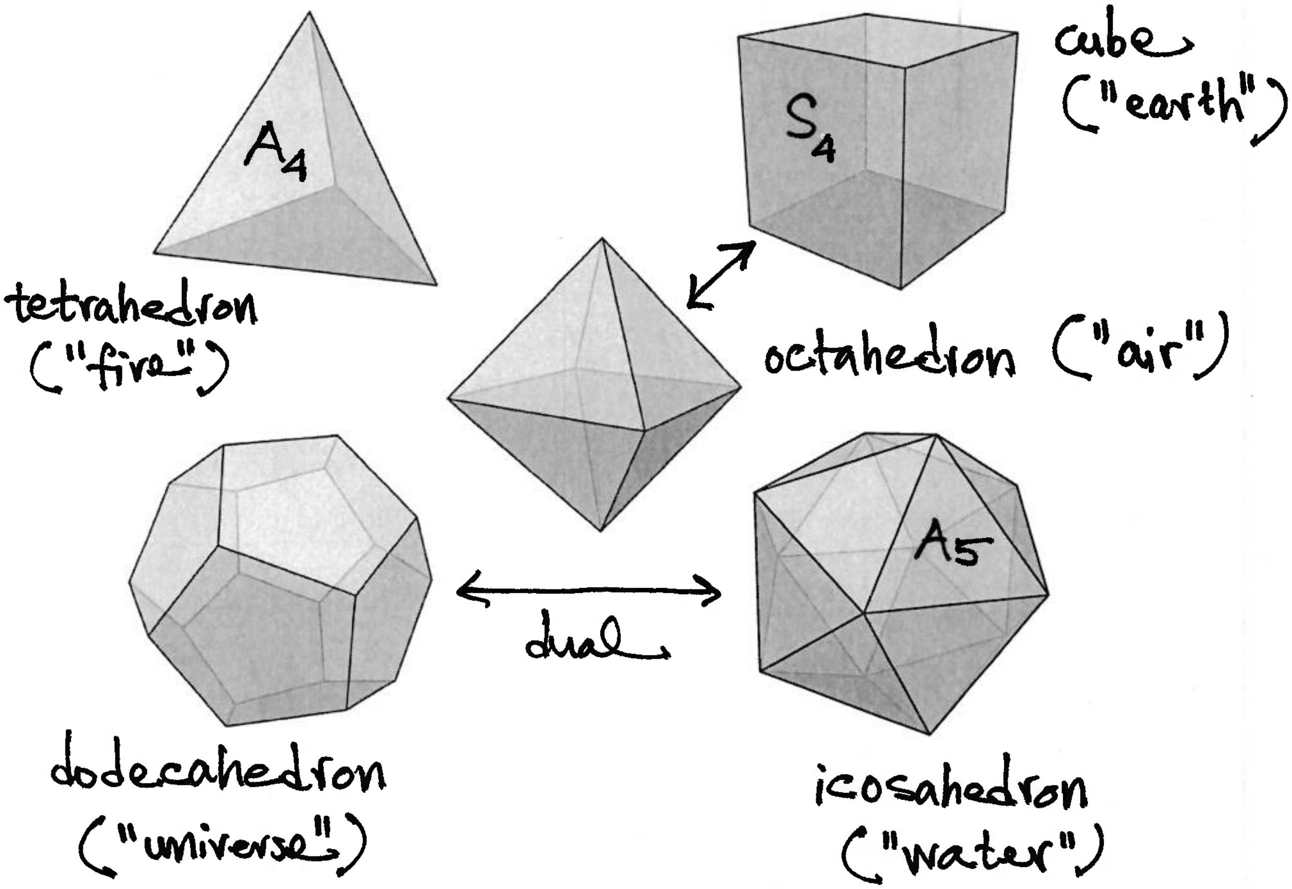
LECTURE 16
(Monday Nov. 4, 2019)

PLATONIC SOLIDS.



$SO_3(\mathbb{R})$ consists of all 3×3 — matrices s.t.

$$\underset{(\text{"orthogonal"})}{A^TA = I} \quad \underline{\text{and}} \quad \underset{(\text{"special"})}{\det(A) = 1.}$$

Note: A fixes some $u \neq 0$ in $\mathbb{R}^3$

(i.e., $Au = u$ meaning $\lambda = 1$ is an eigenvalue)

Why? $\det(A - I) = 0$?

$$\underbrace{\det(A^T)}_{1} \cdot \det(A - I) = \det(A^TA - A^T) = \det(I - A) \overset{(-1)^3 = -1}{=} -\det(A - I).$$

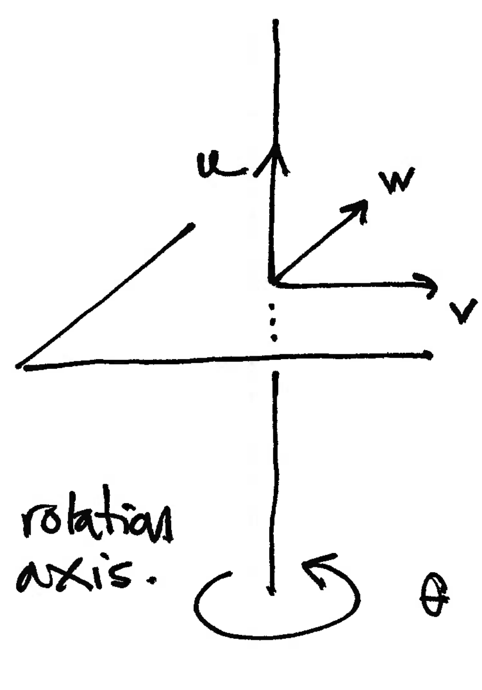
— May normalize $\|u\| = 1$,

extend to an orthonormal basis (ONB): $\{u, v, w\}$.

relative to this basis the linear transformation $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is
$x \longmapsto Ax$
rep. by the matrix:

$$A \sim \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{array}\right)$$

"conjugate"
(cf. MT1)

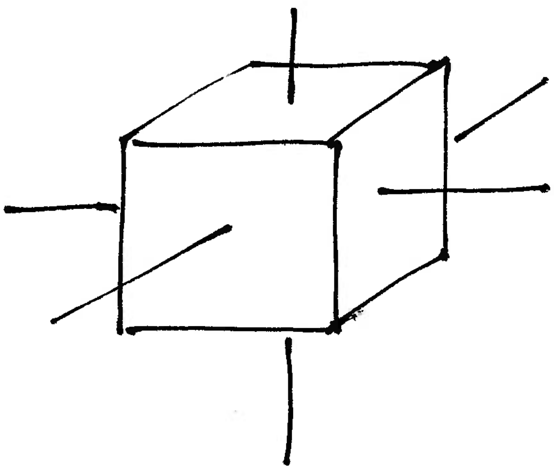


$P =$ regular polyhedron centered at the origin.
(PLATONIC SOLID)

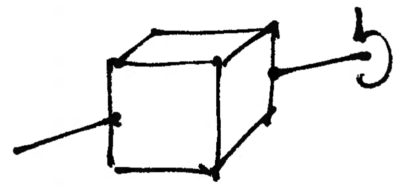
$G = \{ A \in SO_3(\mathbb{R}) : \ A(P) = P \}$,

(rotational) symmetry group of P.

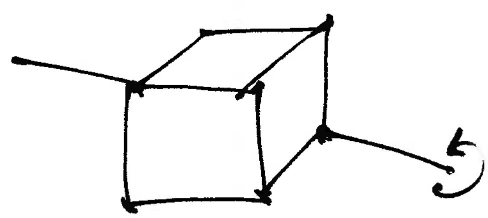
EX $P =$ cube.



- the identity e
- 3 face axes, each contributing 3 rotations $\neq e$
- 6 side axes, —"— —"— 1 rotation $\neq e$



- 4 diagonal axes, —"— —"— 2 rotations $\neq e$



In total:

$|G| = 1 + 3\cdot 3 + 6 \cdot 1 + 4 \cdot 2$

$= \boxed{24}$.

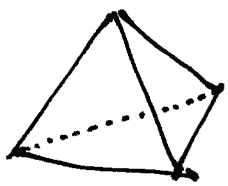
Any $A \in G$ permutes the 4 diagonals

$G \longrightarrow \{$ permutations of the diagonals $\}$ isomorphism

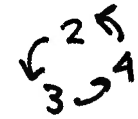
(injective map between sets of size $24 = 4!$)

Shows $G \cong S_4$.

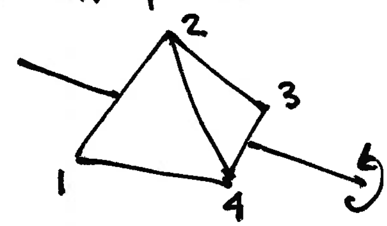
EX $P =$ tetrahedron.



- identity e
- 4 face axes, each giving 2 rotations $\neq e$. (fix 1)



- 3 side axes, —"— —"— 1 rot. $\neq e$



interchanges (1,2) and (3,4)

In total

$$|G| = 1 + 4\cdot 2 + 3\cdot 1 = \boxed{12}$$

Here G permutes the 4 vertices, but the injection

$$G \longrightarrow \left\{\begin{array}{l}\text{permutations of} \\ \text{the vertices}\end{array}\right\}$$

is not an isomorphism.

will show:

$G \cong A_4$ (= THE subgroup of S_4 of size 12)

"even permutations"

NOTE: $P =$ Icosahedron has $G \cong A_5$

— in general $|A_n| = \frac{n!}{2}$

(has size $|A_5| = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{2} = 60$)