LECTURE 17

(Wednesday NOV. 6, 2019)

Def.

A <u>permutation</u> of a set X is a bijective function $\alpha : X \longrightarrow X$
$x \longmapsto \alpha(x)$

RECALL: $(\alpha \circ \beta)(x) = \alpha(\beta(x))$.

The "symmetric group" on X is:

$$S_X = \{\text{all permutations of } X\}$$ — with composition $\circ$ (of functions).

neutral: $e = Id_X$ $(x \mapsto x)$

Usually $X = \{1, 2, 3, \dots, n\}$ (or any finite set), in which case we write S_n instead of $S_{\{1,2,\dots,n\}}$.

<u>Note:</u> $|S_n| = n!$
$(= 1 \cdot 2 \cdot 3 \cdots n)$

Combinatorics:

n possible $\alpha(1)$
$n-1$ — " — $\alpha(2)$ (can't be $\alpha(1)$)
$n-2$ — " — $\alpha(3)$
$\vdots$
1 — " — $\alpha(n)$

<u>Notation</u> ("array form"): A permutation
$\alpha : \{1, 2, \dots, n\} \longrightarrow \{1, 2, \dots, n\}$
is often written

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \alpha(1) & \alpha(2) & \alpha(3) & & \alpha(n) \end{pmatrix}$$

<u>not</u> a matrix!

outputs — no duplicates.
(all numbers $1, \dots, n$ in some order)

EX ($n=3$) S_3 contains the 6 permutations:

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad \beta\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \beta\alpha^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Note:

$$\alpha\beta = \alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

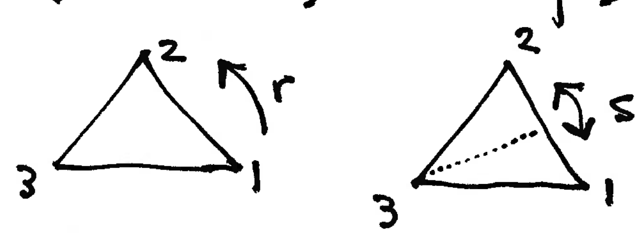
$$\beta\alpha = \beta \circ \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

so non-abelian. CHECK: $\alpha\beta = \beta\alpha^2$

(and $\alpha^3 = \beta^2 = e$)

— may deduce S_3 is isomorphic to D_3

$(r \longleftrightarrow \alpha, \ s \longleftrightarrow \beta)$



* NOT true that $S_n \cong D_n$ for $n > 3$.

$|S_n| = n! > |D_n| = 2n.$

__Notation__ ("cycles"): Suppose $x_1, x_2, \ldots, x_m$ $\quad (m \leq n)$ is an __ordered__ list of distinct elements from $\{1, 2, \ldots, n\}$.

not necessarily in increasing order!

Then: $(x_1\, x_2 \cdots x_m)$ means the permutation of __order__ m:

"m-cycle".

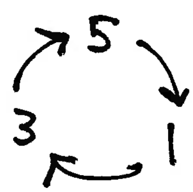
$$x_1 \mapsto x_2 \mapsto x_3 \mapsto \cdots \mapsto x_m$$

__EX__ $\alpha = (5\,1\,3) \in S_5$

is the permutation (3-cycle) (fixing every x __not__ on the list)

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$$

fixed fixed.



__Obs__: $\alpha = (5\,1\,3) = (1\,3\,5) = (3\,5\,1)$. $\quad$ (__not__ $= (5\,3\,1)$)

Try to show:

$$\#\{m\text{-cycles in } S_n\} = (m-1)!\binom{n}{m}.$$

(select $x_1, \ldots, x_m$ in $\binom{n}{m}$ ways; put x_1 first)

$\boxed{m = 2}$: 2-cycles are called "__transpositions__"

($\alpha = (a\,b)$ just __swaps__ a, b and fixes everything else)

$$a \longleftrightarrow b.$$

1)

CYCLE → ARRAY form: Given $\alpha = (1372)(7294)$ in S_9.
express α in array form.

EX: $7 \xmapsto{(7294)} 2 \xmapsto{(1372)} 1$. (same for 4)

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 9 & 7 & 2 & 5 & 6 & 1 & 8 & 4 \end{pmatrix}.$$

2) **ARRAY → CYCLE form:** Given

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 7 & 1 & 6 & 4 & 8 & 2 & 5 & 3 \end{pmatrix}$$

express α as a composition of cycles.
(disjoint)

— work out its "orbits":

$1 \mapsto 9 \mapsto 3 \mapsto 1$ $\quad (193)$

$2 \mapsto 7 \mapsto 2$ $\quad (27)$

$4 \mapsto 6 \mapsto 8 \mapsto 5 \mapsto 4$ $\quad (4685)$

Call others.

conclude:

$$\alpha = (193) \circ (27) \circ (4685)$$

all three cycles are disjoint.