_2, x_3, x_4)$.
For $\alpha = (23)(412)$, because α is odd, $\alpha P(x_1, x_2, x_3, x_4) = -P(x_1, x_2, x_3, x_4)$.

Armstrong.7.11
Write $\alpha = (1234)$, $\beta = (56)$ and denote the subgroup generated by α, β as G in S_6. We first investigate what G looks like. Because $\alpha\beta = \beta\alpha$ can be easily checked true, an arbitrary element $g \in G$ can be written as $g = \alpha^a \beta^b$. Also we have $\alpha^4 = \beta^2 = e$, so we can restrict our consideration for a (resp. b) to take value in $\{0, 1, 2, 3\}$ (resp. $\{0, 1\}$). So we have

$$G = \{e, \alpha, \alpha^2, \alpha^3, \beta, \beta\alpha, \beta\alpha^2, \beta\alpha^3\}.$$

Now we define a homomorphism

$$f : G \to H := \{[1], [3], [9], [7], [11], [13], [17], [19]\} \subset \mathbb{Z}_{20}$$

by $f(\alpha) = [3]$, $f(\beta) = [11]$ and extend through homomorphism property. We can check, $f(\alpha)f(\beta) = f(\beta)f(\alpha)$, $f(\alpha)^4 = [1]$ and $f(\beta)^2 = [1]$. So this homomorphism is well-defined.
Finally we check f is bijective. To do this we can write out the correspondence explicitly, we have

$$e \mapsto [1], \alpha \mapsto [3], \alpha^2 \mapsto [9], \alpha^3 \mapsto [7]$$
$$\beta \mapsto [11], \beta\alpha \mapsto [13], \beta\alpha^2 \mapsto [19], \beta\alpha^3 \mapsto [17],$$

which proves the wanted bijectiveness.

Armstrong.7.12
Write $\alpha = (1234)$, $\beta = (24)$ and denote the subgroup generated by α, β as G in S_4. We first investigate what G looks like. We can check that $\beta\alpha\beta^{-1} = (1432) = \alpha^{-1}$, hence $\beta\alpha = \alpha^{-1}\beta$. We also have $\beta^2 = \alpha^4 = e$. So by theorem in lecture notes 15 (or similar argument as in Armstrong.7.11), we have

$$G = \{e, \alpha, \alpha^2, \alpha^3, \beta, \beta\alpha, \beta\alpha^2, \beta\alpha^3\}.$$

Now we define a homomorphism $f : G \to D_4$ by $f(\alpha) = r$ the rotation, $f(\beta) = s$ the reflection and extend through homomorphism property. We can check $f(\beta)f(\alpha) = f(\alpha)^{-1}f(\beta)$, $f(\alpha)^4 = e$ and $f(\beta)^2 = e$. So this homomorphism is well-defined.
Finally we check f is bijective by explicitly write out the correspondence. Indeed,

$\beta^i\alpha^j \mapsto s^i r^j$ for all $i \in \{0,1\}$ and $j \in \{0,1,2,3\}$, which proves bijectiveness.

Problem.B

(a) Just check $(ab)(bc)$ permutes $a \mapsto b$, $b \mapsto c$, $c \mapsto a$ and fixes every $d \neq a, b, c$.
(b) Just check $(abc)(bcd)$ permutes $a \mapsto b$, $b \mapsto a$, $c \mapsto d$, $d \mapsto c$ and fixes every $e \neq a, b, c, d$.
(c) For an arbitrary $\alpha \in A_n$, we can write

$$\alpha = (a_1 b_1)(a_2 b_2) \dots (a_{2k} b_{2k}).$$

For every $1 \leq i \leq k$, if $(a_{2i-1} b_{2i-1})$ and $(a_{2i} b_{2i})$ are disjoint, then by (b) we can write $(a_{2i-1} b_{2i-1})(a_{2i} b_{2i}) = (a_{2i-1} b_{2i-1} a_{2i})(b_{2i-1} a_{2i} b_{2i})$. If $(a_{2i-1} b_{2i-1})$ and $(a_{2i} b_{2i})$ share an element in common, WLOG say $b_{2i-1} = a_{2i}$, then $(a_{2i-1} b_{2i-1})(a_{2i} b_{2i}) = (a_{2i-1} a_{2i} b_{2i})$. If $(a_{2i-1} b_{2i-1})$ and $(a_{2i} b_{2i})$ share two elements, $(a_{2i-1} b_{2i-1})(a_{2i} b_{2i}) = e$. So we can write α as a product of 3-cycles.

Problem.D

(a) $[S_4 : V] = 6$ and $[A_4 : V] = 3$.
(b) $(123)V = \{(123), (134), (243), (142)\}$ and $(132)V = \{(132), (234), (241), (143)\}$.
(c) Take $\delta = (123)$, by HW6 Problem.C.(c) $\delta V \delta^{-1} = V$. Multiplying δ to both side from the right, we get $\delta V = V\delta$, which is what we want.
(d) $(12)V = \{(12), (34), (1324), (1423)\}$.