LECTURE 15

(Friday Nov. 1, 2019)

(e.g. "horizontal")

- For arbitrary N, choose a preferred axis of symmetry of the N-gon P.

$$\begin{cases} s = \text{reflection across this axis.} \\ r = \text{rotation by } \frac{2\pi}{N} \text{ (counterclockwise)} \end{cases}$$

Satisfy $r^N = e$, $s^2 = e$, and the "Commutation relation":

$$\boxed{rs = sr^{N-1}} . \quad (\star)$$

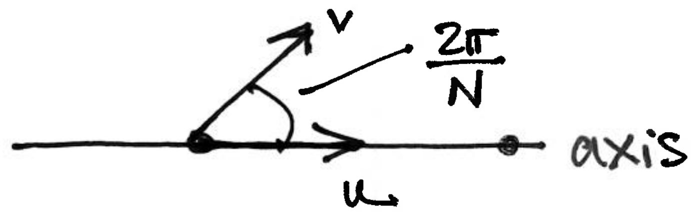
Why? Both sides are linear transformations of $\mathbb{R}^2$.
- check they agree on a basis.

any! Pick $\{u, v\}$

From the geometry:

$$u \xmapsto{s} u \xmapsto{r} v$$

$$v \xmapsto{s} \bar{v} \xmapsto{r} u$$



so the LHS swaps u, v. (both unit vectors say)

For the RHS of $(\star)$: Note $r^{N-1} = r^{-1}$ clockwise rotation by $\frac{2\pi}{N}$.

$$u \xmapsto{r^{-1}} \bar{v} \xmapsto{s} v$$

$$v \xmapsto{r^{-1}} u \xmapsto{s} u$$

also swaps $u \longleftrightarrow v$

so both sides of $(\star)$ represents reflection in the $\frac{\pi}{N}$ — line ✓

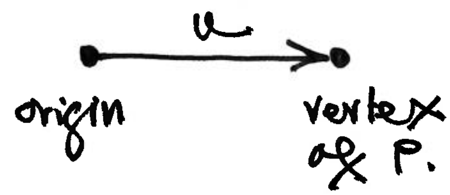
Thm.

$$D_N = \{e, r, r^2, \ldots, r^{N-1}\} \cup \{s, sr, sr^2, \ldots, sr^{N-1}\}$$

(rotations) (reflections)

In particular $|D_N| = 2N$.

PROOF. Suppose A is a symmetry of P. Then Au is one of the vertices: (A preserves length)



Write $Au = r^i u$, some $0 \leq i < N$.

Then: $r^{-i}A$ fixes the vector u

must be s or the identity $(= e)$

$$\Rightarrow A = r^i s \quad \text{or} \quad A = r^i.$$

$$= sr^{-i}$$

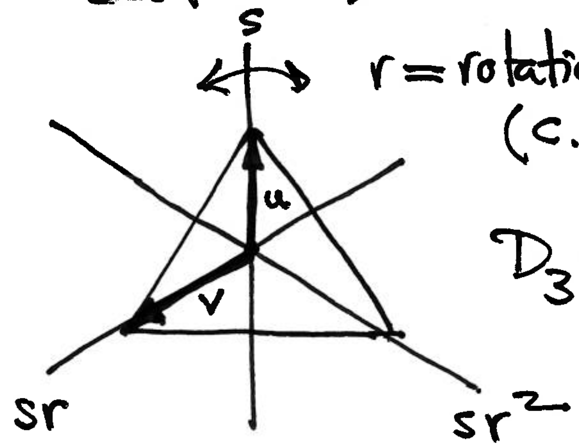
$$= sr^{N-i}. \quad \square$$

EXC: $(sr^i)(sr^j) = s(r^i s)r^j = s(sr^{-i})r^j = r^{j-i}$

reflections

rotation.

EX ($N=3$)



$r =$ rotation by $\frac{2\pi}{3}$ (c.c.)

$$D_3 = \{e, r, r^2\} \cup \{s, sr, sr^2\}$$

(composition table ~ p. 17)

ex. $r(sr) = (sr^{-1})r = s.$

– In this case

$$D_3 \longrightarrow \left\{ \begin{array}{l} \text{permutations of} \\ \text{the 3 vertices} \end{array} \right\} = S_3$$

is bijective (in fact "isomorphism" when S_3 equipped with composition $\circ$)

FAR from true for $N > 3$: $|D_N| = 2N < |S_N| = N!$.

(only injective $D_N \to S_N$)

$SO_3(\mathbb{R})$ consists of all 3×3 — matrices s.t.

$$A^T A = I \quad \underline{\text{and}} \quad \det(A) = 1.$$
("orthogonal") ("special")

<u>Note</u>: A fixes some $u \neq 0$ in $\mathbb{R}^3$

(i.e., $Au = u$ meaning $\lambda = 1$ is an <u>eigenvalue</u>)

<u>Why</u>? $\det(A - I) = 0$?

$$(-1)^3 = -1$$

$$\underbrace{\det(A^T)}_{1} \cdot \det(A - I) = \det(A^T A - A^T) = \det(I - A) = -\det(A - I).$$

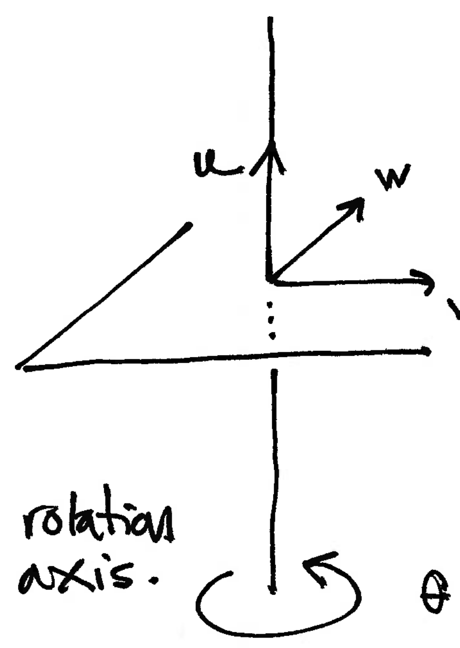
— May normalize $\|u\| = 1$,

extend to an <u>orthonormal</u> <u>basis</u> (ONB): $\{u, v, w\}$.

relative to this basis the linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ is $x \mapsto Ax$ rep. by the matrix:

$$A \sim \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{array}\right)$$

"conjugate" (cf. MT1)



PLATONIC SOLIDS.