Math 103A, Fall 2019

Modern Algebra I, HW 8

Due MONDAY December 2nd at 11:30AM in Ji Zeng's box outside B402A.

From Armstrong's Groups and Symmetry:

- Exercises (Chapter 11, page 60):

  11.3, 11.4

- Exercises (Chapter 15, pages 84–85):

  15.12, 15.14

Problem A. Consider the alternating group A_5, and let $H \subset A_5$ be the subgroup generated by the 5-cycle (12345).

(a) Give the index $[A_5 : H]$.

(b) List all the elements of the left coset $(123) \circ H$ and the right coset $H \circ (123)$. (Decompose all permutations into disjoint cycles.) Are they the same?

(c) Does A_5 have a subgroup of index 18?

Problem B.

(a) Let $(G, *)$ be a group with a subgroup $H \subset G$ of index two. Show that

$$a^2 \in H$$

for all elements $a \in G$. Is H automatically a **normal** subgroup of G?

(b) Use part (a) to prove that A_n does not admit a subgroup of index two for any $n > 2$. (`Hint`: Observe that $(abc) = (acb)^2$ and use the result of Problem B part (c) on Homework 7.)

(c) Find all integers $N > 0$ for which there exists a surjective homomorphism

$$f : A_4 \longrightarrow \mathbb{Z}_N.$$

Problem C. Define a function $f : \mathbb{R} \longrightarrow \mathbb{C}^\times$ by the formula

$$f(x) == \cos(2\pi x) + i\sin(2\pi x).$$

(a) Verify that f is a homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{C}^\times, \bullet)$.

(b) Find $\mathrm{im}(f)$ and $\ker(f)$.

(c) Check that $f(\mathbb{Q}) = \{f(x) : x \in \mathbb{Q}\}$ coincides with the group $\bigcup_{N=1}^{\infty} U_N$ from Exercise 3.4 on page 14 in Armstrong's book (cf. Homework 2).

Problem D.

(a) Check that $[3]$ generates $\mathbb{Z}_7^\times$ and conclude that there is a unique homomorphism $f : \mathbb{Z}_7^\times \longrightarrow \mathbb{Z}_{12}$ such that $f([3]) = [4]$. (Be careful here. $\mathbb{Z}_{12}$ is an additive group, whereas $\mathbb{Z}_7^\times$ is multiplicative.)

(b) Find all the other values of f:

$$f([1]) \qquad f([2]) \qquad f([3]) \qquad f([4]) \qquad f([5]) \qquad f([6]).$$

(c) List all the elements of $\mathrm{im}(f)$ and $\ker(f)$.

Problem E. Please complete your CAPE teaching evaluation forms. The deadline is Monday December 9th at 8AM. Thank you so much, it has been a pleasure teaching this class!