HW8 SOLUTION

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Only problems with provided solutions will be graded. Solutions might be concise for some problems, but please be noticed that they don't reflect the wanted level of detailedness of your answer.

Armstrong.11.3
$H \cap K$ is a subgroup of H and K, so $|H \cap K|$ divides $|H|$ as well as $|K|$. As $|H|$ and $|K|$ are coprime, $|H \cap K| = 1$, so $H \cap K = \{e\}$.

Armstrong.11.4
Write $|G| = pq$ for two primes p, q. Let H be arbitrary proper subgroup of G, so $|H||pq$ and $|H| \neq pq$. So we have $|H| = 1$ or $|H| = p$ or $|H| = q$. So we conclude by the fact that every group of prime order is cyclic.

Armstrong.15.12
A proper normal subgroup of A_4 is $V = \{e, (12)(34), (13)(24), (14)(23)\}$ or the trivial group $\{e\}$.
$(12345)^{-1}(345)^{-1}(12345)(345) = (245)$ and $(12)(34)(345)^{-1}(12)(34)(345) = (354)$.
Let H be a non-trivial normal subgroup of A_5. Pick $e \neq h \in H$. $\forall \delta \in A_5$, as H is normal $\delta^{-1}h\delta \in H$. So $h^{-1}\delta^{-1}h\delta \in H$. Consider the cycle decomposition of h, in A_5 h could only be a 5-cycle or $2, 2$-cycle. If $h = (abcde)$, take $\delta = (cde)$, by above computation, $\delta^{-1}h\delta$ is a 3-cycle. If $h = (ab)(cd)$, take (cde), by above computation, $\delta^{-1}h\delta$ is a 3-cycle. Hence H has a 3-cycle always. Let's call this 3-cycle α.
By Armstrong.14.5, every 3-cycle β is conjugate to $\alpha \in H$, i.e. $\exists \delta \in A_5$ s.t. $\beta = \delta\alpha\delta^{-1}$. As H is normal, $\beta \in H$ as well. By arbitrariness of β, we conclude that H has all the 3-cycles, so by HW7 Problem.B $A_5 \subset H$. So $H = A_5$, contradicting our assumption that H is proper in A_5.

Problem.B
(a) If $a \in H$, then $a^2 \in H$ as H is a subgroup. If $a \notin H$, then $aH \cap H = \emptyset$, as $[|G| : |H|] = 2$ $G = H \cup aH$. So either $a^2 \in H$ or $a^2 \in aH$, but it's impossible for $a^2 \in aH$, otherwise $a^2 = ah$ for some $h \in H$ so $a = h \in H$ a contradiction.
H is normal. For an arbitrary $a \in G$, we want to show $aH = Ha$. If $a \in H$, then $aH = H = Ha$ by group property. If $a \in H$, by above argument, $aH = G \backslash H = Ha$.
(b) Because A_n is generated by 3-cycles, it suffices to prove for any arbitrary 3-cycle (abc), then it's in H. Indeed, $(abc) = (acb)^2$, so apply part.(a) with $a = (abc)$. So $H = A_n$ is not of index two.
(c) By isomorphism theorem we have $\frac{A_4}{\ker(f)} \simeq \frac{\mathbb{Z}}{N\mathbb{Z}}$, so $[A_4 : \ker(f)] = N$ and N must divide $|A_4| = 12$.
$N = 1$: The trivial homomorphism $f : A_4 \to \{e\}$ works.

$N = 2$: No such f exists because otherwise $\ker(f)$ would be of index 2, contradicting part.(b).
$N = 3$: We have a normal subgroup $V \subset A_4$ of order 4, we have the projection homomorphism

$$f : A_4 \to A_4/V, \alpha \mapsto \alpha V.$$

As $|\frac{A_4}{V}| = 3$ is a prime, $\frac{A_4}{V} \simeq \frac{\mathbb{Z}}{3\mathbb{Z}}$.
$N = 4$: No such f exists because otherwise $\ker(f)$ would be a normal subgroup of order 3. As 3 is a prime, $\ker(f)$ is cyclic. The only kind of elements of order 3 in A_4 are 3-cycles. So $\ker(f) = \langle\alpha\rangle$ for some 3-cycle. However, such a group cannot be normal.
$N = 6$: No such f exists because otherwise $\ker(f)$ would be a normal subgroup of order 2. As 2 is a prime, $\ker(f)$ is cyclic. The only kind of elements of order 2 in A_4 are transpositions. So $\ker(f) = \langle\alpha\rangle$ for some transposition. However, such a group cannot be normal.
$N = 12$: No such f exists because such f would be an isomorphism. But A_4 is not abelian, while $\frac{\mathbb{Z}}{12\mathbb{Z}}$ is.

Problem.D
(a) $\langle[3]\rangle = \{[3], [2], [6], [4], [5], [1]\}$. Because we have $6 \cdot [4] = [0]$. So we can define $f : \mathbb{Z}_7^* \to \mathbb{Z}_{12}$ by $f([3]^k) = k[4]$.
(b) $f([1]) = [0]$, $f([2]) = [8]$, $f([4]) = [4]$, $f([5]) = [8]$ and $f([6]) = [0]$.
(c) $\ker(f) = \{[1], [6]\}$ and $\mathrm{im}(f) = \{[0], [4], [8]\}$.