LECTURE 18
(Friday NOV. 8, 2019)

Theorem Every $\alpha \in S_n$ can be written as a composition of disjoint cycles (unique up to reordering the cycles)

$$\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_r$$

disjoint cycles commute.

each α_i is a cycle, of length m_i say — disjoint from α_j for all $j \neq i$.

(OMIT formal proof)

Cor. $\operatorname{ord}(\alpha) = \operatorname{LCM}(m_1, m_2, \ldots, m_r)$

PF. $\alpha^m = \alpha_1^m \circ \cdots \circ \alpha_r^m = e \iff$ all $\alpha_i^m = e$

$\iff m_i \mid m$ for all i. $\square$

EX (previous ex. cont.) $\alpha = (193)(27)(4685)$ has

$$\operatorname{ord}(\alpha) = \operatorname{LCM}(3,2,4) = 12.$$

EX Order of $(1372)(7294)$? (cf. (1) above)

not disjoint!

Found its array form.
— read off it cycle decomposition.

$1 \mapsto 3 \mapsto 7 \mapsto 1$

$2 \mapsto 9 \mapsto 4 \mapsto 2$

$5 \mapsto 5$

$6 \mapsto 6$

$8 \mapsto 8$

$(137)(294)$

disjoint.

$\Rightarrow$ Order $=$ LCM$(3,3) = 3$.

Thm. S_n is generated by transpositions.
(I.e., every $\alpha \in S_n$ can be written as a composition of swaps (a,b).) — Not at all uniquely!

PROOF: May assume α is a cycle (by decomposing into disjoint cycles).
Say
$$\alpha = (x_1 x_2 \cdots x_m).$$
One way to factor α into 2-cycles is:
$$(x_1 x_2 \cdots x_m) = (x_1 x_m)(x_1 x_{m-1}) \cdots (x_1 x_2) \quad (\star)$$
["Transposition Formula"]; why? RHS takes
$$\left.\begin{array}{l} x_1 \mapsto x_2 \\ x_2 \mapsto x_1 \mapsto x_3 \\ x_3 \mapsto x_1 \mapsto x_4 \\ \vdots \\ x_m \mapsto x_1 \end{array}\right\} \text{ same effect as LHS.}$$
$\square$

Ex. $\alpha = (5341) = (51)(54)(53)$
(Also, $\alpha = (5341) = (3415) =$
$(35)(31)(34)$.)
non-unique.

Important special case ($m=3$):
$$(abc) = (ac)(ab).$$

"MAGIC": Any decomposition of α into transpositions has either an EVEN or ODD number of swaps.

$\rightsquigarrow$ say α is an even resp. odd permutation.

EX: By (★) an m-cycle is even iff m odd.
(so 3-cycles, 5-cycles etc. are all EVEN; transpositions (ab) are ODD.)

EX Let $\alpha = (193)(27)(4685) \in S_9$.
$\rightsquigarrow$ Is α even or odd?

$$\alpha = \underbrace{(13)(19)(27)(45)(48)(46)}_{\text{even \# factors.}}$$

Def. $\operatorname{sign}(\alpha) = \begin{cases} +1 & \alpha \text{ even} \\ -1 & \alpha \text{ odd}. \end{cases}$

$\rightsquigarrow$ still have to show "MAGIC" for this to make sense...

Strategy: Show that there's a unique non-trivial homomorphism

$$\varphi = \operatorname{sign} : S_n \longrightarrow \{\pm 1\}$$

i.e., $\operatorname{sign}(\alpha \circ \beta) = \operatorname{sign}(\alpha) \cdot \operatorname{sign}(\beta)$

$\forall \alpha, \beta \in S_n$.

any transposition.

— Must have $\varphi((a,b)) = -1$: Any other transposition is of the form $(\gamma(a), \gamma(b)) = \gamma(ab)\gamma^{-1}$ for some $\gamma \in S_n$. Thus if $\varphi((a,b)) = 1$,

$$\varphi((\gamma(a), \gamma(b))) = \varphi(\gamma)\varphi((a,b))\varphi(\gamma)^{-1} = 1.$$

Therefore $\varphi =$ trivial. [Also shows uniqueness of φ.]

— Factoring $\alpha = \alpha_1\alpha_2\cdots\alpha_r$ w. all α_i being transpositions,

$$\varphi(\alpha) = \varphi(\alpha_1)\varphi(\alpha_2)\cdots\varphi(\alpha_r) = (-1)^r.$$

Shows: 1) The parity r is given by $\varphi(\alpha)$, and therefore independent of decomposition.

2) $\varphi =$ sign "well-defined".

Def. $A_n = \{$<u>even</u> permutations in $S_n\}$ <u>subgroup</u> $\leq S_n$.
"Alternating group".

$\text{sign}(\alpha\beta) =$
$\text{sign}(\alpha)\,\text{sign}(\beta) =$
$(+1)\cdot(+1) = +1$

<u>Note:</u> $\{$<u>odd</u> permutations$\}$
— do <u>not</u> form a subgroup.

observe: α even $\Longleftrightarrow$ $\alpha \circ (12)$ odd.

Therefore

$$\{\text{even permutations}\} \longleftrightarrow \{\text{odd permutations}\}$$

gives a one-to-one correspondence. $\alpha \longmapsto \alpha \circ (12)$

In particular,

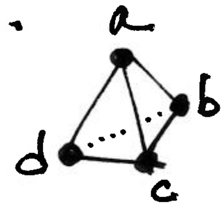
$$|A_n| = |\{\text{even perm.}\}| = |\{\text{odd perm.}\}| = \frac{n!}{2}.$$

<u>Ex</u> $(n=3)$ A_3 cyclic order 3
$(A_3 \simeq \mathbb{Z}_3)$

$$\boxed{|A_n| = \frac{n!}{2}}.$$

n	$\lvert A_n \rvert$
2	1
3	3
4	12
5	60
6	360.

/ cf. rot. symmetries of tetrahedron.



Ex ($n=4$) Possible cycle structures.

- The identity e $\quad (\# = 1)$
- 3-cycles (abc) $\quad \left(\# = 8 = \binom{4}{3} \cdot 2!\right)$
- Two disjoint transpositions $(ab)(cd)$ $\quad \left(\# = 3 = \frac{1}{2}\binom{4}{2}\right)$

  commute

(123)
(132)
(124)
(142)
(134)
(143)
(234)
(243)

$(12)(34)$
$(13)(24)$
$(14)(23)$

Exc $V = \{e, (12)(34), (13)(24), (14)(23)\}$ (all elements of order ≤ 2) form a subgroup of A_4, isomorphic to Klein's $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Note: {3-cycles} not closed under $\circ$. Indeed:
$(123)(124) = (13)(24)$.