LECTURE 14

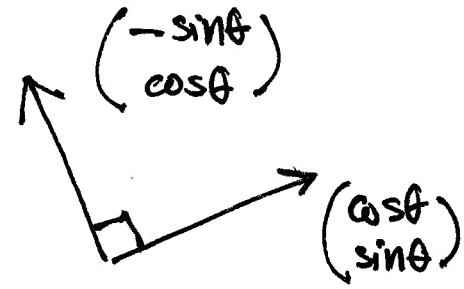
(Wednesday OCT. 30, 2019)

- Recall: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ represents <u>rotation</u> by θ (counterclockwise).

- all matrices satisfying: $A^T A = I$ <u>and</u> $\det(A) = 1$.
  Form a <u>group</u> "orthogonal" "special"

$$SO_2(\mathbb{R}) \quad (= \underline{\text{special}}\ \underline{\text{orthogonal}}\ \underline{\text{group}} \text{ in } GL_2(\mathbb{R})).$$

<u>Example</u> Let $\theta = \frac{2\pi}{N}$ (cf. HW0)



Then A^n ($=$ rotation by $\frac{2\pi}{N} n$)

i.e., $\mathrm{ord}(A) = N$.

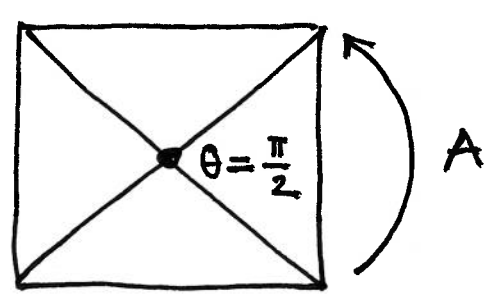
$$G = \{ I, A, A^2, \ldots, A^{N-1} \} = \langle A \rangle$$

finite cyclic subgroup of $SO(2)$.

- geometrically

$$G = \{ \text{rotations preserving regular } N\text{-gon} \}.$$

$N = 4$

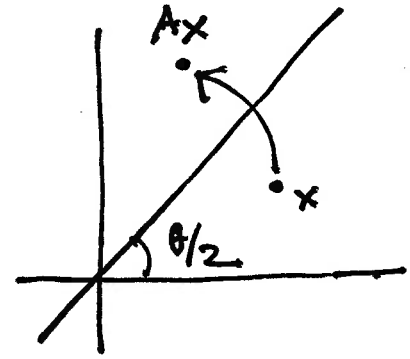


(centered at the origin)

REFLECTIONS: $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ across the $\frac{\theta}{2}$-line

all 2×2-matrices s.t.

$$A^T A = I \text{ and } \det(A) = -1.$$



(exc. ← linear algebra)

Hint: A fixes $\begin{pmatrix} \cos\theta + 1 \\ \sin\theta \end{pmatrix}$ on the line.)

OBSERVE: $A^2 = I$.

The "orthogonal" group in $GL_2(\mathbb{R})$:

$$O_2(\mathbb{R}) = \{A : A^T A = I\}$$

$$= \underset{(\det = +1)}{\{\text{rotations}\}} \cup \underset{(\det = -1)}{\{\text{reflections}\}}$$

Fix a regular N-gon, P centered at the origin.

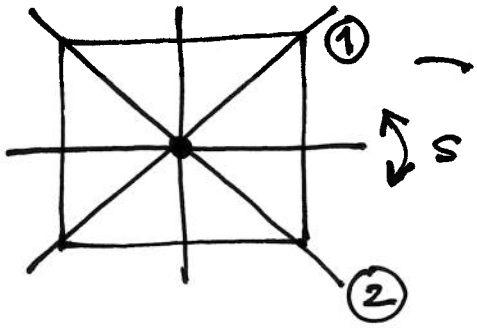
— by the "symmetries" of P we mean all rotations & reflections mapping P to itself.

$$D_N = \{A \in O_2(\mathbb{R}) : A(P) = P\}$$

"dihedral group"

BEWARE: (notation..) $|D_N| = 2N$ (later).

Ex ($N=4$) The square has 4 rotations



— preserving it: $\{e, r, r^2, r^3\}$

$r =$ rotation by $\frac{\pi}{2}$ (counterclockwise)

and 4 axes of symmetry $\rightarrow$ 4 reflections preserving it.

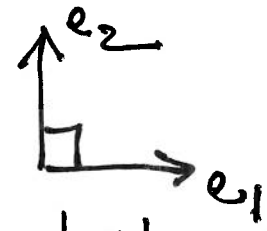
$s =$ reflection across the horizontal axis.

geometry: "clever choice"

— What's rs? What happens on a basis?

$$e_1 \xmapsto{s} e_1 \xmapsto{r} e_2$$

$$e_2 \xmapsto{s} -e_2 \xmapsto{r} e_1$$



(standard basis).

so rs swaps $\{e_1, e_2\}$, i.e.

$rs =$ reflection across the $\frac{\pi}{4}$-line (1)

— How about sr?

$$e_1 \xmapsto{r} e_2 \xmapsto{s} -e_2$$

$$e_2 \xmapsto{r} -e_1 \xmapsto{s} -e_1$$

shows $sr =$ reflection across the $\frac{3\pi}{4}$-line (2)

$$\boxed{rs \neq sr}$$

D_4 non-abelian.