LECTURE 14

(Wednesday Oct. 30, 2019)
Recall: \( A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \) represents \underline{rotation} by \( \theta \) (counterclockwise).

All matrices satisfying: \( A^T A = I \) and \( \det(A) = 1 \).

Form a group "orthogonal" "special"

\[ \text{SO}_2(\mathbb{R}) = \text{special orthogonal group in } \mathbf{GL}_2(\mathbb{R}) \]

Example. Let \( \theta = \frac{2\pi}{N} \) (cf. HW0)

Then \( A^n = \text{rotation by } \frac{2\pi n}{N} \)

i.e., \( \text{ord}(A) = N \).

\[ G = \{ I, A, A^2, \ldots, A^{N-1} \} = \langle A \rangle \]

finite cyclic subgroup of \( \text{SO}(2) \).

Geometrically

\[ G = \{ \text{rotations preserving regular } N \text{-gon} \} \]

\( N = 4 \)

\[ \theta = \frac{\pi}{2} \]

(centered at the origin)
REFLECTIONS:  \[ A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]

all \(2 \times 2\)-matrices s.t.

\[ A^T A = I \text{ and } \det(A) = -1. \]

The "orthogonal" group in \(O_2(\mathbb{R})\):

\[ O_2(\mathbb{R}) = \{ A : A^T A = I \} \]

= \{ rotations \} \cup \{ reflections \}

\[ \text{det} = +1 \quad \text{det} = -1 \]

fix a regular \(N\)-gon, \(P\)

centered at the origin.

— by the "symmetry" of \(P\) we mean

all rotations & reflections mapping \(P\) to itself.

\[ D_N = \{ A \in O_2(\mathbb{R}) : A(P) = P \} \]

"dihedral group"

\[ |D_N| = 2N \quad \text{(later)} \]

BEWARE:  \( |D_N| = 2N \) (later).
The square has 4 rotations preserving it: \( \{ e, r, r^2, r^3 \} \)

- rotation by \( \frac{\pi}{2} \) (counterclockwise)

and 4 axes of symmetry \( \rightarrow 4 \) reflections preserving it.

\( s = \) reflection across the horizontal axis.

- What's \( rs \)? What happens on a basis?

\[
\begin{align*}
e_1 \xrightarrow{s} & \ x \quad e_1 \xrightarrow{r} \ x \\
e_2 \xrightarrow{s} & \ x \quad -e_2 \xrightarrow{r} \ x
\end{align*}
\]

so \( rs \) swaps \( \{ e_1, e_2 \} \), i.e.

\( rs = \) reflection across the \( \frac{\pi}{4} \) line \( \text{(1)} \)

- How about \( sr \)?

\[
\begin{align*}
e_1 \xrightarrow{r} & \ x \quad e_2 \xrightarrow{s} \ x \\
e_2 \xrightarrow{r} & \ x \quad -e_1 \xrightarrow{s} \ x
\end{align*}
\]

\( sr = \) reflection across the \( \frac{3\pi}{4} \) line \( \text{(2)} \)

\( rs \neq sr \)

\( D_4 \) non-abelian.