From Armstrong’s Groups and Symmetry:

- Exercises (Chapter 11, page 60):
  11.3, 11.4
- Exercises (Chapter 15, pages 84–85):
  15.12, 15.14

Problem A. Consider the alternating group $A_5$, and let $H \subset A_5$ be the subgroup generated by the 5-cycle (12345).

(a) Give the index $[A_5 : H]$.

(b) List all the elements of the left coset $(123) \circ H$ and the right coset $H \circ (123)$.
   (Decompose all permutations into disjoint cycles.) Are they the same?

(c) Does $A_5$ have a subgroup of index 18?

Problem B.

(a) Let $(G, \ast)$ be a group with a subgroup $H \subset G$ of index two. Show that

$$a^2 \in H$$

for all elements $a \in G$. Is $H$ automatically a normal subgroup of $G$?

(b) Use part (a) to prove that $A_n$ does not admit a subgroup of index two for any $n > 2$. (Hint: Observe that $(abc) = (acb)^2$ and use the result of Problem B part (c) on Homework 7.)

(c) Find all integers $N > 0$ for which there exists a surjective homomorphism

$$f : A_4 \rightarrow \mathbb{Z}_N.$$
Problem C. Define a function $f : \mathbb{R} \rightarrow \mathbb{C}^\times$ by the formula

$$f(x) = \cos(2\pi x) + i\sin(2\pi x).$$

(a) Verify that $f$ is a homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{C}^\times, \cdot)$.

(b) Find $\text{im}(f)$ and $\ker(f)$.

(c) Check that $f(\mathbb{Q}) = \{ f(x) : x \in \mathbb{Q} \}$ coincides with the group $\bigcup_{N=1}^{\infty} U_N$ from Exercise 3.4 on page 14 in Armstrong’s book (cf. Homework 2).

Problem D.

(a) Check that $[3]$ generates $\mathbb{Z}_7^\times$ and conclude that there is a unique homomorphism $f : \mathbb{Z}_7^\times \rightarrow \mathbb{Z}_{12}$ such that $f([3]) = [4]$. (Be careful here. $\mathbb{Z}_{12}$ is an additive group, whereas $\mathbb{Z}_7^\times$ is multiplicative.)

(b) Find all the other values of $f$:

$$f([1]) \quad f([2]) \quad f([3]) \quad f([4]) \quad f([5]) \quad f([6]).$$

(c) List all the elements of $\text{im}(f)$ and $\ker(f)$.

Problem E. Please complete your CAPE teaching evaluation forms. The deadline is Monday December 9th at 8AM. Thank you so much, it has been a pleasure teaching this class!